2.2. Exact Solution for the Free Particle

For a 1-D free particle,

$$(x_b\, t_b \mid x_a\, t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}'x \int \frac{\mathcal{D}p}{2\pi\hbar} \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} dt \left(p\,\dot{x} - \frac{p^2}{2M}\right)\right] \tag{2.67}$$

or, in the configuration form,

$$(x_b\, t_b \mid x_a\, t_a) = \int \mathcal{D}x \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} dt\, \frac{1}{2} M \dot{x}^2\right) \tag{2.68}$$

where the endpoint limits were omitted as they are obvious from the L.H.S. of the equation.

2.2.1. Direct Solution

Using

$$\int_{-\infty}^{\infty} dx\, e^{i\left(a x^2 + b x\right)} = \sqrt{\frac{i\pi}{a}}\, e^{-b^2/4a} \tag{2.68a}$$

and

$$\frac{1}{A}(x'' - x')^2 + \frac{1}{B}(x' - x)^2 = \frac{A+B}{AB} x'^2 - 2\left(\frac{x''}{A} + \frac{x}{B}\right) x' + \frac{1}{A} x''^2 + \frac{1}{B} x^2$$

$$-\left(\frac{x''}{A} + \frac{x}{B}\right)^2 \frac{AB}{A+B} + \frac{1}{A} x''^2 + \frac{1}{B} x^2 = \frac{1}{A+B}(x'' - x)^2$$

we have

$$\int_{-\infty}^{\infty} dx' \exp\left\{ i\alpha\left[\frac{1}{A}(x'' - x')^2 + \frac{1}{B}(x' - x)^2\right]\right\} = \sqrt{\frac{i\pi AB}{\alpha(A+B)}} \exp\left(i\alpha \frac{(x'' - x)^2}{A+B}\right) \tag{2.69b}$$

According to (2.52-3), the time-sliced version of (2.68) is

$$(x_b\, t_b \mid x_a\, t_a) = \left(\frac{M}{2\pi i\hbar\epsilon}\right)^{(N+1)/2} \int dx_N \exp\left[i\frac{\epsilon M}{2\hbar}\left(\frac{x_{N+1} - x_N}{\epsilon}\right)^2\right] \tag{2.69a}$$

$$\times \ldots \int dx_1 \exp\left[i\frac{\epsilon M}{2\hbar}\left(\frac{x_1 - x_0}{\epsilon}\right)^2\right]$$

which can be evaluated successively using (2.69b) with $\alpha = \frac{M}{2\hbar\epsilon}$.

Starting with the x_1 integration, we have

$$\mathcal{I}_1 = \int dx_1 \exp\left[i\alpha(x_2 - x_1)^2\right] \exp\left[i\alpha(x_1 - x_0)^2\right]$$

$$= \left(\frac{i\pi}{\alpha}\right)^{1/2} \frac{1}{\sqrt{2}} \exp\left(i\alpha \frac{(x_2 - x_0)^2}{2}\right)$$

For the x_2 integration, we have

$$\mathcal{I}_2 = \left(\frac{i\pi}{\alpha}\right)^{1/2} \frac{1}{\sqrt{2}} \int dx_2 \exp\left[i\alpha(x_3 - x_2)^2\right] \exp\left[i\alpha \frac{(x_2 - x_0)^2}{2}\right]$$

$$= \left(\frac{i\pi}{\alpha}\right)^{1/2} \frac{1}{\sqrt{2}} \left(\frac{i\pi}{\alpha}\right)^{1/2} \sqrt{\frac{1\times 2}{1+2}} \exp\left(i\alpha \frac{(x_3 - x_0)^2}{1+2}\right)$$

$$= \left(\frac{i\pi}{\alpha}\right) \frac{1}{\sqrt{3}} \exp\left(i\alpha \frac{(x_3 - x_0)^2}{3} \right)$$

Thus, for the x_n integration, we have

$$\mathcal{I}_n = \left(\frac{i\pi}{\alpha}\right)^{(n-1)/2} \frac{1}{\sqrt{n}} \int d x_n \exp\left[i\alpha (x_{n+1} - x_n)^2 \right] \exp\left[i\alpha \frac{(x_n - x_0)^2}{n} \right]$$

$$= \left(\frac{i\pi}{\alpha}\right)^{(n-1)/2} \frac{1}{\sqrt{n}} \left(\frac{i\pi}{\alpha}\right)^{1/2} \sqrt{\frac{1 \times n}{1+n}} \exp\left(i\alpha \frac{(x_{n+1} - x_0)^2}{1+n} \right)$$

$$= \left(\frac{i\pi}{\alpha}\right)^{n/2} \frac{1}{\sqrt{n+1}} \exp\left(i\alpha \frac{(x_{n+1} - x_0)^2}{n+1} \right)$$

(2.69a) thus becomes

$$(x_b\, t_b \mid x_a\, t_a) = \left(\frac{M}{2\pi i \hbar (N+1)\epsilon} \right)^{1/2} \exp\left(i \frac{M}{2\hbar} \frac{(x_{N+1} - x_0)^2}{(N+1)\epsilon} \right) \tag{2.70}$$

$$= \left(\frac{M}{2\pi i \hbar (t_b - t_a)} \right)^{1/2} \exp\left(i \frac{M}{2\hbar} \frac{(x_b - x_a)^2}{t_b - t_a} \right) \tag{2.71}$$

which agrees with the Schrodinger result (1.337). Note that (2.71) is independent of N.

Consider now replacing M with $M\,g(t)$ so that (2.67) becomes

$$(x_b\, t_b \mid x_a\, t_a) = \int_{x(t_a) = x_a}^{x(t_b) = x_b} \mathcal{D}' x \int \frac{\mathcal{D} p}{2\pi\hbar} \exp\left[\frac{i}{\hbar} \int_{t_a}^{t_b} d t \left(p\,\dot{x} - \frac{p^2}{2 M g(t)} \right) \right] \tag{2.72}$$

The 1-D version of the measure (2.60a) then becomes

$$\int_{x(t_a) = x_a}^{x(t_b) = x_b} \mathcal{D} x \sqrt{g} \equiv \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{(N+1)/2} \sqrt{g(t_b)} \lim_{N \to \infty} \left(\prod_{k=1}^{N} \sqrt{g(t_k)} \int d x_k \right) \tag{2.74}$$

so that the configuration path integral (2.68) takes the form

$$(x_b\, t_b \mid x_a\, t_a) = \int \mathcal{D} x \sqrt{g} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} d t\, \frac{1}{2} M g(t)\, \dot{x}^2 \right) \tag{2.73}$$

The time-sliced version of (2.73) is again evaluated iteratively.

Starting with the x_1 integration, we have

$$\mathcal{I}_1 = \sqrt{g(t_1)} \int d x_1 \exp\left[i\alpha \frac{(x_2 - x_1)^2}{g(t_2)^{-1}} \right] \exp\left[i\alpha \frac{(x_1 - x_0)^2}{g(t_1)^{-1}} \right]$$

$$= \left(\frac{i\pi}{\alpha}\right)^{1/2} \sqrt{\frac{g(t_2)^{-1}}{g(t_2)^{-1} + g(t_1)^{-1}}} \exp\left(i\alpha \frac{(x_2 - x_0)^2}{g(t_2)^{-1} + g(t_1)^{-1}} \right)$$

For the x_2 integration, we have

$$\mathcal{I}_2 = \left(\frac{i\pi}{\alpha}\right)^{1/2} \sqrt{\frac{g(t_2)^{-1}}{g(t_2)^{-1} + g(t_1)^{-1}}} \sqrt{g(t_2)} \int d x_2 \exp\left[i\alpha \frac{(x_3 - x_2)^2}{g(t_3)^{-1}} \right] \exp\left[i\alpha \frac{(x_2 - x_0)^2}{g(t_2)^{-1} + g(t_1)^{-1}} \right]$$

$$= \left(\frac{i\pi}{\alpha}\right)^{1/2} \sqrt{\frac{1}{g(t_2)^{-1} + g(t_1)^{-1}}} \left(\frac{i\pi}{\alpha}\right)^{1/2}$$

$$\times \sqrt{\frac{g(t_3)^{-1}\left[g(t_2)^{-1} + g(t_1)^{-1}\right]}{g(t_3)^{-1} + g(t_2)^{-1} + g(t_1)^{-1}}} \exp\left(i\alpha \frac{(x_3 - x_0)^2}{g(t_3)^{-1} + g(t_2)^{-1} + g(t_1)^{-1}}\right)$$

$$= \left(\frac{i\pi}{\alpha}\right) \sqrt{\frac{g(t_3)^{-1}}{g(t_3)^{-1} + g(t_2)^{-1} + g(t_1)^{-1}}} \exp\left(i\alpha \frac{(x_3 - x_0)^2}{g(t_3)^{-1} + g(t_2)^{-1} + g(t_1)^{-1}}\right)$$

Thus, for the x_n integration, we have

$$\mathcal{I}_n = \left(\frac{i\pi}{\alpha}\right)^{(n-1)/2} \sqrt{\frac{g(t_n)^{-1}}{\sum_{k=1}^{n} g(t_k)^{-1}}} \sqrt{g(t_n)} \int d x_n \exp\left[i\alpha \frac{(x_{n+1} - x_n)^2}{g(t_{n+1})^{-1}}\right] \exp\left[i\alpha \frac{(x_n - x_0)^2}{\sum_{k=1}^{n} g(t_k)^{-1}}\right]$$

$$= \left(\frac{i\pi}{\alpha}\right)^{(n-1)/2} \sqrt{\frac{1}{\sum_{k=1}^{n} g(t_k)^{-1}}} \left(\frac{i\pi}{\alpha}\right)^{1/2} \sqrt{\frac{g(t_{n+1})^{-1} \sum_{k=1}^{n} g(t_k)^{-1}}{\sum_{k=1}^{n+1} g(t_k)^{-1}}} \exp\left(i\alpha \frac{(x_{n+1} - x_0)^2}{\sum_{k=1}^{n+1} g(t_k)^{-1}}\right)$$

$$= \left(\frac{i\pi}{\alpha}\right)^{n/2} \sqrt{\frac{g(t_{n+1})^{-1}}{\sum_{k=1}^{n+1} g(t_k)^{-1}}} \exp\left(i\alpha \frac{(x_{n+1} - x_0)^2}{\sum_{k=1}^{n+1} g(t_k)^{-1}}\right)$$

(2.73) thus becomes

$$(x_b\, t_b \mid x_a\, t_a) = \left(\frac{M}{2\pi i \hbar \epsilon}\right)^{(N+1)/2} \left(\frac{i\pi}{\alpha}\right)^{N/2} \frac{1}{\sqrt{\sum_{k=1}^{N+1} g(t_k)^{-1}}} \exp\left(i\alpha \frac{(x_{N+1} - x_0)^2}{\sum_{k=1}^{N+1} g(t_k)^{-1}}\right)$$

$$= \sqrt{\frac{M}{2\pi i \hbar \epsilon \sum_{k=1}^{N+1} g(t_k)^{-1}}} \exp\left(i \frac{M}{2\hbar} \frac{(x_{N+1} - x_0)^2}{\epsilon \sum_{k=1}^{N+1} g(t_k)^{-1}}\right)$$

$$= \sqrt{\frac{M}{2\pi i \hbar \int_{t_a}^{t_b} d t\, g(t)^{-1}}} \exp\left(i \frac{M}{2\hbar} \frac{(x_b - x_a)^2}{\int_{t_a}^{t_b} d t\, g(t)^{-1}}\right) \tag{2.75}$$

This has the Fourier representation

$$(x_b\, t_b \mid x_a\, t_a) = \int \frac{d p}{2\pi\hbar} \exp\left\{\frac{i}{\hbar}\left[p(x_b - x_a) - \frac{p^2}{2M} \int_{t_a}^{t_b} d t\, g(t)^{-1}\right]\right\} \tag{2.76}$$

which can be verified using (2.68a) to evaluate (2.76).

2.2.2. Fluctuations around the Classical Path

The classical path of a free particle is the straightline

$$x_{\rm cl}(t) = x_a + \frac{x_b - x_a}{t_b - t_a}(t - t_a) \tag{2.77}$$

that is the solution to the equation of motion

$$\ddot{x}_{\rm cl} = 0 \tag{2.78}$$

Paths in the path integral can be written as

$$x(t) = x_{\rm cl}(t) + \delta x(t) \tag{2.79}$$

where $\delta x(t)$ represents quantum fluctuations from the classical path. Obviously

$$\delta x(t_a) = \delta x(t_b) = 0 \tag{2.80}$$

In mathematics, conditions of the type (2.80) are called Dirichlet boundary conditions.

Inserting (2.79) into the free particle action gives

$$\mathcal{A}[x] = \frac{1}{2} M \int_{t_a}^{t_b} d\,t \left\{ \dot{x}_{\text{cl}}^2(t) + 2\, \dot{x}_{\text{cl}}(t)\, \delta\, \dot{x}(t) + [\delta\, \dot{x}(t)]^2 \right\}$$

Using (2.78) and (2.80), we have

$$\int_{t_a}^{t_b} d\,t\, \dot{x}_{\text{cl}}(t)\, \delta\, \dot{x}(t) = \dot{x}_{\text{cl}}(t)\, \delta\, x(t) \Big|_{t_a}^{t_b} - \int_{t_a}^{t_b} d\,t\, \ddot{x}_{\text{cl}}(t)\, \delta\, x(t) \;= 0$$

so that

$$\mathcal{A}[x] = \frac{1}{2} M \int_{t_a}^{t_b} d\,t \left\{ \dot{x}_{\text{cl}}^2(t) + [\delta\, \dot{x}(t)]^2 \right\}$$

This absence of a mixed term is a general consequence of the least action principle,

$$\delta\, \mathcal{A} \Big|_{x(t) = x_{\text{cl}}(t)} = 0 \tag{2.81}$$

so that a fluctuation expansion around the classical action

$$\mathcal{A}_{\text{cl}} \equiv \mathcal{A}[x_{\text{cl}}] \tag{2.82}$$

have no linear terms in $\delta\, x(t)$, i.e.,

$$\mathcal{A} = \mathcal{A}_{\text{cl}} + \frac{1}{2} \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t' \left. \frac{\delta^2 \mathcal{A}}{\delta\, x(t)\, \delta\, x(t\,')} \right|_{x = x_{\text{cl}}} \delta\, x(t)\, \delta\, x(t\,') + \ldots \tag{2.83}$$

$$= \mathcal{A}_{\text{cl}} + \mathcal{A}_{\text{fl}} + \ldots$$

where

$$\mathcal{A}_{\text{fl}} = \frac{1}{2} \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t' \left. \frac{\delta^2 \mathcal{A}}{\delta\, x(t)\, \delta\, x(t\,')} \right|_{x = x_{\text{cl}}} \delta\, x(t)\, \delta\, x(t\,') \tag{2.83a}$$

Keeping only two terms, the amplitude becomes

$$(x_b\, t_b \mid x_a\, t_a) \approx e^{i \mathcal{A}_{\text{cl}} / \hbar}\, F_0(t_b,\, t_a) \tag{2.84}$$

where the fluctuation factor is

$$F_0(t_b,\, t_a) = \int \mathcal{D}\, \delta\, x(t)\; e^{i \mathcal{A}_{\text{fl}} / \hbar} \tag{2.84a}$$

For the free particle,

$$\mathcal{A}_{\text{cl}} = \frac{1}{2} M \int_{t_a}^{t_b} d\,t\, \dot{x}_{\text{cl}}^2(t) \tag{2.85}$$

$$F_0(t_b,\, t_a) = \int \mathcal{D}\, \delta\, x(t)\; \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} d\,t\, \frac{M}{2} [\,\delta\, \dot{x}(t)\,]^2 \right\} \tag{2.86}$$

Using (2.77) on (2.85), we get

$$\mathcal{A}_{\text{cl}} = \frac{1}{2} M \frac{(x_b - x_a)^2}{t_b - t_a} \tag{2.87}$$

Using the time-slice technique, (2.86) becomes

$$F_0^N = \left(\frac{M}{2\, \pi\, i\, \hbar\, \epsilon} \right)^{(N+1)/2} \left(\prod_{n=1}^{N} \int d\, \delta\, x_n \right) e^{i \mathcal{A}_{\text{fl}}^N / \hbar} \tag{2.88}$$

where

$$\mathcal{A}_{\text{fl}}^N = \frac{1}{2} M\, \epsilon \sum_{n=1}^{N+1} \left(\frac{\delta\, x_n - \delta\, x_{n-1}}{\epsilon} \right)^2 \tag{2.89}$$

2.2.3. Fluctuation Factor

The remainder of this section will be devoted to calculating the fluctuation factor (2.88).

For convenience, we shall drop the δ symbol.

In dealing with time-sliced, or finite difference, quantities, it is useful to define the difference operator (or lattice derivative) ∇ and its conjugate $\overline{\nabla}$ as

$$\nabla f(t) \equiv \frac{1}{\epsilon}\left[f(t+\epsilon) - f(t) \right] \qquad \overline{\nabla} f(t) \equiv \frac{1}{\epsilon}\left[f(t) - f(t-\epsilon) \right] \tag{2.90}$$

Note that (or lattice)

$$\nabla \,,\; \overline{\nabla} \underset{\epsilon\to 0}{\longrightarrow} \frac{\partial}{\partial t} \tag{2.91}$$

For the fluctuation factor (2.89), we have $x(t_n) = x_n$ and

$$\nabla x_n = \frac{1}{\epsilon}(x_{n+1} - x_n) \qquad \text{for} \qquad N \geq n \geq 0$$

$$\overline{\nabla} x_n = \frac{1}{\epsilon}(x_n - x_{n-1}) \qquad \text{for} \qquad N+1 \geq n \geq 1 \tag{2.92}$$

so that, with all the δ symbols dropped,

$$\mathcal{A}_{\text{fl}}^N = \frac{1}{2} M\epsilon \sum_{n=0}^{N} (\nabla x_n)^2 = \frac{1}{2} M\epsilon \sum_{n=1}^{N+1} (\overline{\nabla} x_n)^2 \tag{2.93}$$

Using (2.91), we have

$$\lim_{N\to\infty} \mathcal{A}_{\text{fl}}^N = \frac{1}{2} M \int_{t_a}^{t_b} dt\, \dot{x}^2 \tag{2.94}$$

Lattice derivatives have properties quite similar to ordinary derivatives except for the necessity to distinguish between ∇ and $\overline{\nabla}$. For example, analogous to the integration by parts

$$\int_{t_a}^{t_b} dt\, g(t)\, \dot{f}(t) = g(t)\, f(t) \Big|_{t_a}^{t_b} - \int_{t_a}^{t_b} dt\, \dot{g}(t)\, f(t) \tag{2.95}$$

we have the summation by parts,

$$\epsilon \sum_{n=1}^{N+1} g_n \overline{\nabla} f_n = g_n f_n \Big|_0^{N+1} - \epsilon \sum_{n=0}^{N} (\nabla g_n) f_n \tag{2.96}$$

Proof is similar to the rewriting used in (2.35):

$$\begin{aligned} L.H.S. &= \sum_{n=1}^{N+1} g_n (f_n - f_{n-1}) \\ &= g_{N+1} f_{N+1} - g_{N+1} f_N + g_N f_N - g_N f_{N-1} + \ldots + g_1 f_1 - g_1 f_0 \\ &= g_{N+1} f_{N+1} - \sum_{n=0}^{N} (g_{n+1} - g_n) f_n - g_0 f_0 \\ &= R.H.S. \end{aligned}$$

For functions that vanishes at the end points, e.g.,

$$f_0 = f_{N+1} = 0$$

the surface term in (2.96) can be dropped,

$$\begin{aligned} \sum_{n=1}^{N+1} g_n \overline{\nabla} f_n &= - \sum_{n=0}^{N} (\nabla g_n) f_n \\ &= - \sum_{n=1}^{N} (\nabla g_n) f_n = - \sum_{n=1}^{N+1} (\nabla g_n) f_n \end{aligned} \tag{2.97}$$

The same is true if both f and g are periodic, i.e.,

$$f_n = f_{n+N+1} \qquad \text{and} \qquad g_n = g_{n+N+1}$$

$$\rightarrow \quad \sum_{n=1}^{N+1} g_n \overline{\nabla} f_n = -\sum_{n=0}^{N} (\nabla g_n) f_n$$
$$= -\sum_{n=1}^{N+1} (\nabla g_n) f_n + (\nabla g_{N+1}) f_{N+1} - (\nabla g_0) f_0$$
$$= -\sum_{n=1}^{N+1} (\nabla g_n) f_n \qquad (2.98)$$

since

$$(\nabla g_{N+1}) f_{N+1} - (\nabla g_0) f_0 = \frac{1}{\epsilon}(g_{N+2} - g_{N+1}) f_{N+1} - \frac{1}{\epsilon}(g_1 - g_0) f_0 = 0$$

By (2.80), we can use (2.97) to get

$$\sum_{n=1}^{N+1} (\overline{\nabla} x_n)^2 = -\sum_{n=1}^{N+1} (\nabla \overline{\nabla} x_n) x_n \qquad (2.99)$$

Similarly, using (2.96) in reverse, we have

$$\sum_{n=0}^{N} (\nabla x_n)^2 = -\sum_{n=1}^{N+1} x_n (\overline{\nabla} \nabla x_n) \qquad (2.100)$$

Note that by (2.93), both (2.99) & (2.100) are equal to $\frac{2}{M\epsilon}\mathcal{F}^N$.

Using (2.90) or (2.92), we have

$$\nabla \overline{\nabla} f_n = \frac{1}{\epsilon}\nabla(f_n - f_{n-1}) = \frac{1}{\epsilon^2}\left[f_{n+1} - f_n - (f_n - f_{n-1})\right] = \frac{1}{\epsilon^2}(f_{n+1} - 2 f_n + f_{n-1}) \qquad (2.100a)$$

$$\overline{\nabla} \nabla f_n = \frac{1}{\epsilon}\overline{\nabla}(f_{n+1} - f_n) = \frac{1}{\epsilon^2}\left[f_{n+1} - f_n - (f_n - f_{n-1})\right] = \frac{1}{\epsilon^2}(f_{n+1} - 2 f_n + f_{n-1}) \qquad (2.100b)$$

Since this is true for all f,

$$\nabla \overline{\nabla} = \overline{\nabla} \nabla \qquad (2.100c)$$

In order to write (2.100a) in matrix form, we set

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$$

(2.100a) thus becomes

$$\left[(\nabla \overline{\nabla}) \cdot \boldsymbol{f}\right]_n = \frac{1}{\epsilon^2}(f_{n+1} - 2 f_n + f_{n-1})$$
$$= \sum_{k=1}^{N} (\nabla \overline{\nabla})_{n k} f_k$$

with the understanding that $f_0 = f_{N+1} = 0$.

The (non-zero) components of the n^{th} row of the $N \times N$ matrix $\nabla \overline{\nabla}$ are therefore

$$(\nabla \overline{\nabla})_{n, n+1} = \frac{1}{\epsilon^2} \quad (\nabla \overline{\nabla})_{n n} = -\frac{2}{\epsilon^2} \quad (\nabla \overline{\nabla})_{n, n-1} = \frac{1}{\epsilon^2}$$

Explicitly,

$$\nabla \overline{\nabla} = \overline{\nabla} \nabla = \frac{1}{\epsilon^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix} \qquad (2.102)$$

According to (2.91), this is the lattice version of $\frac{\partial^2}{\partial t^2}$ and hence called the lattice Laplacian.

(2.99-100) can now be written as

$$\sum_{n=0}^{N} (\nabla x_n)^2 = -\boldsymbol{x}^T \cdot (\overline{\nabla}\,\nabla)\cdot \boldsymbol{x} = -\sum_{n,k=1}^{N} x_n (\overline{\nabla}\,\nabla)_{nk}\, x_k$$

$$\sum_{n=1}^{N+1} (\overline{\nabla} x_n)^2 = -\boldsymbol{x}^T \cdot (\nabla\,\overline{\nabla})\cdot \boldsymbol{x} = -\sum_{n,k=1}^{N} x_n (\nabla\,\overline{\nabla})_{nk}\, x_k \qquad (2.101)$$

where

$$\boldsymbol{x}^T = (x_1 \ldots x_N)$$

Consider now the Fourier transform (beware of the differences against Kleinert's version)

$$x(t) = \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi}\, e^{i\,\omega\,t}\, x(\omega) \qquad (2.103)$$

Applying the lattice derivative ∇, we have

$$\begin{aligned}
\nabla x(t_n) &= \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi} \frac{1}{\epsilon} \left(e^{i\,\omega\,t_{n+1}} - e^{i\,\omega\,t_n}\right) x(\omega) \\
&= \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi} \frac{1}{\epsilon} \left(e^{i\,\omega\,(t_n+\epsilon)} - e^{i\,\omega\,t_n}\right) x(\omega) \\
&= \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi}\, e^{i\,\omega\,t_n} \left(\frac{e^{i\,\omega\,\epsilon} - 1}{\epsilon}\right) x(\omega) \qquad (2.104) \\
&= \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi}\, e^{i\,\omega\,t_n}\, \nabla x(\omega)
\end{aligned}$$

where

$$\nabla x(\omega) = \frac{e^{i\,\omega\,\epsilon} - 1}{\epsilon}\, x(\omega) \qquad (2.105)$$

describes the difference operator on the Fourier component $x(\omega)$, i.e., $\frac{e^{i\,\omega\,\epsilon} - 1}{\epsilon}$ is the eigenvalue of ∇ in ω-space.

Furthermore

$$\nabla x(\omega) \underset{\epsilon \to 0}{\longrightarrow} i\,\omega\, x(\omega)$$

shows the effect on $x(\omega)$ when $x(t)$ is operated by $\frac{\partial}{\partial t}$. As a reminder of this, we set

$$\Omega(\omega) = -i\, \frac{e^{i\,\omega\,\epsilon} - 1}{\epsilon} \underset{\epsilon \to 0}{\longrightarrow} \omega \qquad (2.106a)$$

so that

$$(-i\,\nabla)\, x(\omega) = \Omega(\omega)\, x(\omega) \qquad (2.106)$$

i.e., $\Omega(\omega)$ is the eigenvalue of $-i\,\nabla$ in ω-space.

For the conjugate lattice derivative $\overline{\nabla}$, we have

$$(-i\,\overline{\nabla})\, x(\omega) = \overline{\Omega}(\omega)\, x(\omega) \qquad (2.107)$$

where, using

$$\begin{aligned}
\overline{\nabla} x(t_n) &= \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi} \frac{1}{\epsilon} \left(e^{i\,\omega\,t_n} - e^{i\,\omega\,t_{n-1}}\right) x(\omega) \\
&= \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi}\, e^{i\,\omega\,t_n} \left(\frac{1 - e^{-i\,\omega\,\epsilon}}{\epsilon}\right) x(\omega)
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi}\, e^{i\,\omega\,t_n}\, \overline{\nabla} x(\omega) \qquad (2.107a)$$

we have

$$\overline{\Omega}\,(\omega) = -i\,\frac{1 - e^{-i\,\omega\,\epsilon}}{\epsilon} = \Omega^*(\omega) \underset{\epsilon \to 0}{\longrightarrow} \omega \qquad (2.107b)$$

Thus,

$$-\nabla\overline{\nabla} x(\omega) = (-i\,\nabla)\left(-i\,\overline{\nabla}\right) x(\omega) = \Omega(\omega)\,\overline{\Omega}(\omega)\, x(\omega) = -\overline{\nabla}\nabla\, x(\omega) \qquad (2.107c)$$

so that $\Omega(\omega)\,\overline{\Omega}(\omega) = |\,\Omega(\omega)\,|^2$ are the eigenvalues of $-\nabla\overline{\nabla} = -\overline{\nabla}\nabla$.

Using (2.106a) & (2.107b), we have

$$\Omega(\omega)\,\overline{\Omega}(\omega) = -i\,\frac{e^{i\,\omega\,\epsilon} - 1}{\epsilon}\left(-i\,\frac{1 - e^{-i\,\omega\,\epsilon}}{\epsilon}\right)$$

$$= \frac{1}{\epsilon^2}\left(2 - e^{i\,\omega\,\epsilon} - e^{-i\,\omega\,\epsilon}\right)$$

$$= \frac{2}{\epsilon^2}\left[\,1 - \cos(\omega\,\epsilon)\,\right] \ge 0 \qquad (2.108)$$

The boundary conditions (2.80) on the quantum fluctuations can be implemented as follows

$$x(t) = \int_0^{\infty} \frac{d\,\omega}{\pi}\, \sin\omega\,(t - t_a)\, x(\omega) \qquad \to \qquad x(t_a) = 0 \qquad (2.109)$$

$$\omega = \nu_m = \frac{m\,\pi}{t_b - t_a} \qquad \to \qquad x(t_b) = 0 \qquad (2.110)$$

Using

$$\int_0^{\infty} d\,\omega \approx \sum_{m=0}^{\infty} \Delta\omega \qquad \Delta\omega = \frac{\pi}{t_b - t_a}\,\Delta\, m = \frac{\pi}{t_b - t_a}$$

we have

$$x(t) = \frac{1}{t_b - t_a} \sum_{m=1}^{\infty} \sin[\,\nu_m\,(t - t_a)\,]\, x(\nu_m) \qquad (2.111)$$

Finally, on the lattice, $t = t_n$ with $n = 0,\ 1,\ \ldots N + 1$. Using $t_b - t_a = (N + 1)\,\epsilon$, (2.111) becomes

$$x\,(t_n) = \frac{1}{N + 1} \sum_{m=1}^{N} \sin[\,\nu_m\,(t_n - t_a)\,]\, x(\nu_m) \qquad (2.112)$$

where, for convenience, a factor ϵ^{-1} was absorbed in a re-defined $x(\nu_m)$ and

$$\nu_m = \frac{m\,\pi}{(N + 1)\,\epsilon} \qquad (2.112a)$$

Note that the summation in (2.112) runs only from $m = 1$ to $m = N$ because

$$\sin[\,\nu_m\,(t_n - t_a)\,] = \sin\left[\frac{m\,\pi}{(N + 1)\,\epsilon}\,n\,\epsilon\right] = \sin\frac{m\,n\,\pi}{N + 1} = \sin[\,\nu_n\,(t_m - t_a)\,]$$

$$\to \qquad \sin[\,\nu_0\,(t_n - t_a)\,] = \sin[\,\nu_{N+1}\,(t_n - t_a)\,] = 0 \qquad \forall\ n \qquad (2.112b)$$

The basis sine functions are orthogonal and complete. Using

$$\sum_{n=1}^{N} \sin\left(n\,\frac{m\,\pi}{N + 1}\right) \sin\left(n\,\frac{k\,\pi}{N + 1}\right) = \frac{N + 1}{2}\,\delta_{m\,k} \qquad (2.112c)$$

we have

$$\sum_{n=1}^{N} \sin[\nu_m\,(t_n - t_a)]\, \sin[\nu_k\,(t_n - t_a)] = \frac{N + 1}{2}\,\delta_{m\,k} \qquad \text{(orthogonality) (2.113)}$$

$$\frac{2}{N+1}\sum_{n=1}^{N}\sin[\nu_n(t_m-t_a)]\sin[\nu_n(t_k-t_a)]=\delta_{mk} \qquad \text{(completeness)} \qquad (2.114)$$

The proof (2.112c) is as follows. Using

$$\text{Re}\left[e^{i(\theta-\phi)}-e^{i(\theta+\phi)}\right]=\cos(\theta-\phi)-\cos(\theta+\phi)$$
$$=2\sin\theta\sin\phi$$

and (2.112b), we have,

$$L.H.S.=\frac{1}{2}\text{Re}\sum_{n=0}^{N+1}\left\{\exp\left[i\,n\frac{(m-k)\pi}{N+1}\right]-\exp\left[i\,n\frac{(m+k)\pi}{N+1}\right]\right\}$$
$$=\frac{1}{2}\text{Re}\left\{\frac{1-e^{i(m-k)\pi}\exp\left[i\frac{(m-k)\pi}{N+1}\right]}{1-\exp\left[i\frac{(m-k)\pi}{N+1}\right]}-\frac{1-e^{i(m+k)\pi}\exp\left[i\frac{(m+k)\pi}{N+1}\right]}{1-\exp\left[i\frac{(m+k)\pi}{N+1}\right]}\right\} \qquad (2.114a)$$

Each term in (2.114a) is of the form

$$\frac{1-\beta\alpha}{1-\alpha}=\frac{(1-\beta\alpha)(1-\alpha^*)}{|1-\alpha|^2}=\frac{1+\beta-\alpha^*-\beta\alpha}{|1-\alpha|^2}$$

where

$$\alpha\alpha^*=1 \qquad \beta=e^{i(m\mp k)\pi}$$

Note that $m-k$ and $m+k$ must be both even or both odd integers.

For $m\neq k$, if $m\mp k$ are both even, $\beta=1$ and we have

$$\frac{1-\beta\alpha}{1-\alpha}=1$$

Both terms in (2.114a) are equal to 1 so $L.H.S.=0$.

For $m\neq k$, if $m\mp k$ both odd, $\beta=-1$ and we have

$$\frac{1-\beta\alpha}{1-\alpha}=\frac{1+\alpha}{1-\alpha}=\frac{(1+\alpha)(1-\alpha^*)}{|1-\alpha|^2}=\frac{\alpha-\alpha^*}{|1-\alpha|^2}=i\frac{2\,\text{Im}\,\alpha}{|1-\alpha|^2}$$

which is purely imaginary. Hence, $L.H.S.=0$.

Therefore, (2.114a) gives

$$L.H.S.=0 \qquad \text{for} \qquad m\neq k$$

For $m=k$, the summation for the $m+k$ terms in (2.114a) does not contribute since it is purely imaginary. The summation of the $m-k$ terms is a sum of 1's. Here, the number of terms in the summation must be determined precisely. Since there should be $N+1$ independent basis functions on a lattice of $N+1$ inequivalent points, we have

$$L.H.S.=\frac{1}{2}\text{Re}\sum_{n=0}^{N}1=\frac{N+1}{2}$$

thus proving (2.112c). Finally, the verity of (2.112c) can be checked explicitly for small N, say, $N=2$ or 3.

Consider now the fluctuation action (2.89)

$$\mathcal{A}_{\text{fl}}^N=\frac{1}{2}M\epsilon\sum_{n=1}^{N+1}\left(\frac{x_n-x_{n-1}}{\epsilon}\right)^2=\frac{1}{2}M\epsilon\sum_{n=1}^{N+1}\left(\overline{\nabla}x_n\right)^2$$
$$=-\frac{1}{2}M\epsilon\sum_{n=1}^{N+1}x_n\nabla\overline{\nabla}\,x_n \qquad [\text{ (2.101) used. }]$$
$$=-\frac{M\epsilon}{2(N+1)^2}\sum_{n,m,k=1}^{N}\sin[\nu_m(t_n-t_a)]\,x(\nu_m)\sin[\nu_k(t_n-t_a)]\,\nabla\overline{\nabla}x(\nu_k)[\text{ (2.112) used. }]$$

$$= \frac{M\epsilon}{2(N+1)^2} \sum_{n,m,k=1}^{N} \sin[\nu_m (t_n - t_a)] \sin[\nu_k (t_n - t_a)]\, x(\nu_m)\, \Omega_k \overline{\Omega}_k\, x(\nu_k) \qquad \text{[(2.107c) used.]}$$

where, according to (2.106a),

$$\Omega_k = \Omega(\nu_k) = -i \frac{e^{i \nu_k \epsilon} - 1}{\epsilon} \qquad \overline{\Omega}_k = \Omega_k^* = i \frac{e^{-i \nu_k \epsilon} - 1}{\epsilon}$$

$$\rightarrow \qquad \Omega_k \overline{\Omega}_k = \frac{2}{\epsilon^2}\left[1 - \cos(\nu_k \epsilon)\right] \qquad \text{[(2.108) used.]}$$

$$= \frac{2}{\epsilon^2}\left[1 - \cos\left(\frac{k\pi}{N+1}\right)\right] \qquad \text{[(2.112a) used.]} \qquad (2.118)$$

Using the orthogonality (2.113), we have

$$\mathcal{A}_{\text{fl}}^N = \frac{M\epsilon}{2(N+1)^2} \sum_{m,k=1}^{N} \frac{N+1}{2} \delta_{mk}\, x(\nu_m)\, \Omega_k \overline{\Omega}_k\, x(\nu_k)$$

$$= \frac{M\epsilon}{4(N+1)} \sum_{m=1}^{N} \Omega_m \overline{\Omega}_m\, x(\nu_m)^2 \qquad (2.118a)$$

The fluctuation factor (2.88) thus becomes

$$F_0^N = \left(\frac{M}{2\pi i \hbar \epsilon}\right)^{(N+1)/2} \left(\prod_{n=1}^{N} \int d x_n\right) \prod_{m=1}^{N} \exp\left(\frac{i}{\hbar} \frac{M\epsilon}{4(N+1)} \Omega_m \overline{\Omega}_m\, x(\nu_m)^2\right) \qquad (2.119)$$

Now, according to (2.113), the orthonormal basis functions are

$$\sqrt{\frac{2}{N+1}}\, \sin[\nu_m (t_n - t_a)]$$

If x_n were expanded using this basis, the Jacobian of the transformation would be 1. Since we've used (2.112) instead, our $x(\nu_m)$ is $\sqrt{2(N+1)}$ times that for the orthonormal basis expansion. The Jacobian of our transformation is therefore $[2(N+1)]^{-N/2}$ so that

$$\prod_{n=1}^{N} d x_n = [2(N+1)]^{-N/2} \prod_{m=1}^{N} d x(\nu_m) \qquad (2.120)$$

(2.119) thus becomes

$$F_0^N = \left(\frac{M}{2\pi i \hbar \epsilon}\right)^{(N+1)/2} \prod_{m=1}^{N} \left\{ \int \frac{d x(\nu_m)}{\sqrt{2(N+1)}} \exp\left(\frac{i}{\hbar} \frac{M\epsilon}{4(N+1)} \Omega_m \overline{\Omega}_m\, x(\nu_m)^2\right) \right\} \qquad (2.120a)$$

$$= \sqrt{\frac{M}{2\pi i \hbar \epsilon}} \prod_{m=1}^{N} \frac{1}{\sqrt{\epsilon^2 \Omega_m \overline{\Omega}_m}} \qquad \text{[(1.333) used.]} \qquad (2.121)$$

Using (2.118), we have

$$\prod_{m=1}^{N} \epsilon^2 \Omega_m \overline{\Omega}_m = \prod_{m=1}^{N} 2\left[1 - \cos\left(\frac{m\pi}{N+1}\right)\right] = N+1 \qquad (2.123)$$

where we've used [see Gradshteyn & Ryzhik, Formula 1.396.2]

$$\prod_{m=1}^{N} \left[1 + x^2 - 2x\cos\left(\frac{m\pi}{N+1}\right)\right] = \frac{x^{2(N+1)} - 1}{x^2 - 1} \qquad (2.122)$$

$$= \frac{e^{2(N+1)\ln x} - 1}{x^2 - 1}$$

$$= \lim_{\epsilon \to 0} \frac{e^{2(N+1)\epsilon} - 1}{(1+\epsilon)^2 - 1} \qquad \text{for } x = 1 + \epsilon \to 1$$
$$= \lim_{\epsilon \to 0} \frac{2(N+1)\epsilon}{2\epsilon} = N + 1$$

(2.121) thus becomes

$$F_0^N = \sqrt{\frac{M}{2\pi i \hbar \epsilon (N+1)}} \tag{2.124}$$

i.e.,

$$F_0(t_b - t_a) = \sqrt{\frac{M}{2\pi i \hbar (t_b - t_a)}} \tag{2.125}$$

The dimension of F_0 is

$$[F_0] = \left[\sqrt{\frac{M}{(M L^2 T^{-1})\, T}}\right] = [L^{-1}]$$

Therefore, the relevant length scale associated with a time interval $t_b - t_a$ is

$$l(t_b - t_a) = \frac{1}{|\, F_0(t_b - t_a)\,|} = \sqrt{\frac{2\pi \hbar (t_b - t_a)}{M}} \tag{2.126}$$

(2.125) becomes

$$F_0(t_b - t_a) = \frac{1}{\sqrt{i}\; l(t_b - t_a)} \tag{2.127}$$

With (2.125) and (2.87), the full time evolution amplitude of a free particle (2.84) is again given by (2.71)

$$(x_b\, t_b \mid x_a\, t_a) = \left(\frac{M}{2\pi i \hbar (t_b - t_a)}\right)^{1/2} \exp\left(i \frac{M}{2\hbar} \frac{(x_b - x_a)^2}{t_b - t_a}\right) \tag{2.128}$$

Generalization to a D-D free particle is trivial. In terms of Cartesian coordiantes $\boldsymbol{x} = (x_1, \ldots, x_D)$, the action is simply

$$\mathcal{A}[\boldsymbol{x}] = \frac{1}{2} M \int_{t_a}^{t_b} dt\, \dot{\boldsymbol{x}}^2 \tag{2.129}$$

and (2.128) becomes

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \left(\frac{M}{2\pi i \hbar (t_b - t_a)}\right)^{1/2} \exp\left(i \frac{M}{2\hbar} \frac{(\boldsymbol{x}_b - \boldsymbol{x}_a)^2}{t_b - t_a}\right) \tag{2.130}$$

in agreement with the quantum mechanical result (1.337).

Another way to evaluate the product

$$\prod_{m=1}^{N} \epsilon^2\, \Omega_m \overline{\Omega}_m \tag{2.131}$$

makes use of the fact that $\Omega_m \overline{\Omega}_m$ are the eigenvalues of $-\epsilon^2 \nabla \overline{\nabla}$ [see (2.107c)]. Therefore

$$\prod_{m=1}^{N} \epsilon^2\, \Omega_m \overline{\Omega}_m = \det_N\left(-\epsilon^2 \nabla \overline{\nabla}\right) \tag{2.132}$$

where $\nabla\overline{\nabla}$ is the tri-diagonal matrix in (2.102). (2.121) thus becomes

$$F_0^N = \sqrt{\frac{M}{2\pi i\hbar\epsilon}}\,\frac{1}{\sqrt{\det_N\left(-\epsilon^2\nabla\overline{\nabla}\right)}} \tag{2.133}$$

The special symmetry of $\nabla\overline{\nabla}$ allows us to evaluate it by iteration. From (2.102), we have

$$\det_N\left(-\epsilon^2\nabla\overline{\nabla}\right) = \det\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}_N \tag{2.133a}$$

Let

$$W_N = \det\begin{pmatrix} -1 & -1 & & & 0 \\ 0 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}_N = -\det_{N-1}\left(-\epsilon^2\nabla\overline{\nabla}\right) \tag{2.133b}$$

then the Laplace expansion of (2.133a) gives

$$\begin{aligned}\det_N\left(-\epsilon^2\nabla\overline{\nabla}\right) &= 2\det_{N-1}\left(-\epsilon^2\nabla\overline{\nabla}\right) + W_{N-1} \\ &= 2\det_{N-1}\left(-\epsilon^2\nabla\overline{\nabla}\right) - \det_{N-2}\left(-\epsilon^2\nabla\overline{\nabla}\right)\end{aligned} \tag{2.136}$$

Starting with

$$\det_1\left(-\epsilon^2\nabla\overline{\nabla}\right) = \det(2) = 2 \tag{2.134}$$

$$\det_2\left(-\epsilon^2\nabla\overline{\nabla}\right) = \det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3 \tag{2.135}$$

we have

$$\det_3\left(-\epsilon^2\nabla\overline{\nabla}\right) = 2\times 3 - 2 = 4$$

This suggests the ansatz

$$\det_N\left(-\epsilon^2\nabla\overline{\nabla}\right) = N+1 \tag{2.137}$$

which indeed satisfies (2.136) since

$$L.H.S. = N+1 \quad R.H.S. = 2N - (N-1) = N+1$$

By (2.132), (2.137) is simply (2.123).

Finally, a simple Fourier transform of the initial and final positions according to the rule (2.37) yields the momentum space evolution amplitude

$$\begin{aligned}(\mathbf{p}_b t_b \mid \mathbf{p}_a t_a) &= \int d\mathbf{x}_b\, e^{-i\mathbf{p}_b\cdot\mathbf{x}_b/\hbar}\int d\mathbf{x}_a\, e^{i\mathbf{p}_a\cdot\mathbf{x}_a/\hbar}\,(\mathbf{x}_b t_b \mid \mathbf{x}_a t_a) \\ &= \left(\frac{M}{2\pi i\hbar(t_b-t_a)}\right)^{D/2}\int d\mathbf{x}_b\, e^{-i\mathbf{p}_b\cdot\mathbf{x}_b/\hbar}\int d\mathbf{x}_a\, e^{i\mathbf{p}_a\cdot\mathbf{x}_a/\hbar}\exp\left(i\frac{M}{2\hbar}\frac{(\mathbf{x}_b-\mathbf{x}_a)^2}{t_b-t_a}\right) \\ &= \left(\frac{M}{2\pi i\hbar(t_b-t_a)}\right)^{D/2}\int d\mathbf{x}_a\, e^{-i(\mathbf{p}_b-\mathbf{p}_a)\cdot\mathbf{x}_a/\hbar}\int d\mathbf{X}\exp\left[\frac{i}{\hbar}\left(\frac{M}{2}\frac{\mathbf{X}^2}{t_b-t_a} - \mathbf{p}_b\cdot\mathbf{X}\right)\right] \\ &= (2\pi\hbar)^D\,\delta(\mathbf{p}_b-\mathbf{p}_a)\exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}_b^2}{2M}(t_b-t_a)\right]\end{aligned} \tag{2.138}$$

2.2.4. Finite Slicing Properties of Free-Particle Amplitude

The time-sliced free-particle time evolution amplitude (2.70) happens to be independent of N. This is also the case for the fluctuation factor (2.124). Let us study its origin.

For the classical case. The difference equation of motion

$$-\overline{\nabla}\nabla\, x(t) = 0 \tag{2.139}$$

has the same linear solution

$$x(t) = A\,t + B \tag{2.140}$$

as in the continuum where $\overline{\nabla}\nabla \to \dfrac{\partial^2}{\partial t^2}$. Thus,

$$x(t_n) = A\,t_n + B$$

$$\to \qquad \nabla x(t_n) = A\left(\frac{t_{n+1} - t_n}{\epsilon}\right) + \frac{B - B}{\epsilon} = A \tag{2.140a}$$

$$\therefore \qquad \overline{\nabla}\nabla\, x(t_n) = 0 \tag{2.140b}$$

Imposing the initial conditions

$$x_{\text{cl}}(t_0) = x_0 = x_a \qquad\qquad x_{\text{cl}}(t_{N+1}) = x_{N+1} = x_b$$

gives

$$x_{\text{cl}}(t_n) = x_a + n\,\frac{x_b - x_a}{N+1} \tag{2.141}$$

Using

$$t_n = t_a + n\,\epsilon = t_a + n\,\frac{t_b - t_a}{N+1} \qquad \to \qquad \frac{n}{N+1} = \frac{t_n - t_a}{t_b - t_a}$$

(2.141) becomes

$$\begin{aligned} x_{\text{cl}}(t_n) &= x_a + (x_b - x_a)\,\frac{t_n - t_a}{t_b - t_a} \\ &= \frac{x_b - x_a}{t_b - t_a}\,t_n + \frac{x_a\,t_b - x_b\,t_a}{t_b - t_a} \end{aligned} \tag{2.141a}$$

Comparing with (2.140) gives

$$A = \frac{x_b - x_a}{t_b - t_a} \qquad\qquad B = \frac{x_a\,t_b - x_b\,t_a}{t_b - t_a} \tag{2.141b}$$

Using the summation by parts (2,96), we have

$$\mathcal{A}_{\text{cl}} = \epsilon \sum_{n=1}^{N+1} \frac{1}{2} M \left(\overline{\nabla} x_{\text{cl}}\right)^2 \tag{2.142}$$

$$\begin{aligned} &= \frac{1}{2} M \left(x_{\text{cl}}\,\nabla x_{\text{cl}} \Big|_{n=0}^{N+1} - \epsilon \sum_{n=0}^{N} x_{\text{cl}}\,\overline{\nabla}\nabla\, x_{\text{cl}} \right) \\ &= \frac{1}{2} M\, x_{\text{cl}}\,\nabla x_{\text{cl}} \Big|_{n=0}^{N+1} && \text{[(2.140b) used.]} \\ &= \frac{1}{2} M\,(\, x_b\,\nabla x_b - x_a\,\nabla\, x_a) \\ &= \frac{1}{2} M A\,(\, x_b - x_a) && \text{[(2.140a) used.]} \\ &= \frac{1}{2} M\,\frac{(x_b - x_a)^2}{t_b - t_a} && \text{[(2.141b) used.]} \end{aligned} \tag{2.142a}$$

which coincides with the continuum result for any N.

In the operator formulation of quantum mechanics, the ϵ-independence of the amplitude of the free particle follows from the fact that in the absence of a potential $V(x)$, the two sides of the Trotter formula (2.26) coincide for any N.