2.3. Exact Solution for Harmonic Oscillator

For the 1-D linear oscillator, the evolution amplitude

$$(x_b\, t_b \mid x_a\, t_a) = \int \mathcal{D}' x \int \frac{\mathcal{D} p}{2\pi\hbar} \exp\left(\frac{i}{\hbar} \mathcal{A}[x, p] \right)$$

$$= \int \mathcal{D} x \exp\left(\frac{i}{\hbar} \mathcal{A}[x] \right) \qquad (2.143)$$

has a canonical action

$$\mathcal{A}[x, p] = \int_{t_a}^{t_b} dt \left(p\,\dot{x} - \frac{p^2}{2M} - \frac{1}{2} M \omega^2 x^2 \right) \qquad (2.144)$$

and a Lagrangian action

$$\mathcal{A}[x] = \int_{t_a}^{t_b} dt \frac{1}{2} M \left(\dot{x}^2 - \omega^2 x^2 \right) \qquad (2.145)$$

2.3.1. Fluctuations around the Classical Path

The time-sliced action is

$$\mathcal{A}^N = \epsilon \frac{M}{2} \sum_{n=1}^{N+1} \left[(\overline{\nabla} x_n)^2 - \omega^2 x_n^2 \right] \qquad (2.146)$$

Direct evaluation of (2.143) is rather involved so the details are relegated to Appendix 2B. Here, we employ the much easier fluctuation expansion. Thus, we write $x = x_{\rm cl} + \delta x$, so that the action is splitted into a classical part

$$\mathcal{A}_{\rm cl} = \int_{t_a}^{t_b} dt \frac{1}{2} M \left(\dot{x}_{\rm cl}^2 - \omega^2 x_{\rm cl}^2 \right) \qquad (2.147)$$

and a fluctuation part

$$\mathcal{A}_{\rm fl} = \int_{t_a}^{t_b} dt \frac{1}{2} M \left[(\delta \dot{x})^2 - \omega^2 (\delta x)^2 \right] \qquad (2.148)$$

with the boundary conditions

$$x(t_a) = x_a \qquad x(t_b) = x_b \qquad (2.149a)$$

$$\delta x(t_a) = \delta x(t_b) = 0 \qquad (2.149)$$

Note that there is no mixed term due to the extremality of $\mathcal{A}_{\rm cl}$. The equation of motion is

$$\ddot{x}_{\rm cl} = -\omega^2 x_{\rm cl} \qquad (2.150)$$

As in the free particle case, we write

$$(x_b\, t_b \mid x_a\, t_a) = \int \mathcal{D} x\, e^{i \mathcal{A}[x]/\hbar} = e^{i \mathcal{A}_{\rm cl}/\hbar} F_\omega(t_b - t_a) \qquad (2.151)$$

The classical orbit satisfying the B.C. (2.149a) is easily found to be

$$x_{\rm cl}(t) = \frac{x_b \sin\omega(t - t_a) + x_a \sin\omega(t_b - t)}{\sin\omega(t_b - t_a)} \qquad (2.152)$$

Note that if

$$\omega (t_b - t_a) = n\pi \qquad n = \text{ integers}$$

(2.152) becomes singular and leads to what is known as caustic phenomena [see Notes & References at the end of the chapter]. Here, we shall simply assume this is not the case.

After an integration by parts, (2.147) becomes

$$\mathcal{A}_{\rm cl} = \frac{1}{2} M x_{\rm cl} \dot{x}_{\rm cl} \Big|_{t_a}^{t_b} - \int_{t_a}^{t_b} dt \frac{1}{2} M (\ddot{x}_{\rm cl} + \omega^2 x_{\rm cl}) x_{\rm cl} \tag{2.153}$$

$$= \frac{1}{2} M \left[x_{\rm cl}(t_b) \dot{x}_{\rm cl}(t_b) - x_{\rm cl}(t_a) \dot{x}_{\rm cl}(t_a) \right] \quad \text{[(2.150) used.]} \tag{2.154}$$

From (2.152), we have

$$\dot{x}_{\rm cl}(t) = \omega \frac{x_b \cos\omega(t - t_a) - x_a \cos\omega(t_b - t)}{\sin\omega(t_b - t_a)}$$

$$\rightarrow \quad \dot{x}_{\rm cl}(t_a) = \omega \frac{x_b - x_a \cos\omega(t_b - t_a)}{\sin\omega(t_b - t_a)} \tag{2.155}$$

$$\dot{x}_{\rm cl}(t_b) = \omega \frac{x_b \cos\omega(t_b - t_a) - x_a}{\sin\omega(t_b - t_a)} \tag{2.156}$$

(2.154) becomes

$$\mathcal{A}_{\rm cl} = \frac{M\omega}{2 \sin\omega(t_b - t_a)} \left[(x_b^2 + x_a^2) \cos\omega(t_b - t_a) - 2 x_b x_a \right] \tag{2.157}$$

2.3. 2. Fluctuation Factor

By (2.88),

$$F_\omega^N(t_b, t_a) = \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{(N+1)/2} \left(\prod_{n=1}^{N} \int d\,\delta x_n \right) e^{i \mathcal{A}_{\rm fl}^N / \hbar} \tag{2.158}$$

where, the time-sliced version of (2.148) is [see (2.101)],

$$\mathcal{A}_{\rm fl}^N = \frac{M}{2} \epsilon \sum_{n,k=1}^{N} \delta x_n \left(-\nabla\overline{\nabla} - \omega^2 \mathbb{I} \right)_{nk} \delta x_k \tag{2.158a}$$

When going to the Fourier components, everything goes as the free particle case except that $\Omega_m \overline{\Omega}_m$ is now replaced by

$$\Omega_m \overline{\Omega}_m - \omega^2 = \frac{2}{\epsilon^2} \left[1 - \cos\left(\frac{m\pi}{N+1} \right) \right] - \omega^2 \tag{2.159}$$

Thus, (2.121) becomes

$$F_\omega^N = \sqrt{\frac{M}{2\pi i \hbar \epsilon}} \prod_{m=1}^{N} \frac{1}{\sqrt{\epsilon^2 (\Omega_m \overline{\Omega}_m - \omega^2)}} \quad \text{for} \quad \Omega_m \overline{\Omega}_m - \omega^2 > 0 \tag{2.160}$$

Reminder: (2.121) was obtained using the Fresnel integral formula (1.333) with $a > 0$.

Setting

$$\frac{\epsilon\omega}{2} = \sin\frac{\epsilon\tilde{\omega}}{2} \tag{2.161}$$

(2.159) becomes

$$\Omega_m \overline{\Omega}_m - \omega^2 = \frac{4}{\epsilon^2} \left[\sin^2\left(\frac{m\pi}{2(N+1)} \right) - \sin^2 \frac{\epsilon\tilde{\omega}}{2} \right]$$

$$= \frac{4}{\epsilon^2} \sin^2\left(\frac{m\pi}{2(N+1)} \right) \left(1 - \frac{\sin^2 \frac{\epsilon\tilde{\omega}}{2}}{\sin^2\left(\frac{m\pi}{2(N+1)} \right)} \right)$$

$$= \Omega_m \overline{\Omega}_m \left(1 - \frac{\sin^2 \frac{\epsilon\tilde{\omega}}{2}}{\sin^2\left(\frac{m\pi}{2(N+1)} \right)} \right) \tag{2.162a}$$

Using (2.123), we have

$$\prod_{m=1}^{N} \epsilon^2 \left(\Omega_m \overline{\Omega}_m - \omega^2 \right) = (N+1) \prod_{m=1}^{N} \left(1 - \frac{\sin^2 \frac{\epsilon \tilde{\omega}}{2}}{\sin^2 \left(\frac{m \pi}{2(N+1)} \right)} \right) \tag{2.162b}$$

Using [see Gradshteyn & Ryzhik, Formula 1.391.1]

$$\sin n x = \frac{n}{2} \sin 2 x \prod_{k=1}^{(n-2)/2} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k \pi}{n}} \right) \tag{2.163}$$

with $n = 2(N+1)$, (2.162b) becomes

$$\begin{aligned} \prod_{m=1}^{N} \epsilon^2 \left(\Omega_m \overline{\Omega}_m - \omega^2 \right) &= \frac{\sin[(N+1) \epsilon \tilde{\omega}]}{\sin \epsilon \tilde{\omega}} \\ &= \frac{\sin \tilde{\omega} (t_b - t_a)}{\sin \epsilon \tilde{\omega}} \qquad (2.164) \\ &= \det_N \left[\epsilon^2 \left(-\nabla \overline{\nabla} - \omega^2 \mathbb{1} \right) \right] \end{aligned}$$

From (2.162a), we have

$$\Omega_m \overline{\Omega}_m - \omega^2 > 0 \qquad \to \qquad \sin^2 \left(\frac{m \pi}{2(N+1)} \right) > \sin^2 \frac{\epsilon \tilde{\omega}}{2}$$

Since $\epsilon << 1$ and $N >> 1$, this mean

$$\frac{m \pi}{2(N+1)} > \frac{\epsilon \tilde{\omega}}{2} \qquad \to \qquad m \frac{\pi}{\tilde{\omega}} > (N+1) \epsilon = t_b - t_a \tag{2.164a}$$

In order for this to hold for $m = 1, \ldots, N$, we must have

$$\frac{\pi}{\tilde{\omega}} > t_b - t_a \tag{2.164b}$$

Using (2.164 & b) on (2.160) gives

$$F_\omega^N = \sqrt{\frac{M}{2 \pi i \hbar}} \sqrt{\frac{\sin \epsilon \tilde{\omega}}{\epsilon \sin \tilde{\omega} (t_b - t_a)}} \qquad \text{for} \qquad \frac{\pi}{\tilde{\omega}} > t_b - t_a \tag{2.165}$$

2.3.3. The $i\,\eta$-Prescription and Maslov-Morse Index

(2.165) was derived assuming

$$\pi > \tilde{\omega} (t_b - t_a) > 0 \tag{2.166}$$

In this time interval, all eigenvalues $\Omega_m \overline{\Omega}_m - \omega^2$ are positive. According to (2.164a), as $t_b - t_a$ grows, the first eigenvalue to become negative is $\Omega_1 \overline{\Omega}_1 - \omega^2$. The integration over the associated Fourier component must then be evaluated using the Fresnel formula with $a < 0$, which gives an extra phase factor $e^{-i \pi/2} = -i$. As $t_b - t_a$ continues to grow, the same thing happens again when $\Omega_2 \overline{\Omega}_2 - \omega^2$ becomes negative. And so on....

Thus, (2.165) becomes

$$F_\omega^N = \sqrt{\frac{M}{2 \pi i \hbar}} \sqrt{\frac{\sin \epsilon \tilde{\omega}}{\epsilon \, | \sin \tilde{\omega} (t_b - t_a) |}} \, e^{-i \nu \pi/2} \tag{2.167}$$

where

$$\nu = \text{Floor} \left(\frac{\tilde{\omega} (t_b - t_a)}{\pi} \right)$$

is called the Maslov-Morse index of the trajectory and the Floor operator is defined as

$$0 \le \text{Floor}(x) = n \le x < n+1 \qquad \text{for some } n = 0, 1, 2, \ldots$$

Actually, the same result is obtained if we simply remove the restriction on $\tilde{\omega}$ from (2.165). Thus, using

$$\sqrt{\sin(\tilde{\omega}\,\tau + \nu\,\pi)} = \sqrt{\sin\tilde{\omega}\,\tau\cos\nu\,\pi} = \sqrt{\sin\tilde{\omega}\,\tau}\,(-)^{\nu/2} = \sqrt{\sin\tilde{\omega}\,\tau}\,e^{i\nu\pi/2}$$

$$= \sqrt{|\sin(\tilde{\omega}\,\tau + \nu\,\pi)|}\,e^{i\nu\pi/2} \qquad \text{if } 0 < \tilde{\omega}\,\tau < \pi$$

the unrestricted (2.165) becomes (2.167) with $\tau = t_b - t_a$.

Unfortunately, I don't understand how the $i\,\eta$-prescription works.

2.3.4. Continuum Limit

In the continuum limit, $\epsilon \to 0$ so that (2.161) gives $\tilde{\omega} \to \omega$.

(2.164) & (2.165) become, respectively

$$\det_N[\epsilon^2(-\nabla\overline{\nabla} - \omega^2\,\mathbb{1})] \underset{\epsilon\to 0}{\longrightarrow} \frac{\sin\omega\,(t_b - t_a)}{\epsilon\,\omega} \tag{2.168}$$

$$F_\omega(t_b - t_a) = \sqrt{\frac{M}{2\,\pi\,i\,\hbar}}\sqrt{\frac{\omega}{\sin\omega\,(t_b - t_a)}} \tag{2.169}$$

where the phase associated with $\sqrt{\sin\omega\,(t_b - t_a)}$ is to be treated as described in §2.3.3.

Note that

$$\lim_{\omega\to 0} F_\omega(t_b - t_a) = \sqrt{\frac{M}{2\,\pi\,i\,\hbar\,(t_b - t_a)}}$$

which is simply $F_0(t_b - t_a)$ of the free particle [see (2.125)].

(2.162a) can be written as

$$\frac{\Omega_m\,\overline{\Omega}_m - \omega^2}{\Omega_m\,\overline{\Omega}_m} = 1 - \frac{\sin^2\frac{\epsilon\tilde{\omega}}{2}}{\sin^2\left(\frac{m\,\pi\,\epsilon}{2\,(t_b - t_a)}\right)}$$

$$\underset{\epsilon\to 0}{\longrightarrow} 1 - \frac{\left(\frac{\epsilon\,\omega}{2}\right)^2}{\left(\frac{m\,\pi\,\epsilon}{2\,(t_b - t_a)}\right)^2} = 1 - \frac{\omega^2 (t_b - t_a)^2}{m^2\,\pi^2} \tag{2.170}$$

Using [see Gradshteyn & Ryzhik, Formula 1.431.1]

$$\sin x = x\prod_{m=1}^{\infty}\left(1 - \frac{x^2}{m^2\,\pi^2}\right) \tag{2.171}$$

we have, with $x = \omega\,(t_b - t_a)$,

$$\prod_{m=1}^{\infty}\frac{\Omega_m\,\overline{\Omega}_m - \omega^2}{\Omega_m\,\overline{\Omega}_m} \underset{\epsilon\to 0}{\longrightarrow} \frac{\sin\omega\,(t_b - t_a)}{\omega\,(t_b - t_a)}$$

$$\to \prod_{m=1}^{\infty}\frac{\Omega_m\,\overline{\Omega}_m}{\Omega_m\,\overline{\Omega}_m - \omega^2} \underset{\epsilon\to 0}{\longrightarrow} \frac{\omega\,(t_b - t_a)}{\sin\omega\,(t_b - t_a)} \tag{2.172}$$

Together with (2.160), one obtains again the continuum fluctuation factor (2.169).

Combining (2.151) , (2.157) & (2.169) give

$$(x_b\, t_b \mid x_a\, t_a) = \sqrt{\frac{M}{2\pi i \hbar}}\, \sqrt{\frac{\omega}{\sin\omega\,(t_b - t_a)}} \quad (2.173)$$

$$\times \exp\left\{ \frac{i}{\hbar} \frac{M\omega}{2\sin\omega(t_b - t_a)} \left[\left(x_b^2 + x_a^2\right) \cos\omega(t_b - t_a) - 2\, x_b\, x_a \right] \right\}$$

This result is readily extended to arbitrary D-D for which the action (2.145) is generalized to

$$\mathcal{A} = \int_{t_a}^{t_b} d\,t\, \frac{1}{2} M \left(\dot{\boldsymbol{x}}^2 - \omega^2 \boldsymbol{x}^2 \right) \quad (2.174)$$

(2.173) then becomes

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \left(\frac{M}{2\pi i \hbar} \right)^{D/2} \left(\frac{\omega}{\sin\omega\,(t_b - t_a)} \right)^{D/2} \quad (2.175)$$

$$\times \exp\left\{ \frac{i}{\hbar} \frac{M\omega}{2\sin\omega(t_b - t_a)} \left[\left(\boldsymbol{x}_b^2 + \boldsymbol{x}_a^2\right) \cos\omega(t_b - t_a) - 2\, \boldsymbol{x}_b \cdot \boldsymbol{x}_a \right] \right\}$$

the phase associated with $\sqrt{\sin\omega\,(t_b - t_a)}$ is to be treated as described in §2.3.3.

2.3.5. Useful Fluctuation Formulas

From (2.91), we have

$$\nabla\overline{\nabla} = \overline{\nabla}\nabla \xrightarrow[\epsilon\to 0]{} \frac{\partial^2}{\partial t^2}$$

while (2.107c) shows that $\Omega_m \overline{\Omega}_m$ are the eigenvalues of $-\nabla\overline{\nabla}$.

Thus, (2.172) can be written as

$$\frac{\det\left(-\partial_t^2 - \omega^2\right)}{\det\left(-\partial_t^2\right)} = \lim_{\epsilon\to 0} \frac{\prod_{m=1}^{\infty} \left(\Omega_m \overline{\Omega}_m - \omega^2\right)}{\prod_{m=1}^{\infty} \Omega_m \overline{\Omega}_m} = \frac{\sin\omega\,(t_b - t_a)}{\omega\,(t_b - t_a)} \quad (2.176)$$

The solutions to the eigen-equation

$$-\partial_t^2\, x = \lambda\, x$$

are

$$x(t) = A \sin\nu\, t + B \cos\nu\, t \qquad \lambda = \nu^2$$

Imposing the boundary conditions for fluctuations

$$x(t_b) = x(t_a) = 0$$

turns the spectrum into a discrete one with

$$x_m(t) = A \sin\nu_m\,(t - t_a) \qquad \nu_m = \frac{m\pi}{t_b - t_a} \qquad \lambda_m = \nu_m^2 \quad (2.177)$$

Hence,

$$\frac{\det\left(-\partial_t^2 - \omega^2\right)}{\det\left(-\partial_t^2\right)} = \prod_{m=1}^{\infty} \frac{\nu_m^2 - \omega^2}{\nu_m^2} = \prod_{m=1}^{\infty} \left(1 - \frac{\omega^2 (t_b - t_a)^2}{m^2 \pi^2} \right) \quad (2.178)$$

which is simply (2.170) and hence evaluated to (2.176).

Note however that

$$\prod_{m=1}^{\infty} m \to \infty \quad (2.178a)$$

so that both $\det\left(-\partial_t^2\right)$ and $\det\left(-\partial_t^2 - \omega^2\right)$ diverge and only their ratio is meaningful.

Thus,

$$\det\left(-\partial_t^2 - \omega^2\right) = \prod_{m=1}^{\infty} \left(\nu_m^2 - \omega^2\right) \quad (2.181)$$

$$= \prod_{k=1}^{\infty} \nu_k^2 \prod_{m=1}^{\infty} \frac{\nu_m^2 - \omega^2}{\nu_m^2} = \det(-\partial_t^2) \prod_{m=1}^{\infty} \frac{\nu_m^2 - \omega^2}{\nu_m^2}$$

$$= \left\{ \prod_{k=1}^{\infty} \left(\frac{k\pi}{t_b - t_a} \right)^2 \right\} \frac{\sin \omega (t_b - t_a)}{\omega (t_b - t_a)} \qquad \text{[(2.176-7) used.]} \qquad (2.181a)$$

which diverges due to (2.718a).

Hence, the fluctuation factor [see (2.158)]

$$F_\omega(t_b - t_a) = \lim_{N \to \infty} \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{(N+1)/2} \left(\prod_{n=1}^{N} \int d\,\delta x_n \right)$$

$$\times \exp\left[i \frac{M}{2\hbar} \epsilon \sum_{n,k=1}^{N} \delta x_n \left(-\nabla\overline{\nabla} - \omega^2 \mathbb{1} \right)_{nk} \delta x_k \right]$$

$$= \int \mathcal{D}\,\delta x \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \frac{M}{2} \left[(\delta \dot{x})^2 - \omega^2 (\delta x)^2 \right] \right\} \qquad (2.179)$$

cannot be calculated from

$$\lim_{\epsilon \to 0} \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{1/2} \frac{1}{\sqrt{\det(-\partial_t^2 - \omega^2)}} \qquad (2.180)$$

The correct way to do so is to write (2.179) as [see (2.160)]

$$F_\omega^N(t_b - t_a) = \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{(N+1)/2} \left(\prod_{n=1}^{N} \int d\,\delta x_n \right)$$

$$\times \exp\left[i \frac{M}{2\hbar\epsilon} \sum_{n,k=1}^{N} \delta x_n \left(-\epsilon^2 \nabla\overline{\nabla} - \epsilon^2 \omega^2 \mathbb{1} \right)_{nk} \delta x_k \right]$$

$$= \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{1/2} \frac{1}{\sqrt{\det_N(-\epsilon^2 \nabla\overline{\nabla} - \epsilon^2 \omega^2)}} \qquad (2.183)$$

which was evaluated successfully in (2.164).

Alternatively, one can use [see (2.123)]

$$\prod_{m=1}^{N} \epsilon^2 \Omega_m \overline{\Omega}_m = N + 1 = \frac{t_b - t_a}{\epsilon}$$

to write (2.183) as

$$F_\omega^N(t_b - t_a) = \left(\frac{M}{2\pi i \hbar (t_b - t_a)} \right)^{1/2} \left(\frac{\det_N(-\epsilon^2 \nabla\overline{\nabla} - \epsilon^2 \omega^2)}{\det_N(-\epsilon^2 \nabla\overline{\nabla})} \right)^{-1/2} \qquad (2.184)$$

$$= \sqrt{\frac{M}{2\pi i \hbar}} \sqrt{\frac{\sin \epsilon \tilde{\omega}}{\epsilon \sin \tilde{\omega} (t_b - t_a)}} \qquad \text{[See (2.165)]} \; (2.185)$$

In the continuum limit, (2.184) becomes

$$F_\omega(t_b - t_a) = \left(\frac{M}{2\pi i \hbar (t_b - t_a)} \right)^{1/2} \left(\frac{\det(-\partial_t^2 - \omega^2)}{\det(-\partial_t^2)} \right)^{-1/2} \qquad (2.186a)$$

$$= \left(\frac{M}{2\pi i \hbar (t_b - t_a)} \right)^{1/2} \sqrt{\frac{\omega (t_b - t_a)}{\sin \omega (t_b - t_a)}} \qquad \text{[(2.176) used.]} \; (2.186)$$

In other words, (2.180) should be evaluated by turning it into the ratio form (2.186a) first.

Using (2.175), the evolution amplitude in momentum space becomes

$$(\boldsymbol{p}_b\, t_b \mid \boldsymbol{p}_a\, t_a) = \int d\,\boldsymbol{x}_b\, e^{-i\,\boldsymbol{p}_b\cdot\boldsymbol{x}_b/\hbar} \int d\,\boldsymbol{x}_a\, e^{i\,\boldsymbol{p}_a\cdot\boldsymbol{x}_a/\hbar}\,(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) \qquad \text{[See (2.138).]}$$
$$= \left(\frac{M}{2\,\pi\, i\,\hbar}\right)^{D/2}\left(\frac{\omega}{\sin\omega\,(t_b - t_a)}\right)^{D/2} \int d\,\boldsymbol{x}_b\, e^{-i\,\boldsymbol{p}_b\cdot\boldsymbol{x}_b/\hbar}\int d\,\boldsymbol{x}_a\, e^{i\,\boldsymbol{p}_a\cdot\boldsymbol{x}_a/\hbar}$$
$$\times \exp\left\{\frac{i}{\hbar}\,\frac{M\,\omega}{2\sin\omega(t_b - t_a)}\left[(\boldsymbol{x}_b^2 + \boldsymbol{x}_a^2)\cos\omega(t_b - t_a) - 2\,\boldsymbol{x}_b\cdot\boldsymbol{x}_a\right]\right\}$$

Evaluation of the Gauss integrals are straightforward but tedious. Using *Mathematica* [see file "2.03._Code.nb"], we get

$$(\boldsymbol{p}_b\, t_b \mid \boldsymbol{p}_a\, t_a) = \left(\frac{2\,\pi\,\hbar}{i\,M\,\omega\sin\omega\,(t_b - t_a)}\right)^{D/2} \tag{2.187}$$
$$\times \exp\left\{\frac{i}{\hbar}\,\frac{1}{2\,M\,\omega\sin\omega(t_b - t_a)}\left[(\boldsymbol{p}_b^2 + \boldsymbol{p}_a^2)\cos\omega(t_b - t_a) - 2\,\boldsymbol{p}_b\cdot\boldsymbol{p}_a\right]\right\}$$

For $\omega \to 0$,

$$\frac{1}{2\,M\,\omega\sin\omega(t_b - t_a)}\left[(\boldsymbol{p}_b^2 + \boldsymbol{p}_a^2)\cos\omega(t_b - t_a) - 2\,\boldsymbol{p}_b\cdot\boldsymbol{p}_a\right]$$
$$\to \frac{1}{2\,M\,\omega\,[\,\omega(t_b - t_a) + \ldots]}\left\{(\boldsymbol{p}_b^2 + \boldsymbol{p}_a^2)\left[1 - \frac{1}{2}\,\omega^2\,(t_b - t_a)^2 + \ldots\right] - 2\,\boldsymbol{p}_b\cdot\boldsymbol{p}_a\right]\right\}$$
$$= \frac{1}{2\,M\,\omega^2\,(t_b - t_a)}\left[(\boldsymbol{p}_b - \boldsymbol{p}_a)^2 - \frac{1}{2}\,(\boldsymbol{p}_b^2 + \boldsymbol{p}_a^2)\,\omega^2(t_b - t_a)^2 + \ldots\right] \tag{2.188}$$

Hence, (2.187) becomes

$$(\boldsymbol{p}_b\, t_b \mid \boldsymbol{p}_a\, t_a) \to \left(\frac{2\,\pi\,\hbar}{i\,M\,\omega^2\,(t_b - t_a)}\right)^{D/2}\exp\left(\frac{i}{\hbar}\,\frac{(\boldsymbol{p}_b - \boldsymbol{p}_a)^2}{2\,M\,\omega^2\,(t_b - t_a)}\right)\exp\left(-\frac{i}{\hbar}\,\frac{\boldsymbol{p}_b^2 + \boldsymbol{p}_a^2}{4\,M}\,(t_b - t_a)\right)$$
$$= (2\,\pi\,\hbar)^D\,\delta\,(\boldsymbol{p}_b - \boldsymbol{p}_a)\exp\left(-\frac{i}{\hbar}\,\frac{\boldsymbol{p}_b^2}{2\,M}\,(t_b - t_a)\right) \qquad \text{[(1.531) used.]} \tag{2.189}$$

which is simply the free particle amplitude (2.138).

2.3.6. Oscillator Amplitude on Finite Time Lattice

Let us calculate the exact time evolution amplitude for a finite number of time slices.

Recall that for a given N, there are $N+2$ time points t_0, t_1, ..., t_N, t_{N+1}. In order to make the Lagrangian look the same at the end points, i.e., to keep time reversal invariance, we need to rewrite the action of (2.146)

$$\mathcal{A}^N = \epsilon\,\frac{M}{2}\sum_{n=1}^{N+1}\left[(\overline{\nabla}x_n)^2 - \omega^2\,x_n^2\right]$$

either as

$$\mathcal{A}^N = \epsilon\,\frac{M}{2}\sum_{n=1}^{N+1}\left[(\overline{\nabla}x_n)^2 - \frac{1}{2}\,\omega^2(x_n^2 + x_{n-1}^2)\right] \tag{2.190}$$

or

$$\mathcal{A}^N = \epsilon\,\frac{M}{2}\sum_{n=0}^{N}\left[(\nabla x_n)^2 - \frac{1}{2}\,\omega^2(x_{n+1}^2 + x_n^2)\right] \tag{2.191}$$

Applying the summation by part formula (2.96) to (2.190-1) leads to trouble at the end points. We therefore start from the beginning:

$$\epsilon \sum_{n=1}^{N+1} (\overline{\nabla} x_n)^2 = \frac{1}{\epsilon} \sum_{n=1}^{N+1} (x_n - x_{n-1})^2$$
$$= \frac{1}{\epsilon}\left[(x_{N+1} - x_N)^2 + \ldots + (x_1 - x_0)^2 \right]$$
$$= \frac{1}{\epsilon}\left(x_{N+1}^2 - 2\, x_{N+1}\, x_N + 2\, x_N^2 - 2\, x_N\, x_{N-1} + \ldots + 2\, x_1^2 - 2\, x_1\, x_0 + x_0^2 \right)$$
$$\epsilon \sum_{n=1}^{N} x_n\, \nabla\overline{\nabla} x_n = \frac{1}{\epsilon} \sum_{n=1}^{N} x_n (x_{n+1} - 2\, x_n + x_{n-1})$$
$$= \frac{1}{\epsilon}\left[x_N\, (x_{N+1} - 2\, x_N + x_{N-1}) + \ldots + x_1\, (x_2 - 2\, x_1 + x_0) \right]$$
$$= \frac{1}{\epsilon}\left(x_{N+1}\, x_N - 2\, x_N^2 + 2\, x_N\, x_{N-1} - 2\, x_{N-1}^2 + \ldots + 2\, x_1\, x_2 - 2\, x_1^2 + x_1\, x_0 \right)$$
$$\rightarrow \qquad \epsilon \sum_{n=1}^{N+1} (\overline{\nabla} x_n)^2 = \frac{1}{\epsilon} \left(x_{N+1}^2 - x_{N+1}\, x_N - x_1\, x_0 + x_0^2 \right) - \epsilon \sum_{n=1}^{N} x_n\, \nabla\overline{\nabla} x_n$$
$$= x_{N+1}\, \overline{\nabla} x_{N+1} - x_0\, \nabla x_0 - \epsilon \sum_{n=1}^{N} x_n\, \nabla\overline{\nabla} x_n$$
$$= x_b\, \overline{\nabla} x_b - x_a\, \nabla x_a - \epsilon \sum_{n=1}^{N} x_n\, \nabla\overline{\nabla} x_n \qquad (2.192)$$

(2.190-1) thus become

$$\mathcal{A}^N = \frac{M}{2} \left(x_b\, \overline{\nabla} x_b - x_a\, \nabla x_a \right) - \epsilon\, \frac{M}{4}\, \omega^2 \left(x_b^2 + x_a^2 \right) - \epsilon\, \frac{M}{2} \sum_{n=1}^{N} x_n \left(\nabla\overline{\nabla} + \omega^2 \right) x_n \qquad (2.193)$$

Following (2.151), we write

$$(x_b\, t_b \mid x_a\, t_a) = e^{i \mathcal{A}_{\rm cl}^N / \hbar}\, F_\omega^N(t_b - t_a) \qquad (2.193a)$$

Since the variation of $\mathcal{A}^N$ is carried out at fixed endpoints x_a & x_b, $F_\omega^N(t_b - t_a)$ is the same as that given in (2.158-a). The classical equation of motion obtained by the variation of (2.195) gives

$$\left(\nabla\overline{\nabla} + \omega^2 \right) x_{\rm cl}\, (t_n) = 0 \qquad (2.194)$$

The solution of (2.194) can be obtained as follows. From

$$\epsilon^2\, \nabla\overline{\nabla} \sin \omega\, t_n = \sin \omega\, t_{n+1} - 2 \sin \omega\, t_n + \sin \omega\, t_{n-1}$$
$$= 2 \sin\omega\, t_n \cos\omega\epsilon - 2 \sin\omega\, t_n$$
$$= 2 \sin\omega\, t_n\, (\cos\omega\epsilon - 1)$$
$$= -4 \sin^2 \frac{\omega\, \epsilon}{2} \sin\omega\, t_n \qquad (2.194a)$$

we see that $\sin\omega\, t_n$ is the eigenfunction of $\nabla\overline{\nabla}$ with eigenvalue $-\frac{4}{\epsilon^2} \sin^2 \frac{\omega\, \epsilon}{2}$.

(2.194) can therefore be satisfied by

$$x_{\rm cl}(t_n) \propto \sin \tilde{\omega}\, t_n \qquad (2.194b)$$

with

$$\omega^2 = \frac{4}{\epsilon^2} \sin^2 \frac{\tilde{\omega}\, \epsilon}{2}$$
$$\rightarrow \qquad \frac{\epsilon\, \omega}{2} = \sin \frac{\epsilon\, \tilde{\omega}}{2} \qquad (2.194c)$$

which is simply (2.161). It is easy to see that the foregoing derivation also applies to

$$x_{\rm cl}(t_n) = A \sin \tilde{\omega}\, (t_n - B) \qquad (2.194d)$$

where A, B are constants.

Imposing the boundary conditions (2.149a), we have

$$x_{\text{cl}}(t_n) = \frac{x_b \sin\tilde{\omega}(t_n - t_a) + x_a \sin\tilde{\omega}(t_b - t_n)}{\sin\tilde{\omega}(t_b - t_a)} \tag{2.195}$$

Using (2.194) on (2.193), we have

$$\mathcal{A}_{\text{cl}}^N = \frac{M}{2}\left(x_b \overline{\nabla} x_b - x_a \nabla x_a\right) - \epsilon \frac{M}{4}\omega^2\left(x_b^2 + x_a^2\right) \tag{2.196a}$$

Now,

$$\begin{aligned} x_b \overline{\nabla} x_b - x_a \nabla x_a &= \frac{1}{\epsilon}\left\{ x_b\left[x_b - x_{\text{cl}}(t_N)\right] - x_a\left(x_{\text{cl}}(t_1) - x_a\right)\right\} \\ &= \frac{1}{\epsilon}\left[x_b^2 - x_b \frac{x_b \sin\tilde{\omega}(t_N - t_a) + x_a \sin\tilde{\omega}(t_b - t_N)}{\sin\tilde{\omega}(t_b - t_a)} \right. \\ &\qquad \left. - x_a \frac{x_b \sin\tilde{\omega}(t_1 - t_a) + x_a \sin\tilde{\omega}(t_b - t_1)}{\sin\tilde{\omega}(t_b - t_a)} + x_a^2 \right] \end{aligned} \tag{2.196b}$$

Using

$$\sin\tilde{\omega}(t_N - t_a) = \sin\tilde{\omega}(t_b - t_a - \epsilon) = \sin\tilde{\omega}(t_b - t_a)\cos\tilde{\omega}\epsilon - \cos\tilde{\omega}(t_b - t_a)\sin\tilde{\omega}\epsilon$$

$$\sin\tilde{\omega}(t_b - t_N) = \sin\tilde{\omega}\epsilon$$

$$\sin\tilde{\omega}(t_1 - t_a) = \sin\tilde{\omega}\epsilon$$

$$\sin\tilde{\omega}(t_b - t_1) = \sin\tilde{\omega}(t_b - t_a - \epsilon)$$

(2.196b) becomes

$$\begin{aligned} x_b \overline{\nabla} x_b - x_a \nabla x_a &= \frac{1}{\epsilon}\left\{ x_b^2 + x_a^2 - \frac{2 x_a x_b \sin\tilde{\omega}\epsilon}{\sin\tilde{\omega}(t_b - t_a)} \right. \\ &\qquad \left. - \frac{x_b^2 + x_a^2}{\sin\tilde{\omega}(t_b - t_a)}\left[\sin\tilde{\omega}(t_b - t_a)\cos\tilde{\omega}\epsilon - \cos\tilde{\omega}(t_b - t_a)\sin\tilde{\omega}\epsilon\right]\right\} \\ &= \frac{1}{\epsilon}\left\{ \left(x_b^2 + x_a^2\right)(1 - \cos\tilde{\omega}\epsilon) \right. \\ &\qquad \left. - \frac{\sin\tilde{\omega}\epsilon}{\sin\tilde{\omega}(t_b - t_a)}\left[\left(x_b^2 + x_a^2\right)\cos\tilde{\omega}(t_b - t_a) - 2 x_a x_b\right]\right\} \end{aligned} \tag{2.196c}$$

Also, by (2.194c),

$$\begin{aligned} \epsilon \frac{M}{4}\omega^2\left(x_b^2 + x_a^2\right) &= M\left(x_b^2 + x_a^2\right)\sin^2\frac{\epsilon\tilde{\omega}}{2} \\ &= \frac{M}{2}\left(x_b^2 + x_a^2\right)(1 - \cos\tilde{\omega}\epsilon) \end{aligned} \tag{2.196d}$$

Using (2.196c-d) on (2.196a) gives

$$\mathcal{A}_{\text{cl}}^N = \frac{M \sin\tilde{\omega}\epsilon}{2\epsilon \sin\tilde{\omega}(t_b - t_a)}\left[\left(x_b^2 + x_a^2\right)\cos\tilde{\omega}(t_b - t_a) - 2 x_b x_a\right] \tag{2.196}$$

As stated in (2.193a),

$$(x_b t_b \mid x_a t_a) = e^{i \mathcal{A}_{\text{cl}}^N/\hbar} F_\omega^N(t_b - t_a) \tag{2.197}$$