## 2.7. General Time-Dependent Harmonic Action

A further generalization of (2.259) is to allow for a time-dependent mass so that the Lagrangian becomes

$$L = \frac{1}{2} M \left[ g(t)\,\dot{x}^2(t) - \Omega^2(t)\,x^2(t) \right] \tag{2.305a}$$

$$\rightarrow \qquad p = \frac{\partial L}{\partial \dot{x}} = M g \dot{x} \qquad\qquad H = \frac{p^2}{2Mg} - \frac{1}{2} M \Omega^2 x^2 \tag{2.203b}$$

The presence of $g(t)$ in the kinetic energy requires a re-calculation of the $\mathcal{D}\,p$ integration that leads to the configuration path integral (2.60).

The canonical integral and action (2.27-9) now read

$$(x_b\, t_b \mid x_a\, t_a) = \int \mathcal{D}'x \int \frac{\mathcal{D}\,p}{2\pi\hbar} \exp\left( \frac{i}{\hbar} \mathcal{A}[x, p] \right) \tag{2.306}$$

$$\mathcal{A}[x, p] = \int_{t_a}^{t_b} dt \left( p\,\dot{x} - \frac{p^2}{2Mg} - \frac{1}{2} M \Omega^2 x^2 \right) \tag{2.307}$$

$$= \int_{t_a}^{t_b} dt \left[ -\frac{1}{2Mg} (p - g M \dot{x})^2 + \frac{g}{2} M \dot{x}^2 - \frac{1}{2} M \Omega^2 x^2 \right]$$

Thus, (2.50) becomes

$$\mathcal{A}^N = \sum_{n=1}^{N+1} \epsilon \left[ -\frac{1}{2Mg} \left( p_n - g M \frac{x_n - x_{n-1}}{\epsilon} \right)^2 + \frac{g}{2} M \left( \frac{x_n - x_{n-1}}{\epsilon} \right)^2 - \frac{1}{2} M \Omega^2 x_n^2 \right]$$

The momentum integration is therefore

$$\int \frac{d p_n}{2\pi\hbar} \exp\left[ -\frac{i\epsilon}{2M\hbar g} \left( p_n - g M \frac{x_n - x_{n-1}}{\epsilon} \right)^2 \right] = \left( \frac{Mg}{2\pi i \hbar \epsilon} \right)^{1/2}$$

Hence, the configuration path integral becomes

$$(x_b\, t_b \mid x_a\, t_a) = \int \overline{\mathcal{D}}x \, \exp\left( \frac{i}{\hbar} \mathcal{A}[x] \right) \tag{2.309}$$

with

$$\int \overline{\mathcal{D}}x = \left( \frac{Mg}{2\pi i \hbar \epsilon} \right)^{(N+1)/2} \lim_{N\to\infty} \left( \prod_{k=1}^{N} \int d x_k \right) \tag{2.308a}$$

$$\equiv \int \mathcal{D}x \sqrt{g} \tag{2.309a}$$

$$\mathcal{A}[x] = \int_{t_a}^{t_b} dt \, \frac{1}{2} M \left( g \dot{x}^2 - \Omega^2 x^2 \right) = \int_{t_a}^{t_b} dt \, L \tag{2.309b}$$

where $L$ is given by (2.305a).

The Lagrange eq. is

$$\frac{d}{dt} (g \dot{x}) + \Omega^2 x = 0 \tag{2.310}$$

Setting

$$x = \frac{y}{\sqrt{g}} \qquad \rightarrow \qquad \dot{x} = -\frac{1}{2 g^{3/2}} \dot{g}\, y + \frac{\dot{y}}{\sqrt{g}} \tag{2.311a}$$

$$\therefore \qquad g \dot{x} = -\frac{1}{2 g^{1/2}} \dot{g}\, y + \sqrt{g}\, \dot{y}$$

$$\frac{d}{dt}(g\dot{x}) = \frac{1}{4g^{3/2}}\dot{g}^2 y - \frac{1}{2g^{1/2}}\ddot{g}y + \sqrt{g}\,\ddot{y}$$

and (2.310) becomes

$$\sqrt{g}\left[\ddot{y} + \frac{1}{g}\left(\Omega^2 + \frac{1}{4g}\dot{g}^2 - \frac{1}{2}\ddot{g}\right)y\right] = 0$$

$$\to \qquad \ddot{y} + \tilde{\Omega}^2 y = 0 \qquad (2.312)$$

where

$$\tilde{\Omega}^2 = \frac{1}{g}\left(\Omega^2 + \frac{1}{4g}\dot{g}^2 - \frac{1}{2}\ddot{g}\right) \qquad (2.311b)$$

(2.238-9) thus become

$$\int \overline{\mathcal{D}}x = \int \mathcal{D}y = \left(\frac{M}{2\pi i\hbar\epsilon}\right)^{(N+1)/2} \lim_{N\to\infty}\left(\prod_{k=1}^{N}\int dy_k\right)$$

$$L = \frac{1}{2}M\left[g\left(-\frac{1}{2g^{3/2}}\dot{g}y + \frac{\dot{y}}{\sqrt{g}}\right)^2 - \Omega^2\frac{y^2}{g}\right]$$

$$= \frac{1}{2}M\left[\dot{y}^2 - \frac{\dot{g}}{g}y\dot{y} - \frac{1}{g}\left(\Omega^2 - \frac{1}{4g}\dot{g}^2\right)y^2\right] \qquad (2.311c)$$

Using

$$\int_{t_a}^{t_b} dt\,\frac{\dot{g}}{g}y\dot{y} = \int_{t_a}^{t_b} dt\,\frac{\dot{g}}{2g}\frac{dy^2}{dt} = \frac{\dot{g}}{2g}y^2\bigg|_{t_a}^{t_b} - \int_{t_a}^{t_b} dt\,y^2\frac{d}{dt}\left(\frac{\dot{g}}{2g}\right)$$

$$= \frac{\dot{g}_b}{2g_b}y_b^2 - \frac{\dot{g}_a}{2g_a}y_a^2 - \int_{t_a}^{t_b} dt\,\frac{1}{2}\left(\frac{\ddot{g}}{g} - \frac{\dot{g}^2}{g^2}\right)y^2$$

with (2.311c), (2.309b) becomes

$$\mathcal{A}[y] = -\frac{1}{4}M\left(\frac{\dot{g}_b}{g_b}y_b^2 - \frac{\dot{g}_a}{g_a}y_a^2\right) + \int_{t_a}^{t_b} dt\,\frac{1}{2}M\left(\dot{y}^2 - \tilde{\Omega}^2 y^2\right) \qquad (2.313a)$$

$$= \mathcal{B} + \tilde{\mathcal{A}}[y]$$

where

$$\mathcal{B} = -\frac{1}{4}M\left(\frac{\dot{g}_b}{g_b}y_b^2 - \frac{\dot{g}_a}{g_a}y_a^2\right) \qquad (2.313b)$$

$$\tilde{\mathcal{A}}[y] = \int_{t_a}^{t_b} dt\,\frac{1}{2}M\left(\dot{y}^2 - \tilde{\Omega}^2 y^2\right) \qquad (2.313c)$$

(2.309) thus becomes

$$(x_b t_b \mid x_a t_a) = (y_b t_b \mid y_a t_a)_{y=\sqrt{g}\,x} = \int \mathcal{D}y\,\exp\left(\frac{i}{\hbar}\mathcal{A}[y]\right) \qquad (2.313d)$$

Since (2.313c,d) are in the form (2.258-9), the results of §2.5 apply.

Following (2.260-1,4,6,7), we have

$$(x_b t_b \mid x_a t_a) = F_\Omega(t_b, t_a)\exp\left(\frac{i}{\hbar}\mathcal{A}_{\rm cl}\right) \qquad (2.313c)$$

$$= e^{i\mathcal{B}/\hbar}\tilde{F}_\Omega(t_b, t_a)\exp\left(\frac{i}{\hbar}\tilde{\mathcal{A}}_{\rm cl}\right)$$

where

$$\tilde{F}_\Omega(t_b, t_a) = \sqrt{\frac{M}{2\pi i\hbar}}\, \frac{1}{\sqrt{\tilde{D}_{\text{ren}}}} \qquad (2.314)$$

$$\tilde{\mathcal{A}}_{\text{cl}} = \frac{1}{2} M\,(\, y_b\, \dot{y}_b - y_a\, \dot{y}_a\,) \qquad (2.315a)$$

$$= \frac{1}{2\,\tilde{D}_a(t_b)} M\left(y_b^2\, \dot{\tilde{D}}_a(t_b) - y_a^2\, \dot{\tilde{D}}_b(t_a) - 2\, y_b\, y_a\right) \qquad (2.315b)$$

$$\tilde{D}_{\text{ren}} = \tilde{D}_a(t_b) = \tilde{D}_b(t_a) = -M\left(\frac{\partial^2 \tilde{\mathcal{A}}_{\text{cl}}}{\partial\, y_b\, \partial\, y_a}\right)^{-1} = \frac{\partial\, y_b}{\partial\, \dot{y}_a} = -\frac{\partial\, y_a}{\partial\, \dot{y}_b} \qquad (2.316a)$$

The analog of (2.227-8) are

$$\left(\frac{d^2}{d\,t^2} + \tilde{\Omega}^2(t)\right)\tilde{D}_a(t) = 0 \text{ with I.C.} \qquad \tilde{D}_a(t_a) = 0 \qquad \dot{\tilde{D}}_a(t_a) = 1 \qquad (2.316b)$$

$$\left(\frac{d^2}{d\,t^2} + \tilde{\Omega}^2(t)\right)\tilde{D}_b(t) = 0 \text{ with I.C.} \qquad \tilde{D}_b(t_b) = 0 \qquad \dot{\tilde{D}}_b(t_b) = -1 \qquad (2.317a)$$

(2.311a) implies

$$\dot{y} = \sqrt{g}\;\dot{x} + \frac{1}{2\sqrt{g}}\,\dot{g}\,x \quad y\,\dot{y} = g\,x\,\dot{x} + \frac{1}{2}\,\dot{g}\,x^2 \qquad \frac{\dot{g}}{2\,g}\,y^2 = \frac{1}{2}\,\dot{g}\,x^2 \quad (2.315c)$$

so that (2.313b & 2.315a) give

$$\mathcal{A}_{\text{cl}} = \mathcal{B} + \tilde{\mathcal{A}}_{\text{cl}} = \frac{1}{2} M\,(\, g_b\, x_b\, \dot{x}_b - g_a\, x_a\, \dot{x}_a\,) \qquad (2.315)$$

Inverting (2.311a) & (2.315c) gives

$$x = \frac{y}{\sqrt{g}} \qquad \dot{x} = \frac{1}{\sqrt{g}}\left(\dot{y} - \frac{1}{2\,g}\,\dot{g}\,y\right)$$

$$\rightarrow \quad \frac{\partial\, x}{\partial\, y} = \frac{1}{\sqrt{g}} \qquad \frac{\partial\, x}{\partial\, \dot{y}} = 0$$

$$\frac{\partial\, \dot{x}}{\partial\, y} = -\frac{\dot{g}}{2\, g^{3/2}} \qquad \frac{\partial\, \dot{x}}{\partial\, \dot{y}} = \frac{1}{\sqrt{g}}$$

(2.246) gives

$$y(t) = \frac{y_a}{\tilde{D}_b(t_a)}\,\tilde{D}_b(t) + \frac{y_b}{\tilde{D}_a(t_b)}\,\tilde{D}_a(t) \qquad (2.318a)$$

From (2.311a), we have

$$D_a(t) = \frac{\tilde{D}_a(t)}{\sqrt{g(t)}} \qquad D_b(t) = \frac{\tilde{D}_b(t)}{\sqrt{g(t)}} \qquad (2.318b)$$

$$\rightarrow \quad \dot{D}_a = \frac{\dot{\tilde{D}}_a}{\sqrt{g}} - \frac{1}{2\, g^{3/2}}\,\dot{g}\,\tilde{D}_a \qquad \dot{D}_b = \frac{\dot{\tilde{D}}_b}{\sqrt{g}} - \frac{1}{2\, g^{3/2}}\,\dot{g}\,\tilde{D}_b \qquad (2.318c)$$

Therefore, (2.318a) becomes

$$\sqrt{g(t)}\;\, x(t) = \frac{\sqrt{g_a}\;\, x_a}{\sqrt{g_a}\;\, D_b(t_a)}\,\sqrt{g(t)}\;\, D_b(t) + \frac{\sqrt{g_b}\;\, x_b}{\sqrt{g_b}\;\, D_a(t_b)}\,\sqrt{g(t)}\;\, D_a(t)$$

$$\rightarrow \qquad x(t) = \frac{x_a}{D_b(t_a)} D_b(t) + \frac{x_b}{D_a(t_b)} D_a(t) \tag{2.318}$$

From (2.316a), we have

$$\tilde{D}_{\text{ren}} = \sqrt{g_b}\; D_a(t_b) = \sqrt{g_a}\; D_b(t_a) = -M \left( \frac{\partial^2 \tilde{\mathcal{A}}_{\text{cl}}}{\partial y_b \, \partial y_a} \right)^{-1} \tag{2.320a}$$

which disagrees with Kleinert's (2.320).

(2.318) can be written as

$$x(t) = \frac{1}{\tilde{D}_{\text{ren}}} \left[ \sqrt{g_a}\; x_a D_b(t) + \sqrt{g_b}\; x_b D_a(t) \right] \tag{2.318a}$$

$$\dot{x}(t) = \frac{x_a}{D_b(t_a)} \dot{D}_b(t) + \frac{x_b}{D_a(t_b)} \dot{D}_a(t) \tag{2.318b}$$

With the I.C.s (2.320a & b), (2.318b, c) gives

$$D_a(t_a) = 0, \qquad \dot{D}_a(t_a) = \frac{\dot{\tilde{D}}_a(t_a)}{\sqrt{g_a}} - \frac{1}{2 g_a^{3/2}} \dot{g}_a \tilde{D}_a(t_a) = \frac{1}{\sqrt{g_a}}$$

$$D_b(t_b) = 0, \qquad \dot{D}_b(t_b) = \frac{\dot{\tilde{D}}_b(t_b)}{\sqrt{g_b}} - \frac{1}{2 g_b^{3/2}} \dot{g}_b \tilde{D}_b(t_b) = -\frac{1}{\sqrt{g_b}}$$

Together with (2.310), we have

$$\left( \frac{d}{dt} g(t) \frac{d}{dt} + \Omega^2(t) \right) D_a(t) = 0 \quad \text{with} \quad D_a(t_a) = 0 \qquad \dot{D}_a(t_a) = \frac{1}{\sqrt{g_a}} \tag{2.316c}$$

$$\left( \frac{d}{dt} g(t) \frac{d}{dt} + \Omega^2(t) \right) D_b(t) = 0 \quad \text{with} \quad D_b(t_b) = 0 \qquad \dot{D}_b(t_b) = -\frac{1}{\sqrt{g_b}} \tag{2.317b}$$

which differ from Kleinert's (2.316-7).

(2.318b) then gives

$$\dot{x}_b = -\frac{x_a}{\sqrt{g_b}\; D_b(t_a)} + \frac{x_b}{D_a(t_b)} \dot{D}_a(t_b) = \frac{1}{\tilde{D}_{\text{ren}}} \left[ -\sqrt{\frac{g_a}{g_b}}\; x_a + \sqrt{g_b}\; x_b \dot{D}_a(t_b) \right]$$

$$\dot{x}_a = \frac{x_a}{D_b(t_a)} \dot{D}_b(t_a) + \frac{x_b}{\sqrt{g_a}\; D_a(t_b)} = \frac{1}{\tilde{D}_{\text{ren}}} \left[ \sqrt{g_a}\; x_a \dot{D}_b(t_a) + \sqrt{\frac{g_b}{g_a}}\; x_b \right]$$

(2.315) thus becomes

$$\mathcal{A}_{\text{cl}} = \frac{M}{2 \tilde{D}_{\text{ren}}} \left\{ g_b x_b \left[ -\sqrt{\frac{g_a}{g_b}}\; x_a + \sqrt{g_b}\; x_b \dot{D}_a(t_b) \right] - g_a x_a \left[ \sqrt{g_a}\; x_a \dot{D}_b(t_a) + \sqrt{\frac{g_b}{g_a}}\; x_b \right] \right\}$$

$$= \frac{M}{2 \tilde{D}_{\text{ren}}} \left[ g_b^{3/2} x_b^2 \dot{D}_a(t_b) - g_a^{3/2} x_a^2 \dot{D}_b(t_a) - 2 \sqrt{g_b g_a}\; x_b x_a \right] \tag{2.319a}$$

$$= \frac{M}{2 D_a(t_b)} \left[ g_b x_b^2 \dot{D}_a(t_b) - g_a \sqrt{\frac{g_a}{g_b}}\; x_a^2 \dot{D}_b(t_a) - 2 \sqrt{g_a}\; x_b x_a \right]$$

which differs from Kleinert's (2.319). Thus,

$$\frac{\partial^2 \mathcal{A}_{cl}}{\partial x_b \partial x_a} = -\frac{M\sqrt{g_b g_a}}{\tilde{D}_{ren}} \qquad \rightarrow \qquad \tilde{D}_{ren} = -M\sqrt{g_b g_a}\left(\frac{\partial^2 \mathcal{A}_{cl}}{\partial x_b \partial x_a}\right)^{-1} \tag{2.320b}$$

The fluctuation factor (2.314) can then be written as

$$\tilde{F}_\Omega(t_b, t_a) = \frac{1}{\sqrt{2\pi i \hbar}} \sqrt{-\frac{1}{\sqrt{g_b g_a}} \frac{\partial^2 \mathcal{A}_{cl}}{\partial x_b \partial x_a}} \tag{2.321a}$$

For a free particle, (2.316c) becomes

$$\frac{d}{dt} g(t) \frac{d}{dt} D_a(t) = 0 \qquad \text{with} \qquad D_a(t_a) = 0, \qquad \dot{D}_a(t_a) = \frac{1}{\sqrt{g_a}}$$

$$\rightarrow \qquad D_a(t) = C\int_{t_a}^{t} dt' \frac{1}{g(t')} \text{ so that } \qquad D_a(t_a) = 0$$

$$\dot{D}_a(t) = C\frac{1}{g(t)} \quad \rightarrow \qquad C = \sqrt{g_a} \qquad \text{so that } \dot{D}_a(t_a) = \frac{1}{\sqrt{g_a}}$$

$$\therefore \qquad D_a(t) = \sqrt{g_a}\int_{t_a}^{t} dt' \frac{1}{g(t')} \qquad \dot{D}_a(t) = \frac{\sqrt{g_a}}{g(t)} \tag{2.322a}$$

Similarly, (2.317b) gives

$$D_b(t) = \sqrt{g_b}\int_{t}^{t_b} dt' \frac{1}{g(t')} \qquad \dot{D}_b(t) = -\frac{\sqrt{g_b}}{g(t)} \tag{2.322b}$$

$$\rightarrow \qquad \tilde{D}_{ren} = \sqrt{g_a g_b}\int_{t_a}^{t_b} dt' \frac{1}{g(t')} = \sqrt{g_a g_b}\; G \tag{2.322c}$$

where

$$G = \int_{t_a}^{t_b} dt' \frac{1}{g(t')} \tag{2.322d}$$

The classical action (2.319a) becomes

$$\begin{aligned} \mathcal{A}_{cl} &= \frac{M}{2\sqrt{g_a g_b}\; G}\left[ g_b^{3/2} x_b^2 \frac{\sqrt{g_a}}{g_b} + g_a^{3/2} x_a^2 \frac{\sqrt{g_b}}{g_a} - 2\sqrt{g_b g_a}\; x_b x_a \right] \\ &= \frac{M}{2G}\left( x_b^2 + x_a^2 - 2 x_b x_a \right) \\ &= \frac{M}{2G}\left( x_b - x_a \right)^2 \end{aligned} \tag{2.323}$$

which has the same value as Kleinert's result. Note however that his definition of $D_a(t)$ and $D_b(t)$ cannot be reconciled with ours.

A further generalization that still keeps the Lagrangian quadratic is

$$L = \frac{1}{2} M\left[ g(t)\,\dot{x}^2(t) + 2\,b(t)\,x(t)\,\dot{x}(t) - \Omega^2(t)\,x^2(t) \right] \tag{2.324a}$$

$$\rightarrow \qquad p = \frac{\partial L}{\partial \dot{x}} = M(g\dot{x} + b x) \tag{2.324b}$$

$$H = \frac{(p - M b x)^2}{2Mg} + \frac{1}{2} M \Omega^2 x^2 \tag{2.324c}$$

The associated Lagrange eq. is

$$\frac{d}{dt}(g\dot{x}+b x)-b\dot{x}+\Omega^2 x=0$$

$$\rightarrow \quad \frac{d}{dt}(g\dot{x})+\dot{b}x+\Omega^2 x=0 \tag{2.325}$$

Solution to this is therefore the same as before with $\Omega^2$ replaced by $\Omega^2+\dot{b}$. The same conclusion can also be obtained by a integration by part so that

$$\mathcal{A}=\frac{1}{2}M b x^2\Big|_{t_a}^{t_b}-\int_{t_a}^{t_b} dt\,\frac{1}{2}M\left[g\dot{x}^2-\left(\Omega^2+\dot{b}\right)x^2\right] \tag{2.325a}$$

thus leading to an extra term

$$\frac{M}{2}\left(b_b x_b^2-b_a x_a^2\right)$$

to $\mathcal{B}$ in (2.313b).