2.5. Harmonic Oscillator with Time-Dependent Frequency

2.5.1. Coordinate Space

Consider the path integral for a 1-D harmonic oscillator with a time-depandent frequency

$$(x_b\, t_b \mid x_a\, t_a) = \int \mathcal{D}\, x \exp\left(\frac{i}{\hbar} \mathcal{A}[x] \right) \tag{2.258}$$

where

$$\mathcal{A}[x] = \frac{1}{2} M \int_{t_a}^{t_b} d\, t \left[\dot{x}^2(t) - \Omega^2(t)\, x^2(t) \right] \tag{2.259}$$

Following (2.14), we write

$$(x_b\, t_b \mid x_a\, t_a) = F_\Omega(t_b,\, t_a) \exp\left(\frac{i}{\hbar} \mathcal{A}_{\rm cl} \right) \tag{2.260}$$

Note that although Kleinert had spent some time to explain why (2.180) was theoretically unsound, in practice, the correct ratio form (2.186) reduced to it when (2.215) was applied.

Thus,

$$F_\Omega(t_b,\, t_a) = \sqrt{\frac{M}{2\,\pi\, i\, \hbar\,(t_b - t_a)}} \left(\frac{\det\left(-\partial_t^2 - \Omega^2\right)}{\det\left(-\partial_t^2\right)} \right)^{-1/2}$$

$$= \sqrt{\frac{M}{2\,\pi\, i\, \hbar}} \frac{1}{\sqrt{\det\left(-\partial_t^2 - \Omega^2\right)}} \qquad \text{[(2.215) used.]}$$

$$= \sqrt{\frac{M}{2\,\pi\, i\, \hbar}} \frac{1}{\sqrt{D_{\rm ren}}} = \sqrt{\frac{M}{2\,\pi\, i\, \hbar}} \frac{1}{\sqrt{D_a(t_b)}} \tag{2.261}$$

$$= \sqrt{\frac{M}{2\,\pi\, i\, \hbar}} \left(\frac{\partial\, x_b}{\partial\, \dot{x}_a} \right)^{-1/2} = \sqrt{\frac{M}{2\,\pi\, i\, \hbar}} \left(\frac{\partial\, \dot{x}_a}{\partial\, x_b} \right)^{1/2} \qquad \text{[(2.241) used.]} \tag{2.262}$$

$$= \sqrt{\frac{M}{2\,\pi\, i\, \hbar}} \left(-\frac{\partial\, x_a}{\partial\, \dot{x}_b} \right)^{-1/2} = \sqrt{\frac{M}{2\,\pi\, i\, \hbar}} \left(-\frac{\partial\, \dot{x}_b}{\partial\, x_a} \right)^{1/2} \qquad \text{[(2.243) used.]} \tag{2.263}$$

Since the classical equation of motion takes the form of (2.150)

$$\ddot{x}_{\rm cl} = -\Omega^2\, x_{\rm cl} \tag{2.264a}$$

the classical action also takes the form of (2.154),

$$\mathcal{A}_{\rm cl} = \frac{1}{2} M \left(x_b\, \dot{x}_b - x_a\, \dot{x}_a \right) \tag{2.264}$$

(2.248-9) show that $\dot{x}_a$ & $\dot{x}_b$ are both linear functions of x_a & x_b, therefore

$$\sum_k \frac{\partial\, \dot{x}_j}{\partial\, x_k} x_k = \dot{x}_j \quad \text{where} \quad j,\, k = a,\, b \tag{2.265a}$$

(2.264) can therefore be written as

$$\mathcal{A}_{\rm cl} = \frac{1}{2} M \left[x_b \left(\frac{\partial\, \dot{x}_b}{\partial\, x_a} x_a + \frac{\partial\, \dot{x}_b}{\partial\, x_b} x_b \right) - x_a \left(\frac{\partial\, \dot{x}_a}{\partial\, x_a} x_a + \frac{\partial\, \dot{x}_a}{\partial\, x_b} x_b \right) \right]$$

$$= \frac{1}{2} M \left(x_b \frac{\partial \dot{x}_b}{\partial x_b} x_b - x_a \frac{\partial \dot{x}_a}{\partial x_a} x_a + x_b \frac{\partial \dot{x}_b}{\partial x_a} x_a - x_a \frac{\partial \dot{x}_a}{\partial x_b} x_b \right) \tag{2.265}$$

Calculating the partial derivatives from (2.248-9), or simply substituting them into (2.264), we have

$$\mathcal{A}_{\rm cl} = \frac{1}{2} M \left(x_b \frac{\dot{D}_a(t_b)}{D_a(t_b)} x_b - x_a \frac{\dot{D}_b(t_a)}{D_b(t_a)} x_a - x_b \frac{1}{D_b(t_a)} x_a - x_a \frac{1}{D_a(t_b)} x_b \right)$$

$$= \frac{1}{2 D_a(t_b)} M \left(x_b^2 \dot{D}_a(t_b) - x_a^2 \dot{D}_b(t_a) - 2 x_b x_a \right) \qquad \text{[(2.228) used.]} \tag{2.266}$$

Thus,

$$\frac{\partial^2 \mathcal{A}_{\rm cl}}{\partial x_b \partial x_a} = -\frac{M}{D_a(t_b)}$$

which provides yet another way to calculate $D_{\rm ren}$:

$$D_{\rm ren} = D_a(t_b) = D_b(t_a) = -M \left(\frac{\partial^2 \mathcal{A}_{\rm cl}}{\partial x_b \partial x_a} \right)^{-1} \tag{2.267}$$

For $\Omega(t) = \omega$, (2.266) reduces to (2.157) since $D_a(t)$ has the property (2.229) due to time-reversal invariance.

The D-D generalization of (2.262) is simply

$$F_\Omega(t_b, t_a) = \left(\frac{M}{2 \pi i \hbar} \right)^{-D/2} \left(\det \frac{\partial x_{b i}}{\partial \dot{x}_{a j}} \right)^{-1/2}$$

$$= \left(\frac{M}{2 \pi i \hbar} \right)^{-D/2} \left(\det \frac{\partial \dot{x}_{a i}}{\partial x_{b j}} \right)^{1/2} \tag{2.262}$$

All formulas for fluctuation factors hold initially only for sufficiently short times $t_b - t_a$. For larger times, they carry phase factors determined as before in (2.167). Hence,

$$F_\Omega(t_b, t_a) = \left(\frac{M}{2 \pi i \hbar} \right)^{-D/2} \left| \det \frac{\partial x_{b i}}{\partial \dot{x}_{a j}} \right|^{-1/2} e^{-i \nu \pi/2}$$

$$= \left(\frac{M}{2 \pi i \hbar} \right)^{-D/2} \left| \det \frac{\partial \dot{x}_{a i}}{\partial x_{b j}} \right|^{1/2} e^{-i \nu \pi/2} \tag{2.269}$$

where ν is the Maslov-Morse index, which gives the weighted number of zeros of $\det \frac{\partial x_{b i}}{\partial \dot{x}_{a j}}$ along the trajectory. The weight of a zero is equal to the reduction in the rank of $\frac{\partial x_{b i}}{\partial \dot{x}_{a j}}$ it causes. In the special case of 1-D, ν counts the turning points of the trajectory. In this context, ν is also called the Morse index of the trajectory.

The zeros of the functional determinant are also called conjugate points. They are generalizations of the turning points in 1-D systems. The surfaces in $\boldsymbol{x}$-space, on which the determinant vanishes, are called caustics. The conjugate points are the places where the orbits touch the caustics [see M.C.Gutzwiller, “Chaos in Classical and Quantum Mechanics”].

Note that for infinitesimal time intervals, all F_Ω & $\mathcal{A}$ coincide with those of a free particle.

With

$$\tau = t_b - t_a \to 0$$

we have

$$\boldsymbol{x}_b \approx \boldsymbol{x}_a + \dot{\boldsymbol{x}}_a \tau \qquad\qquad \boldsymbol{x}_a \approx \boldsymbol{x}_b - \dot{\boldsymbol{x}}_b \tau \tag{2.270}$$

$$\rightarrow \qquad \frac{\partial x_{b\,i}}{\partial \dot{x}_{a\,j}} = \delta_{ij}\,\tau \qquad\qquad \frac{\partial x_{a\,i}}{\partial \dot{x}_{b\,j}} = -\delta_{ij}\,\tau \tag{2.271}$$

Similarly,

$$\dot{\boldsymbol{x}}_a \approx \frac{\boldsymbol{x}_b - \boldsymbol{x}_a}{\tau} = \frac{\boldsymbol{x}_a - \boldsymbol{x}_b}{-\tau} \approx \dot{\boldsymbol{x}}_b \tag{2.272}$$

$$\rightarrow \qquad \frac{\partial \dot{x}_{a\,i}}{\partial x_{b\,j}} = \frac{\delta_{ij}}{\tau} \qquad\qquad \frac{\partial \dot{x}_{b\,i}}{\partial x_{a\,j}} = -\frac{\delta_{ij}}{\tau} \tag{2.273}$$

Inserting these into the D-D version of the classical action (2.264) gives

$$\mathcal{A}_{\text{cl}} \approx \frac{1}{2} M \dot{\boldsymbol{x}}_a \cdot (\boldsymbol{x}_b - \boldsymbol{x}_a) \approx \frac{1}{2} M \frac{(\boldsymbol{x}_b - \boldsymbol{x}_a)^2}{\tau} = \frac{1}{2} M \frac{(\boldsymbol{x}_b - \boldsymbol{x}_a)^2}{t_b - t_a} \tag{2.274}$$

2.5.2. Momentum Space

The action (2.265) can be written as a quadratic

$$\mathcal{A}_{\text{cl}} = \frac{1}{2} M \begin{pmatrix} x_b & x_a \end{pmatrix} \begin{pmatrix} \frac{\partial \dot{x}_b}{\partial x_b} & \frac{\partial \dot{x}_b}{\partial x_a} \\ -\frac{\partial \dot{x}_a}{\partial x_b} & -\frac{\partial \dot{x}_a}{\partial x_a} \end{pmatrix} \begin{pmatrix} x_b \\ x_a \end{pmatrix}$$

$$\equiv \frac{1}{2} M \boldsymbol{X}^T \mathbb{A} \boldsymbol{X} \tag{2.275}$$

where

$$\boldsymbol{X} = \begin{pmatrix} x_b \\ x_a \end{pmatrix} \qquad\qquad \mathbb{A} = \begin{pmatrix} \frac{\partial \dot{x}_b}{\partial x_b} & \frac{\partial \dot{x}_b}{\partial x_a} \\ -\frac{\partial \dot{x}_a}{\partial x_b} & -\frac{\partial \dot{x}_a}{\partial x_a} \end{pmatrix} \tag{2.276}$$

The inverse of $\mathbb{A}$ is

$$\mathbb{A}^{-1} = \begin{pmatrix} \frac{\partial x_b}{\partial \dot{x}_b} & -\frac{\partial x_b}{\partial \dot{x}_a} \\ \frac{\partial x_a}{\partial \dot{x}_b} & -\frac{\partial x_a}{\partial \dot{x}_a} \end{pmatrix} \tag{2.277}$$

since

$$\mathbb{A}\,\mathbb{A}^{-1} = \begin{pmatrix} \frac{\partial \dot{x}_b}{\partial x_b}\frac{\partial x_b}{\partial \dot{x}_b} + \frac{\partial \dot{x}_b}{\partial x_a}\frac{\partial x_a}{\partial \dot{x}_b} & -\frac{\partial \dot{x}_b}{\partial x_b}\frac{\partial x_b}{\partial \dot{x}_a} - \frac{\partial \dot{x}_b}{\partial x_a}\frac{\partial x_a}{\partial \dot{x}_a} \\ -\frac{\partial \dot{x}_a}{\partial x_b}\frac{\partial x_b}{\partial \dot{x}_b} - \frac{\partial \dot{x}_a}{\partial x_a}\frac{\partial x_a}{\partial \dot{x}_b} & \frac{\partial \dot{x}_a}{\partial x_b}\frac{\partial x_b}{\partial \dot{x}_a} + \frac{\partial \dot{x}_a}{\partial x_a}\frac{\partial x_a}{\partial \dot{x}_a} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial \dot{x}_b}{\partial \dot{x}_b} & -\frac{\partial \dot{x}_b}{\partial \dot{x}_a} \\ -\frac{\partial \dot{x}_a}{\partial \dot{x}_b} & \frac{\partial \dot{x}_a}{\partial \dot{x}_b} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}$$

The partial derivatives $\frac{\partial \dot{x}_j}{\partial x_k}$ in $\mathbb{A}$ can be calculated using (2.248-9). For $\frac{\partial x_j}{\partial \dot{x}_k}$ in $\mathbb{A}^{-1}$, we need the solution of (2.216) specified in terms of $\dot{x}_a$ & $\dot{x}_b$. Using (2.239a), we have

$$\dot{x}(t) = A\,\dot{D}_a(t) + B\,\dot{D}_b(t)$$

The I.C.s are therefore

$$\dot{x}_a = A\,\dot{D}_a(t_a) + B\,\dot{D}_b(t_a) = A + B\,\dot{D}_b(t_a) \qquad [(2.226)\text{ used.}]$$

$$\dot{x}_b = A\,\dot{D}_a(t_b) + B\,\dot{D}_b(t_b) = A\,\dot{D}_a(t_b) - B \qquad [(2.227)\text{ used.}]$$

$$\rightarrow \qquad A = \frac{\dot{x}_a + \dot{D}_b(t_a)\,\dot{x}_b}{1 + \dot{D}_a(t_b)\,\dot{D}_b(t_a)}$$

$$B = \frac{\dot{D}_a(t_b)\,\dot{x}_a - \dot{x}_b}{1 + \dot{D}_a(t_b)\,\dot{D}_b(t_a)}$$

(2.239a) thus becomes

$$x(t) = \frac{\left[\dot{x}_a + \dot{D}_b(t_a)\,\dot{x}_b\right] D_a(t) + \left[\dot{D}_a(t_b)\,\dot{x}_a - \dot{x}_b\right] D_b(t)}{1 + \dot{D}_a(t_b)\,\dot{D}_b(t_a)}$$

$$= \left(\left[D_a(t) + \dot{D}_a(t_b)\,D_b(t)\right]\dot{x}_a + \left[-D_b(t) + \dot{D}_b(t_a)\,D_a(t)\right]\dot{x}_b\right) \Big/ \left(1 + \dot{D}_a(t_b)\,\dot{D}_b(t_a)\right)$$

(2.278)

Hence,

$$x_a = \frac{\left[\dot{x}_a + \dot{D}_b(t_a)\,\dot{x}_b\right] D_a(t_a) + \left[\dot{D}_a(t_b)\,\dot{x}_a - \dot{x}_b\right] D_b(t_a)}{1 + \dot{D}_a(t_b)\,\dot{D}_b(t_a)}$$

$$= \frac{\left[\dot{D}_a(t_b)\,\dot{x}_a - \dot{x}_b\right] D_b(t_a)}{1 + \dot{D}_a(t_b)\,\dot{D}_b(t_a)} \qquad [(2.226)\text{ used.}] \qquad (2.279)$$

$$x_b = \frac{\left[\dot{x}_a + \dot{D}_b(t_a)\,\dot{x}_b\right] D_a(t_b) + \left[\dot{D}_a(t_b)\,\dot{x}_a - \dot{x}_b\right] D_b(t_b)}{1 + \dot{D}_a(t_b)\,\dot{D}_b(t_a)}$$

$$= \frac{\left[\dot{x}_a + \dot{D}_b(t_a)\,\dot{x}_b\right] D_a(t_b)}{1 + \dot{D}_a(t_b)\,\dot{D}_b(t_a)} \qquad [(2.227)\text{ used.}] \qquad (2.280)$$

(2.277) becomes

$$\mathbb{A}^{-1} = \frac{D_{\text{ren}}}{1 + \dot{D}_a(t_b)\,\dot{D}_b(t_a)} \begin{pmatrix} \dot{D}_b(t_a) & -1 \\ -1 & -\dot{D}_a(t_b) \end{pmatrix} \qquad (2.281)$$

From (2.276), we see that

$$\det \mathbb{A} = -\frac{\partial(\dot{x}_b, \dot{x}_a)}{\partial(x_b, x_a)} = \frac{1}{\det \mathbb{A}^{-1}}$$

$$= \left(\left[\frac{D_{\text{ren}}}{1 + \dot{D}_a(t_b)\,\dot{D}_b(t_a)}\right]^2 \left[-\dot{D}_b(t_a)\,\dot{D}_a(t_b) - 1\right]\right)^{-1}$$

$$= -\frac{1 + \dot{D}_a(t_b)\,\dot{D}_b(t_a)}{D_{\text{ren}}^2} \qquad (2.282)$$

Using (2.260-1), the Fourier transform of the time evolution amplitude becomes

$$(p_b\,t_b \mid p_a\,t_a) = \int d\,x_b\, e^{-i\,p_b\,x_b/\hbar} \int d\,x_a\, e^{-i\,p_a\,x_a/\hbar} (x_b\,t_b \mid x_a\,t_a) \qquad (2.283)$$

$$= \sqrt{\frac{M}{2\,\pi\,i\,\hbar}} \int d\,x_b\, e^{-i\,p_b\,x_b/\hbar} \int d\,x_a\, e^{-i\,p_a\,x_a/\hbar} \frac{1}{\sqrt{D_{\text{ren}}}} \exp\!\left(\frac{i}{\hbar}\mathcal{A}_{\text{cl}}\right)$$

$$= \sqrt{\frac{M}{2\,\pi\,i\,\hbar\,D_{\text{ren}}}} \int d\,\boldsymbol{X} \exp\left[\frac{i}{\hbar}\left(-\boldsymbol{p}^T \boldsymbol{X} + \frac{1}{2} M\,\boldsymbol{X}^T\,\mathbb{A}\,\boldsymbol{X}\right)\right] \qquad (2.283a)$$

where (2.275-6) were used.

(2.283a) can be evaluated using the Gaussian integral formula [see J.Zinn-Justin, "Quantum Field Theory & Critical Phenomena, §1.1],

$$\int d^n x \, \exp\left(\frac{1}{2} \boldsymbol{x}^T \mathbb{B} \, \boldsymbol{x} - \boldsymbol{b}^T \boldsymbol{x} \right) = \sqrt{\frac{(2\pi)^n}{\det \mathbb{B}}} \, \exp\left(\frac{1}{2} \boldsymbol{b}^T \mathbb{B}^{-1} \boldsymbol{b} \right)$$

with

$$\boldsymbol{b} = \frac{i}{\hbar} \boldsymbol{p} \qquad \mathbb{B} = \frac{i}{\hbar} M \mathbb{A}$$

$$\to \qquad \mathbb{B}^{-1} = \frac{\hbar}{i M} \mathbb{A}^{-1} \qquad \boldsymbol{b}^T \mathbb{B}^{-1} \boldsymbol{b} = \frac{i}{\hbar M} \boldsymbol{P}^T \mathbb{A}^{-1} \boldsymbol{P} \qquad \boldsymbol{P} = \begin{pmatrix} p_b \\ p_a \end{pmatrix}$$

$$\det \mathbb{B} = \left(\frac{i}{\hbar} M \right)^2 \det \mathbb{A}$$

Thus, (2.83a) becomes

$$(p_b \, t_b \mid p_a \, t_a) = \sqrt{-\frac{2\pi\hbar}{i M D_{\text{ren}} \det \mathbb{A}}} \, \exp\left(\frac{i}{2\hbar M} \boldsymbol{P}^T \mathbb{A}^{-1} \boldsymbol{P} \right) \tag{2.283b}$$

Using (2.281), we have

$$\begin{aligned} \boldsymbol{P}^T \mathbb{A}^{-1} \boldsymbol{P} &= \frac{D_{\text{ren}}}{1 + \dot{D}_a(t_b) \dot{D}_b(t_a)} \begin{pmatrix} p_b & p_a \end{pmatrix} \begin{pmatrix} \dot{D}_b(t_a) & -1 \\ -1 & -\dot{D}_a(t_b) \end{pmatrix} \begin{pmatrix} p_b \\ p_a \end{pmatrix} \\ &= \frac{D_{\text{ren}}}{1 + \dot{D}_a(t_b) \dot{D}_b(t_a)} \left[\dot{D}_b(t_a) \, p_b^2 - \dot{D}_a(t_b) \, p_a^2 - 2 \, p_b \, p_a \right] \end{aligned}$$

With (2.282), (2.283b) becomes

$$\begin{aligned} (p_b \, t_b \mid p_a \, t_a) &= \sqrt{\frac{2\pi\hbar}{i M}} \sqrt{\frac{D_{\text{ren}}}{1 + \dot{D}_a(t_b) \dot{D}_b(t_a)}} \\ &\quad \times \exp\left\{ \frac{i}{2\hbar M} \frac{D_{\text{ren}}}{1 + \dot{D}_a(t_b) \dot{D}_b(t_a)} \left[\dot{D}_b(t_a) \, p_b^2 - \dot{D}_a(t_b) \, p_a^2 - 2 \, p_b \, p_a \right] \right\} \end{aligned} \tag{2.283c}$$

Generalization to D-D is straightforward. First of all, (2.275) retains the same form

$$\mathcal{A}_{\text{cl}} = \frac{1}{2} M \boldsymbol{X}^T \mathbb{A} \, \boldsymbol{X} \tag{2.284}$$

but with

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}_b \\ \boldsymbol{x}_a \end{pmatrix} \qquad \mathbb{A} = \begin{pmatrix} \frac{\partial \dot{\boldsymbol{x}}_b}{\partial \boldsymbol{x}_b} & \frac{\partial \dot{\boldsymbol{x}}_b}{\partial \boldsymbol{x}_a} \\ -\frac{\partial \dot{\boldsymbol{x}}_a}{\partial \boldsymbol{x}_b} & -\frac{\partial \dot{\boldsymbol{x}}_a}{\partial \boldsymbol{x}_a} \end{pmatrix} \tag{2.285}$$

where $\boldsymbol{x} = (x_1, \ldots, x_D)^T$ so that $\mathbb{A}$ is a $2D \times 2D$ matrix.

The determinant of a block matrix

$$\mathbb{A} = \begin{pmatrix} \mathbb{a} & \mathbb{b} \\ \mathbb{c} & \mathbb{d} \end{pmatrix} \tag{2.286}$$

can be calculated after a triangular decomposition

$$\mathbb{A} = \begin{pmatrix} \mathbb{a} & \mathbb{0} \\ \mathbb{c} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & \mathbb{a}^{-1} \mathbb{b} \\ \mathbb{0} & \mathbb{d} - \mathbb{c} \, \mathbb{a}^{-1} \mathbb{b} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{1} & \mathbb{b} \\ \mathbb{0} & \mathbb{d} \end{pmatrix} \begin{pmatrix} \mathbb{a} - \mathbb{b}\,\mathbb{d}^{-1}\,\mathbb{c} & \mathbb{0} \\ \mathbb{d}^{-1}\,\mathbb{c} & \mathbb{1} \end{pmatrix} \tag{2.287}$$

which is easily proved by doing the matrix multiplication.

$$\begin{aligned} \rightarrow \qquad \det \mathbb{A} &= (\det \mathbb{a}) \det\left(\mathbb{d} - \mathbb{c}\,\mathbb{a}^{-1}\,\mathbb{b}\right) \\ &= (\det \mathbb{d}) \det\left(\mathbb{a} - \mathbb{b}\,\mathbb{d}^{-1}\,\mathbb{c}\right) \end{aligned} \tag{2.288}$$

The inverse of a lower (upper) triangular matrix is still a lower (upper) triangular matrix. For example, let

$$\mathbb{B} = \begin{pmatrix} \mathbb{a} & \mathbb{0} \\ \mathbb{b} & \mathbb{c} \end{pmatrix}$$

Using

$$\begin{pmatrix} \mathbb{a} & \mathbb{0} \\ \mathbb{b} & \mathbb{c} \end{pmatrix} \begin{pmatrix} \mathbb{a}^{-1} & \mathbb{0} \\ \mathbb{x} & \mathbb{c}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{b}\,\mathbb{a}^{-1} + \mathbb{c}\,\mathbb{x} & \mathbb{1} \end{pmatrix}$$

we have

$$\mathbb{B}^{-1} = \begin{pmatrix} \mathbb{a}^{-1} & \mathbb{0} \\ -\mathbb{c}^{-1}\,\mathbb{b}\,\mathbb{a}^{-1} & \mathbb{c}^{-1} \end{pmatrix}$$

Similarly,

$$\mathbb{C} = \begin{pmatrix} \mathbb{a} & \mathbb{b} \\ \mathbb{0} & \mathbb{c} \end{pmatrix} \qquad \rightarrow \qquad \mathbb{C}^{-1} = \begin{pmatrix} \mathbb{a}^{-1} & -\mathbb{a}^{-1}\,\mathbb{b}\,\mathbb{c}^{-1} \\ \mathbb{0} & \mathbb{c}^{-1} \end{pmatrix}$$

Thus, the inverse of $\mathbb{A}$ in (2.287) is

$$\begin{aligned} \mathbb{A}^{-1} &= \begin{pmatrix} \mathbb{1} & \mathbb{a}^{-1}\,\mathbb{b} \\ \mathbb{0} & \mathbb{d} - \mathbb{c}\,\mathbb{a}^{-1}\,\mathbb{b} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{a} & \mathbb{0} \\ \mathbb{c} & \mathbb{1} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \mathbb{1} & -\mathbb{a}^{-1}\,\mathbb{b}\left(\mathbb{d} - \mathbb{c}\,\mathbb{a}^{-1}\,\mathbb{b}\right)^{-1} \\ \mathbb{0} & \left(\mathbb{d} - \mathbb{c}\,\mathbb{a}^{-1}\,\mathbb{b}\right)^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{a}^{-1} & \mathbb{0} \\ -\mathbb{c}\,\mathbb{a}^{-1} & \mathbb{1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{1} & -\mathbb{a}^{-1}\,\mathbb{b}\,\mathbb{x} \\ \mathbb{0} & \mathbb{x} \end{pmatrix} \begin{pmatrix} \mathbb{a}^{-1} & \mathbb{0} \\ -\mathbb{c}\,\mathbb{a}^{-1} & \mathbb{1} \end{pmatrix} \qquad \mathbb{x} = \left(\mathbb{d} - \mathbb{c}\,\mathbb{a}^{-1}\,\mathbb{b}\right)^{-1} \\ &= \begin{pmatrix} \mathbb{a}^{-1} + \mathbb{a}^{-1}\,\mathbb{b}\,\mathbb{x}\,\mathbb{c}\,\mathbb{a}^{-1} & -\mathbb{a}^{-1}\,\mathbb{b}\,\mathbb{x} \\ -\mathbb{x}\,\mathbb{c}\,\mathbb{a}^{-1} & \mathbb{x} \end{pmatrix} \end{aligned} \tag{2.289}$$

(2.283b) thus generalizes to

$$\begin{aligned} (\boldsymbol{p}_b\, t_b \mid \boldsymbol{p}_a\, t_a) &= \int d\,\boldsymbol{x}_b\, e^{-i\,\boldsymbol{p}_b \cdot \boldsymbol{x}_b / \hbar} \int d\,x_a\, e^{-i\,\boldsymbol{p}_a \cdot \boldsymbol{x}_a / \hbar} (\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) \\ &= \sqrt{\frac{2\,\pi}{i\,\hbar\, M\, D_{\text{ren}} \det \mathbb{A}}}\ \exp\left(\frac{i}{2\,M\,\hbar}\, \boldsymbol{P}^T\, \mathbb{A}^{-1}\, \boldsymbol{P}\right) \end{aligned} \tag{2.290}$$

where

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{p}_b \\ \boldsymbol{p}_a \end{pmatrix}$$