2.4. Gelfand-Yaglom Formula

Consider now a harmonic oscillator with a time-dependent frequency $\Omega(t)$. The associated fluctuation factor is

$$F(t_b, t_a) = \int \mathcal{D}\,\delta x(t) \exp\left(\frac{i}{\hbar}\mathcal{A}\right) \tag{2.198}$$

where

$$\mathcal{A} = \int_{t_a}^{t_b} d\,t\,\frac{1}{2} M\left[(\delta\dot{x})^2 - \Omega^2(t)\,(\delta x)^2\right] \tag{2.199}$$

If $\Omega(t)$ is not translationally invariant in time, the fluctuation factor will not be a function of merely $t_b - t_a$. (2.184) thus generalizes to

$$F_\omega^N(t_b, t_a) = \left(\frac{M}{2\,\pi\,i\,\hbar\,(t_b - t_a)}\right)^{1/2} \left(\frac{\det_N\left(-\epsilon^2\,\nabla\overline{\nabla} - \epsilon^2\,\Omega^2\right)}{\det_N\left(-\epsilon^2\,\nabla\overline{\nabla}\right)}\right)^{-1/2} \tag{2.200}$$

where

$$\Omega^2(t) = \begin{pmatrix} \Omega_N^2 & & \\ & \ddots & \\ & & \Omega_1^2 \end{pmatrix} \qquad \Omega_n = \Omega(t_n) \tag{2.201}$$

2.4.1. Recursive Calculation of Fluctuation Determinant

Gelfand & Yaglom [J.Math.Phys.1, 48 (1960)] developed a finite difference equation for calculating the determinant of a tri-diagonal matrix without knowing its eigenvalues. In fact, we've already used a simplified version of it to derive (2.136) in §2.2.3.

Consider the determinant of the $N \times N$ fluctuation matrix

$$D_N \equiv \det_N\left(-\epsilon^2\,\nabla\overline{\nabla} - \epsilon^2\,\Omega^2\right) \tag{2.202}$$

$$= \begin{vmatrix} 2 - \epsilon^2\,\Omega_N^2 & -1 & 0 & \cdots\cdots & 0 & 0 & 0 \\ -1 & 2 - \epsilon^2\,\Omega_{N-1}^2 & -1 & \cdots\cdots & 0 & 0 & 0 \\ 0 & -1 & 2 - \epsilon^2\,\Omega_{N-2}^2 & \cdots\cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots\cdots & 2 - \epsilon^2\,\Omega_3^2 & -1 & 0 \\ 0 & 0 & 0 & \cdots\cdots & -1 & 2 - \epsilon^2\,\Omega_2^2 & -1 \\ 0 & 0 & 0 & \cdots\cdots & 0 & -1 & 2 - \epsilon^2\,\Omega_1^2 \end{vmatrix}$$

Laplace expand by the 1st row gives [see (2.136)]

$$D_N = \left(2 - \epsilon^2\,\Omega_N^2\right) D_{N-1} - D_{N-2} \tag{2.203}$$

$$\to \qquad \frac{D_N - 2\,D_{N-1} + D_{N-2}}{\epsilon^2} + \Omega_N^2\,D_{N-1} = 0 \tag{2.204}$$

With the help of (2.100a), (2.204) becomes

$$\left(\nabla\overline{\nabla} + \Omega_N^2\right) D_{N-1} = 0 \tag{2.205}$$

which is known as the Gelfand-Yaglom formula.

(2.205), or (2.203), is to be solved iteratively with the initial conditions

$$D_1 = 2 - \epsilon^2\,\Omega_1^2$$
$$D_2 = \begin{vmatrix} 2-\epsilon^2\,\Omega_2^2 & -1 \\ -1 & 2-\epsilon^2\,\Omega_1^2 \end{vmatrix} = \left(2-\epsilon^2\,\Omega_2^2\right)\left(2-\epsilon^2\,\Omega_1^2\right) - 1 \qquad (2.206)$$

2.4.2. Examples

Consider the known case $\Omega(t) = \omega$ for which the Gelfand-Yaglom formula (2.205) reads

$$\left(\nabla\overline{\nabla}+\omega^2\right)_N D_{N-1} = 0 \qquad (2.207)$$

with initial conditions

$$D_1 = 2 - \epsilon^2\,\omega^2 = 2\left(1 - 2\sin^2\frac{\tilde{\omega}\,\epsilon}{2}\right) = 2\cos\tilde{\omega}\,\epsilon$$
$$D_2 = \left(2-\epsilon^2\,\omega^2\right)^2 - 1 = 4\cos^2\tilde{\omega}\,\epsilon - 1 \qquad (2.209)$$

The solution of (2.207) was already given in (2.194d). For convenience, we'll use the equivalent form

$$D_{N-1} = A\sin\tilde{\omega}\,t_N + B\cos\tilde{\omega}\,t_N = A\sin N\,\tilde{\omega}\,\epsilon + B\cos N\,\tilde{\omega}\,\epsilon \qquad (2.208a)$$

The initial conditions (2.208) then becomes

$$A\sin 2\,\tilde{\omega}\,\epsilon + B\cos 2\,\tilde{\omega}\,\epsilon = 2\cos\tilde{\omega}\,\epsilon$$
$$A\sin 3\,\tilde{\omega}\,\epsilon + B\cos 3\,\tilde{\omega}\,\epsilon = 4\cos^2\tilde{\omega}\,\epsilon - 1$$

The 1st eq. gives

$$2\,A\sin\tilde{\omega}\,\epsilon\cos\tilde{\omega}\,\epsilon - B\left(\cos^2\tilde{\omega}\,\epsilon - \sin^2\tilde{\omega}\,\epsilon\right) = 2\cos\tilde{\omega}\,\epsilon$$

Since all trigonometric functions are independent of each other, we have

$$B = 0 \qquad A = \frac{1}{\sin\tilde{\omega}\,\epsilon} \qquad (2.208b)$$

To check the verity of (2.208b), we consider the 2nd eq., which becomes

$$L.H.S. = \frac{\sin 3\,\tilde{\omega}\,\epsilon}{\sin\tilde{\omega}\,\epsilon} = 3 - 4\sin^2\tilde{\omega}\,\epsilon = R.H.S.$$

Thus, the solution to (2.207) is indeed

$$D_{N-1} = \frac{\sin N\,\tilde{\omega}\,\epsilon}{\sin\tilde{\omega}\,\epsilon} \qquad (2.208)$$

The Gelfand-Yaglom formula (2.205) is much simplified in the continuum limit $\epsilon \to 0$.

To begin, the initial conditions (2.206) simplify to

$$D_1 = 2 - \epsilon^2\,\Omega_1^2 \xrightarrow[\epsilon\to 0]{} 2 \qquad (2.210a)$$
$$D_2 = \left(2-\epsilon^2\,\Omega_2^2\right)\left(2-\epsilon^2\,\Omega_1^2\right) - 1 \xrightarrow[\epsilon\to 0]{} 3 \qquad (2.210b)$$

Defining the renormalized function by

$$D_{\text{ren}}(t_N) = \epsilon\,D_N \qquad (2.210)$$

then (2.210a-b) give rise to initial conditions on D_{ren} at t_a :

$$D_{\text{ren}}(t_a) \approx D_{\text{ren}}(t_1) = \epsilon\,D_1 = 2\,\epsilon = 0 \qquad (2.211)$$
$$\dot{D}_{\text{ren}}(t_a) \approx \epsilon\,\nabla D_1 = \epsilon\,\frac{D_2 - D_1}{\epsilon} = 3 - 2 = 1 \qquad (2.212)$$

The difference eq. (2.205) itself becomes a differential eq.

$$\left(\frac{\partial^2}{\partial\,t^2} + \Omega^2(t)\right) D_{\text{ren}}(t) = 0 \qquad (2.213)$$

The situation is pictured in Fig. 2.2. The determinant D_N is ϵ^{-1} times $D_{\text{ren}}(t_b)$, which is found by solving (2.213) starting from t_a with value 0 and slope 1.

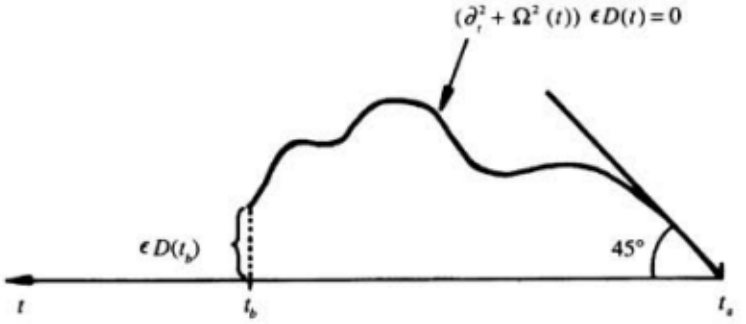

Fig.2.2

Using the oscillator with fixed frequency ω as example, it is easily check that

$$D_{\text{ren}}(t) = \frac{1}{\omega} \sin \omega(t - t_a) \tag{2.214}$$

satisfies (2.211-3). Thus, (2.202) gives

$$D_N = \det_N\left(-\epsilon^2\, \nabla\overline{\nabla} - \epsilon^2\, \omega^2\right) \xrightarrow[\epsilon \to 0]{} \frac{1}{\epsilon} D_{\text{ren}}(t_b) = \frac{1}{\epsilon\, \omega} \sin \omega(t - t_a) \tag{2.215}$$

in agreement with the $\epsilon \to 0$ limit of (2.208).

Setting $\omega = 0$ gives us the results for a free particle:

$$D_{\text{ren}}(t) = t - t_a \qquad \text{and} \qquad D_N = \det_N\left(-\epsilon^2\, \nabla\overline{\nabla}\right) = \frac{t - t_a}{\epsilon}$$

For a time-dependent $\Omega(t)$, an analytic solution of the Gelfand-Yaglom initial-value problem (2.211-3) can be found only for a few cases. In fact, (2.213) has the form of a Schrodinger eq. for a particle in a potential $\Omega^2(t)$. Cases for which analytic solutions can be obtained are well-known but few.

2.4.3. Calculation on Unsliced Time Axis

Since

$$\left(\frac{\partial^2}{\partial t^2} + \Omega^2(t) \right) x(t) = 0 \tag{2.216}$$

is a 2nd order equation, there are two independent solutions, say, $\xi(t)$ and $\eta(t)$.

The solution to (2.213) can therefore be written as the linear combination

$$D_{\text{ren}}(t) = \alpha\, \xi(t) + \beta\, \eta(t) \tag{2.217}$$

where α, β are constants determined by the initial conditions (2.212), namely,

$$\begin{aligned} &\alpha\, \xi(t_a) + \beta\, \eta(t_a) = 0 \\ &\alpha\, \dot{\xi}(t_a) + \beta\, \dot{\eta}(t_a) = 1 \end{aligned} \tag{2.218}$$

Hence,

$$D_{\text{ren}}(t) = \frac{\eta(t_a)\, \xi(t) - \xi(t_a)\, \eta(t)}{\dot{\xi}(t_a)\, \eta(t_a) - \dot{\eta}(t_a)\, \xi(t_a)} \tag{2.219}$$

$$= -\frac{\eta(t_a)\, \xi(t) - \xi(t_a)\, \eta(t)}{W[\xi(t_a),\, \eta(t_a)]} \tag{2.219a}$$

where

$$W[\xi(t),\, \eta(t)] \equiv \xi(t) \overleftrightarrow{\partial} \eta(t) = \dot{\eta}(t)\, \xi(t) - \dot{\xi}(t)\, \eta(t) \tag{2.220}$$

$$= \begin{vmatrix} \xi(t) & \eta(t) \\ \dot{\xi}(t) & \dot{\eta}(t) \end{vmatrix}$$

is the Wronski determinant (or Wronskian) of ξ & η at time t.

It is well-known that for a 2nd order differential equation of the Sturm-Liouville type

$$\frac{d}{d\,t}\left(a(t)\,\frac{d\,y(t)}{d\,t}\right) + b(t)\,y(t) = 0 \tag{2.221}$$

the Wronkian for any pair of independent solutions is simply [see Arfken]

$$W(t) = \frac{C}{a(t)} \tag{2.221a}$$

where C is some constant whose value depends on the choice of solutions.

Since (2.216) is a Sturm-Liouville eq. with $a(t) = 1$, its Wronskian is time-independent. We therefore write (2.219a) as

$$D_{\text{ren}}(t) = -\frac{1}{W}\left[\,\eta(t_a)\,\xi(t) - \xi(t_a)\,\eta(t)\,\right] \tag{2.222}$$

The determinant required for the fluctuation factor is therefore

$$D_{\text{ren}} \equiv D_{\text{ren}}(t_b) = -\frac{1}{W}\left[\,\eta(t_a)\,\xi\,(t_b) - \xi(t_a)\,\eta\,(t_b)\,\right] \tag{2.223}$$

Obviously, we can start from the other end at t_b and work backwards to t_a. Thus, we define

$$\tilde{D}_n = \begin{vmatrix} 2-\epsilon^2\,\Omega_{N-n+1}^2 & -1 & 0 & \cdots\cdots & 0 & 0 & 0 \\ -1 & 2-\epsilon^2\,\Omega_{N-n+2}^2 & -1 & \cdots\cdots & 0 & 0 & 0 \\ 0 & -1 & 2-\epsilon^2\,\Omega_{N-n+3}^2 & \cdots\cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots\cdots & 2-\epsilon^2\,\Omega_{N-2}^2 & -1 & 0 \\ 0 & 0 & 0 & \cdots\cdots & -1 & 2-\epsilon^2\,\Omega_{N-1}^2 & -1 \\ 0 & 0 & 0 & \cdots\cdots & 0 & -1 & 2-\epsilon^2\,\Omega_N^2 \end{vmatrix} \tag{2.223a}$$

Laplace expand by the 1st row gives

$$\tilde{D}_n = \left(2-\epsilon^2\,\Omega_{N-n+1}^2\right)\tilde{D}_{n-1} - \tilde{D}_{n-2} \tag{2.223b}$$

$$\rightarrow \qquad \frac{\tilde{D}_n - 2\,\tilde{D}_{n-1} + \tilde{D}_{n-2}}{\epsilon^2} + \Omega_{N-n+1}^2\,\tilde{D}_{n-1} = 0 \tag{2.223c}$$

The Gelfand-Yaglom formula becomes

$$\left(\,\nabla\overline{\nabla} + \Omega_{N-n+1}^2\,\right)\tilde{D}_{n-1} = 0 \tag{2.223d}$$

with the initial conditions

$$\tilde{D}_1 = 2-\epsilon^2\,\Omega_N^2 \xrightarrow[\epsilon\to 0]{} 2$$

$$\tilde{D}_2 = \begin{vmatrix} 2-\epsilon^2\,\Omega_{N-1}^2 & -1 \\ -1 & 2-\epsilon^2\,\Omega_N^2 \end{vmatrix} \xrightarrow[\epsilon\to 0]{} 3 \tag{2.223e}$$

For the renormalized version:

$$\tilde{D}_{\text{ren}}(t_{N-n+1}) = \epsilon\,\tilde{D}_n \tag{2.223f}$$

with initial conditions at t_b :

$$\tilde{D}_{\text{ren}}(t_b) \approx \tilde{D}_{\text{ren}}(t_N) = \epsilon\;\tilde{D}_1 = 2\,\epsilon\; = 0 \tag{2.225a}$$

$$\dot{\tilde{D}}_{\text{ren}}(t_b) \approx -\epsilon\,\nabla\,\tilde{D}_1\; = -\epsilon\,\frac{\tilde{D}_2 - \tilde{D}_1}{\epsilon} = -1 \tag{2.225}$$

Note that $\frac{d}{d\,t} = -\nabla$ since increasing t corresponds to decreasing n.

(2.219) is then replaced by

$$\tilde{D}_{\text{ren}}(t) = -\frac{1}{W}\left[\,\xi(t_b)\,\eta(t) - \eta(t_b)\,\xi(t)\,\right] \tag{2.224}$$

so that

$$D_{\text{ren}} = \tilde{D}_{\text{ren}}(t_a) = D_{\text{ren}}(t_b) \tag{2.224a}$$

The notations can be streamlined by setting

$$D_a(t) = D_{\text{ren}}(t) \qquad\qquad D_b(t) = \tilde{D}_{\text{ren}}(t) \tag{2.224b}$$

so that

$$\left(\frac{\partial^2}{\partial\,t^2} + \Omega^2(t)\right) D_a(t) = 0 \text{ with I.C.} \qquad D_a(t_a) = 0 \qquad \dot{D}_a(t_a) = 1 \tag{2.226}$$

$$\left(\frac{\partial^2}{\partial\,t^2} + \Omega^2(t)\right) D_b(t) = 0 \text{ with I.C.} \qquad D_b(t_b) = 0 \qquad \dot{D}_b(t_b) = -1 \tag{2.227}$$

and

$$D_{\text{ren}} = D_a\,(t_b) = D_b\,(t_a) \tag{2.228}$$

The values of $\dot{D}_a$ & $\dot{D}_b$ at the respective endpoints can be related as follows.

$\dot{D}_b \times$(2.226) + $\dot{D}_a \times$(2.227) gives

$$\left(\dot{D}_b\,\ddot{D}_a + \dot{D}_a\,\ddot{D}_b\right) + \Omega^2\left(\dot{D}_b\,D_a + \dot{D}_a\,D_b\right) = 0$$

$$\rightarrow \qquad \frac{d}{d\,t}\left(\dot{D}_a\,\dot{D}_b\right) = -\Omega^2\,\frac{d}{d\,t}\,(D_a\,D_b)$$

$$\therefore \qquad \dot{D}_a\,\dot{D}_b\Big|_{t_a}^{t_b} = -\int_{t_a}^{t_b} d\,t\,\Omega^2\,\frac{d}{d\,t}\,(D_a\,D_b) \tag{2.228a}$$

$$= -\Omega^2\,D_a\,D_b\Big|_{t_a}^{t_b} + 2\int_{t_a}^{t_b} d\,t\,\Omega\,\dot{\Omega}\,D_a\,D_b$$

With the I.C. in (2.222-7), we have

$$-\dot{D}_a(t_b) - \dot{D}_b(t_a) = 2\int_{t_a}^{t_b} d\,t\,\Omega\,\dot{\Omega}\,D_a\,D_b \tag{2.230}$$

By shifting the time origin to the midpoint of the interval (t_a, t_b), the right side of (2.228a) becomes

$$\int_{t_a}^{t_b} d\,t\,\Omega^2\,\frac{d}{d\,t}\,(D_a\,D_b) = \int_{-(t_b-t_a)/2}^{(t_b-t_a)/2} d\,t\,\Omega^2\,\frac{d}{d\,t}\,(D_a\,D_b) \tag{2.229a}$$

In case $\Omega(t)$ is time reversal invariant, i.e.,

$$\Omega(t) = \Omega(-t)$$

then (2.226-7) imply D_a & D_b are also time reversal invariant. This means (2.229a) must vanish since it is odd in t.

Hence, (2.230) becomes

$$\dot{D}_a(t_b) = -\dot{D}_b(t_a) \text{if} \qquad \Omega(t) = \Omega(-t) \tag{2.229}$$

For the linear oscillator with fixed frequency, we can choose

$$\xi(t) = \cos\omega\,t \qquad\qquad \eta(t) = \sin\omega\,t \tag{2.231}$$

Using (2.220), we have

$$W = \begin{vmatrix} \cos\omega\,t & \sin\omega\,t \\ -\omega\sin\omega\,t & \omega\cos\omega\,t \end{vmatrix} = \omega \tag{2.232}$$

which is indeed time-independent.

The fluctuation determinant (2.223) becomes

$$D_{\text{ren}} = -\frac{1}{\omega}\left(\cos\omega t_b \sin\omega t_a - \cos\omega t_a \sin\omega t_b\right)$$
$$= \frac{\sin\omega\,(t_b - t_a)}{\omega} \tag{2.233}$$

as expected.

2.4.4. D'Alembert's Construction

Given one solution $\xi(t)$ of the differential eq. (2.216),

$$\left(\frac{\partial^2}{\partial t^2} + \Omega^2(t)\right) x(t) = 0$$

the other independent solution is given by D'Alembert construction as

$$\eta(t) = w\,\xi(t) \int^t \frac{d\,t'}{\xi^2(t')} \tag{2.234}$$

where w is some constant.

Proof is by direct evaluation. Thus,

$$\dot{\eta} = \frac{\dot{\xi}}{\xi}\eta + \frac{w}{\xi} \tag{2.235a}$$

$$\ddot{\eta} = \frac{\ddot{\xi}}{\xi}\eta - \frac{\dot{\xi}^2}{\xi^2}\eta + \frac{\dot{\xi}}{\xi}\left(\frac{\dot{\xi}}{\xi}\eta + \frac{w}{\xi}\right) - \frac{w}{\xi^2}\dot{\xi}$$
$$= \frac{\ddot{\xi}}{\xi}\eta = -\Omega^2\,\eta \tag{2.235}$$

so that η is indeed a solution to (2.216).

Using (2.235a) to evaluate the Wronskian, we have

$$W = \xi\left(\frac{\dot{\xi}}{\xi}\eta + \frac{w}{\xi}\right) - \eta\,\dot{\xi} = w \tag{2.236}$$

Thus,

$$D_{\text{ren}}(t) = -\frac{1}{W}\left[\eta(t_a)\,\xi(t) - \xi(t_a)\,\eta(t)\right] \qquad \text{[See (2.222).]}$$
$$= -\left[\xi(t)\,\xi(t_a)\int^{t_a}\frac{d\,t'}{\xi^2(t')} - \xi(t_a)\,\xi(t)\int^{t}\frac{d\,t'}{\xi^2(t')}\right]$$
$$= \xi(t)\,\xi(t_a)\int_{t_a}^{t}\frac{d\,t'}{\xi^2(t')} \tag{2.237a}$$

Similarly,

$$\tilde{D}_{\text{ren}}(t) = -\frac{1}{W}\left[\xi(t_b)\,\eta(t) - \eta(t_b)\,\xi(t)\right] \qquad \text{[See (2.224).]}$$
$$= -\left[\xi(t_b)\,\xi(t)\int^{t}\frac{d\,t'}{\xi^2(t')} - \xi(t)\,\xi(t_b)\int^{t_b}\frac{d\,t'}{\xi^2(t')}\right]$$
$$= \xi(t)\,\xi(t_b)\int_{t}^{t_b}\frac{d\,t'}{\xi^2(t')} \tag{2.237}$$

$$\rightarrow \qquad D_{\text{ren}} = \xi(t_a)\,\xi(t_b)\int_{t_a}^{t_b}\frac{d\,t'}{\xi^2(t')} \tag{2.238}$$

2.4.5. Another Simple Formula

Given the solutions $D_a(t)$ & $D_b(t)$, another solution $x(t)$ to the differential eq. (2.216) with the I.C.

$$x(t_a) = x_a \qquad \dot{x}(t_a) = \dot{x}_a$$

can be written as

$$x(t) = A\, D_a(t) + B\, D_b(t) \tag{2.239a}$$

where A, B are constants given by

$$x_a = A\, D_a\,(t_a) + B\, D_b\,(t_a) \;= B\, D_b(t_a)$$
$$\dot{x}_a = A\, \dot{D}_a\,(t_a) + B\, \dot{D}_b\,(t_a) \;= A + B\, \dot{D}_b\,(t_a)$$

where the I.C. at t_a, (2.226), was used.

$$\rightarrow \qquad B = \frac{x_a}{D_b(t_a)}$$
$$A = \dot{x}_a - \frac{x_a}{D_b(t_a)}\dot{D}_b\,(t_a)$$

(2.239a) thus becomes

$$\begin{aligned} x(t) &= \left(\dot{x}_a - \frac{x_a}{D_b(t_a)}\dot{D}_b(t_a)\right) D_a(t) + \frac{x_a}{D_b(t_a)} D_b(t) \\ &= \frac{x_a}{D_b(t_a)}\left[D_b(t) - \dot{D}_b(t_a)\, D_a(t)\right] + \dot{x}_a\, D_a(t) \\ &\equiv x(\,x_a,\, \dot{x}_a\,;\; t\,) \end{aligned} \tag{2.239}$$

The Gelfand-Yaglom function [see (2.224b)] $D_{\text{ren}}(t) = D_a(t)$ can therefore be obtained from (2.239) as

$$D_{\text{ren}}(t) = D_a(t) = \frac{\partial\, x(\,x_a,\, \dot{x}_a\,;\; t\,)}{\partial\, \dot{x}_a} \tag{2.240}$$

where we've treated x_a & $\dot{x}_a$ as the only independent variables so that

$$\frac{\partial\, x_a}{\partial\, \dot{x}_a} = 0 \qquad \frac{\partial\, \dot{x}_a}{\partial\, \dot{x}_a} = 1 \tag{2.240a}$$

Obviously, $\dfrac{\partial\, x(\,x_a,\, \dot{x}_a\,;\; t\,)}{\partial\, \dot{x}_a}$ satisfies the same I.C. (2.226) as $D_a(t)$.

The fluctuation determinant is then given by

$$\begin{aligned} D_{\text{ren}} = D_a(t_b) &= \frac{\partial\, x(\,x_a,\, \dot{x}_a\,;\; t_b\,)}{\partial\, \dot{x}_a} \\ &= \frac{\partial\, x_b}{\partial\, \dot{x}_a} \end{aligned} \tag{2.241}$$

where

$$x_b \equiv x(\,x_a,\, \dot{x}_a\,;\; t_b\,) \tag{2.241a}$$

The fore-going results can be adapted to the I.C. at t_b, (2.227), by the transform

$$a \leftrightarrow b \qquad A \rightarrow -A$$

(2.239) thus becomes

$$\begin{aligned} x(t) &= -\left(\dot{x}_b - \frac{x_b}{D_a(t_b)}\dot{D}_a(t_b)\right) D_b(t) + \frac{x_b}{D_a(t_b)} D_a(t) \\ &= \frac{x_b}{D_a(t_b)}\left[D_a(t) + \dot{D}_a(t_b)\, D_b(t)\right] - \dot{x}_b\, D_b(t) \equiv x(\,x_b,\, \dot{x}_b\,;\; t\,) \end{aligned} \tag{2.242}$$

$$\rightarrow \qquad \tilde{D}_{\text{ren}}(t) = D_b(t) = -\frac{\partial\, x(x_b, \dot{x}_b;\, t)}{\partial\, \dot{x}_b}$$

$$D_{\text{ren}} = D_b(t_a) = -\frac{\partial\, x(x_b, \dot{x}_b;\, t_a)}{\partial\, \dot{x}_b} = -\frac{\partial\, x_a}{\partial\, \dot{x}_b} \tag{2.243}$$

These results readily generalize to the case where Ω^2 is a $D \times D$ matrix $\mathbf{\Omega}^2$. Differential eq. (2.216) thus becomes

$$\left(\frac{\partial^2}{\partial\, t^2} \mathbb{I} + \mathbf{\Omega}^2(t) \right) \cdot \boldsymbol{x}(t) = 0 \tag{2.244a}$$

or, in component form

$$\left(\frac{\partial^2}{\partial\, t^2} \delta_{ij} + \Omega^2_{ij}(t) \right) x_j(t) = 0 \tag{2.244b}$$

where $\Omega^2_{ij} \equiv \left(\mathbf{\Omega}^2\right)_{ij}$.

Given 2 independent solutions, $\boldsymbol{\xi}(t) = (\,\xi_i(t)\,)$ & $\boldsymbol{\eta}(t) = (\,\eta_j(t)\,)$ to (2.244a), the recipe (2.219) then produces a matrix Gelfand-Yaglom function $\mathbb{D}_a(t) = \left(D^a_{ij}(t) \right)$ satisfying the I.C.

$$\mathbb{D}_a\,(t_a) = \mathbb{0} = \left(D^a_{ij}(t_a) \right) \qquad\qquad \dot{\mathbb{D}}_a\,(t_a) = \mathbb{I} = \left(\dot{D}^a_{ij}(t_a) \right) \tag{2.244c}$$

Thus,

$$D_{\text{ren}} = \det\left(-\frac{\partial^2}{\partial\, t^2} \mathbb{I} - \mathbf{\Omega}^2(t) \right) = \det \mathbb{D}_a(t_b) \tag{2.244}$$

From a solution $\boldsymbol{x}(t)$ that satisfies the I.C.

$$\boldsymbol{x}(t_a) = \boldsymbol{x}_a \qquad\qquad \dot{\boldsymbol{x}}(t_a) = \boldsymbol{x}_a$$

we then get [see (2.240)]

$$D^a_{ij}(t) = \frac{\partial\, x_i(\boldsymbol{x}_a, \dot{\boldsymbol{x}}_a;\, t)}{\partial\, \dot{x}_{a\,j}} \quad \text{or} \qquad \mathbb{D}_a(t) = \frac{\partial\, \boldsymbol{x}\,(\boldsymbol{x}_a, \dot{\boldsymbol{x}}_a;\, t)}{\partial\, \dot{\boldsymbol{x}}_a}$$

so that (2.244) becomes

$$D_{\text{ren}} = \det \frac{\partial\, x_i(\boldsymbol{x}_a, \dot{\boldsymbol{x}}_a;\, t_b)}{\partial\, \dot{x}_{a\,j}} = \det \frac{\partial\, x_{b\,i}}{\partial\, \dot{x}_{a\,j}} \tag{2.245}$$

Working with $\mathbb{D}_b(t)$, we expect from (2.243) that

$$D_{\text{ren}} = \det\left(-\frac{\partial\, x_{a\,i}}{\partial\, \dot{x}_{b\,j}} \right) \tag{2.245a}$$

Finally, instead of the I.C.'s, one can impose a boundary condition (B.C.)

$$x(t_a) = x_a \qquad\qquad x(t_b) = x_b \tag{2.246a}$$

The solution is easily constructed using (2.226-7),

$$x(t) = \frac{x_a}{D_b(t_a)} D_b(t) + \frac{x_b}{D_a(t_b)} D_a(t) \;\equiv x(x_b,\, x_a;\, t) \tag{2.246}$$

Treating x_a, x_b as the only independent variables, we have

$$\frac{\partial\, x(x_b,\, x_a;\, t)}{\partial\, x_b} = \frac{D_a(t)}{D_a(t_b)} \qquad\qquad \frac{\partial\, x(x_b,\, x_a;\, t)}{\partial\, x_a} = \frac{D_b(t)}{D_b(t_a)} \tag{2.247}$$

which gives the Gelfand-Yaglom functions as

$$D_a(t) = D_a(t_b)\, \frac{\partial\, x(x_b,\, x_a;\, t)}{\partial\, x_b} \qquad\qquad D_b(t) = D_b(t_a)\, \frac{\partial\, x(x_b,\, x_a;\, t)}{\partial\, x_a} \tag{2.247a}$$

(2.246) gives

$$\dot{x}(t) = \frac{x_a}{D_b(t_a)} \dot{D}_b(t) + \frac{x_b}{D_a(t_b)} \dot{D}_a(t)$$

so that at the endpoints

$$\dot{x}_a \equiv \dot{x}(t_a) = \frac{x_a}{D_b(t_a)}\,\dot{D}_b(t_a) + \frac{x_b}{D_a(t_b)}\,\dot{D}_a(t_a)$$

$$= \frac{x_a}{D_b(t_a)}\,\dot{D}_b\,(t_a) + \frac{x_b}{D_a(t_b)} \qquad \text{[(2.226) used.]} \qquad (2.248)$$

$$\dot{x}_b \equiv \dot{x}(t_b) = \frac{x_a}{D_b(t_a)}\,\dot{D}_b(t_b) + \frac{x_b}{D_a(t_b)}\,\dot{D}_a(t_b)$$

$$= -\frac{x_a}{D_b(t_a)} + \frac{x_b}{D_a(t_b)}\,\dot{D}_a(t_b) \qquad \text{[(2.227) used.]} \qquad (2.249)$$

Since $D_{\text{ren}} = D_a(t_b) = D_b(t_a)$ [see (2.228)], we have

$$\frac{1}{D_{\text{ren}}} = \frac{\partial\,\dot{x}_a}{\partial\,x_b} = -\frac{\partial\,\dot{x}_b}{\partial\,x_a} \qquad (2.250)$$

Comparing (2.250) with, say, (2.241), we have

$$D_{\text{ren}} = \frac{\partial\,x_b}{\partial\,\dot{x}_a} = \frac{\partial\,x(\,x_a,\,\dot{x}_a\,;\,t_b\,)}{\partial\,\dot{x}_a} = \left(\frac{\partial\,\dot{x}(x_b,\,x_a;\,t)}{\partial\,x_b}\right)^{-1} = \left(\frac{\partial\,\dot{x}_a}{\partial\,x_b}\right)^{-1} \qquad (2.250a)$$

Writing the independent variables explicitly, we have

$$\left(\frac{\partial\,x_b(x_a,\,\dot{x}_a)}{\partial\,\dot{x}_a}\right)_{x_a} = \left[\left(\frac{\partial\,\dot{x}_a(x_b,\,x_a)}{\partial\,x_b}\right)_{x_a}\right]^{-1}$$

or simply

$$\left(\frac{\partial\,x_b}{\partial\,\dot{x}_a}\right)_{x_a} = \left[\left(\frac{\partial\,\dot{x}_a}{\partial\,x_b}\right)_{x_a}\right]^{-1} \qquad (2.251)$$

where the subscript x_a indicates that x_a is kept constant in the derivative. Note that (2.251) is just the mathematical identity

$$\left(\frac{\partial\,x(y,\,z)}{\partial\,y}\right)_z = \left[\left(\frac{\partial\,y(x,\,z)}{\partial\,x}\right)_z\right]^{-1}$$

or

$$\left(\frac{\partial\,x}{\partial\,y}\right)_z = \left[\left(\frac{\partial\,y}{\partial\,x}\right)_z\right]^{-1} \qquad \text{where } f(x,\,y,\,z) = 0$$

Note that all functional determinants calculated in this Chapter apply to the fluctuation factor of paths with fixed endpoints, i.e., for the Dirichlet B.C. Calculations for periodic B.C. are discussed in §§2.11 & 3.27.

See §§4.3, 17.4 & 17.11 for applications to fluctuations around known classical orbit.

2.4.6. Generalization to *D*-Dimensions

Generalizing to *D*-D is straightforward. Starting with (2.199), we have

$$\mathcal{A} = \int_{t_a}^{t_b} d\,t\,\frac{1}{2}\,M\left[\,(\delta\,\dot{\boldsymbol{x}})^2 - (\delta\,\boldsymbol{x})^T \cdot \boldsymbol{\Omega}^2(t)\cdot\delta\,\boldsymbol{x}\,\right] \qquad (2.252)$$

where $\boldsymbol{x} = (x_1,\ \ldots,\ x_D)^T = (x_i)$ is a column matrix and $\boldsymbol{\Omega}^2 = \left(\Omega_{ij}^2\right)$ is a $D\times D$ matrix.

The fluctuation factor (2.200) generalizes to

$$F_\omega^N(t_b,\,t_a) = \left(\frac{M}{2\,\pi\,i\,\hbar\,(t_b - t_a)}\right)^{D/2}\left(\frac{\det_N\!\left(-\epsilon^2\,\nabla\overline{\nabla} - \epsilon^2\,\boldsymbol{\Omega}^2\right)}{\det_N\!\left(-\epsilon^2\,\nabla\overline{\nabla}\right)}\right)^{-1/2} \qquad (2.253)$$

From (2.244), we have

$$D_{\text{ren}} = \det \mathbb{D}_a(t_b) = \det \mathbb{D}_b(t_a) \qquad (2.254)$$

while (2.244a) & (2.244c) give

$$\left(\frac{\partial^2}{\partial t^2}\mathbb{I} + \mathbf{\Omega}^2(t)\right)\cdot \mathbb{D}_a(t) = 0 \qquad \mathbb{D}_a(t_a) = \mathbb{0} \qquad \dot{\mathbb{D}}_a(t_a) = \mathbb{I} \tag{2.255}$$

The analog for $\mathbb{D}_b$ is obviously

$$\left(\frac{\partial^2}{\partial t^2}\mathbb{I} + \mathbf{\Omega}^2(t)\right)\cdot \mathbb{D}_b(t) = 0 \qquad \mathbb{D}_b(t_b) = \mathbb{0} \qquad \dot{\mathbb{D}}_b(t_b) = -\mathbb{I} \tag{2.256}$$

From (2.245-a), the *D*-D version of (2.250a) is

$$D_{\text{ren}} = \left[\det\left(\frac{\partial \dot{x}_{a\,i}}{\partial x_{b\,j}}\right)_{\boldsymbol{x}_a}\right]^{-1} = \left[\det\left(-\frac{\partial \dot{x}_{b\,i}}{\partial x_{a\,j}}\right)_{\boldsymbol{x}_b}\right]^{-1} \tag{2.257}$$