2.10. Quantum Statistics of the Harmonic Oscillator

For the 1-D harmonic oscillator, we have

$$Z_\omega^N = \left(\frac{M}{2\pi\hbar\epsilon}\right)^{N+1} \left(\prod_{n=0}^{N} \int_{-\infty}^{\infty} d\,x_n\right) \exp\left(-\frac{1}{\hbar}\mathcal{A}_e^N\right) \tag{2.382}$$

where

$$\begin{aligned}
\mathcal{A}_e^N &= -i\,\mathcal{A}^N \Big|_{t\to -i\tau} \\
&= -i\,\frac{1}{2} M \sum_{n=1}^{N+1} \epsilon\left[\left(\frac{x_n - x_{n-1}}{\epsilon}\right)^2 - \omega^2 x_n^2\right]_{\epsilon\to -i\epsilon} \\
&= \frac{1}{2} M \sum_{n=1}^{N+1} \epsilon\left[\left(\frac{x_n - x_{n-1}}{\epsilon}\right)^2 + \omega^2 x_n^2\right] \\
&= \frac{1}{2} M \sum_{n=1}^{N+1} \epsilon\left[\left(\overline{\nabla} x_n\right)^2 + \omega^2 x_n^2\right] && \text{[(2.92) used.]} \\
&= \frac{1}{2\epsilon} M \sum_{n=1}^{N+1} \epsilon^2\left(-x_n \nabla\overline{\nabla} x_n + \omega^2 x_n^2\right) && \text{[(2.99) used.]}
\end{aligned} \tag{2.383}$$

Since the periodic B.C. allows for a Fourier series expansion for x_n [see (2.378)], the analysis leading to (2.133) or (2.183) applies. Since there is an extra integration in (2.382), we have

$$Z_\omega^N = \frac{1}{\sqrt{\det_{N+1}\left(-\epsilon^2\nabla\overline{\nabla} + \epsilon^2\omega^2\right)}} \tag{2.384}$$

Following (2.107c, 2.118), we have

$$-\nabla\overline{\nabla}\, x(\omega_m) = \Omega_m \overline{\Omega}_m\, x(\omega_m) \tag{2.384a}$$

$$\Omega_m = -i\,\frac{e^{i\omega_m\epsilon} - 1}{\epsilon} \qquad \overline{\Omega}(\omega) = -i\,\frac{1 - e^{-i\omega_m\epsilon}}{\epsilon} \tag{2.384b}$$

$$\Omega_m \overline{\Omega}_m = \frac{2}{\epsilon^2}\left[1 - \cos(\omega_m \epsilon)\right] \tag{2.384c}$$

where $\omega_m = \dfrac{2\pi m}{\beta\hbar}$ are the Matsubara frequencies given in (2.379). Hence, the eigenvalues of $-\epsilon^2\nabla\overline{\nabla} + \epsilon^2\omega^2$ are

$$\epsilon^2\Omega_m\overline{\Omega}_m + \epsilon^2\omega^2 = 2\left[1 - \cos(\omega_m\epsilon)\right] + \epsilon^2\omega^2 \tag{2.385}$$

Since $\dfrac{\partial}{\partial\tau} = \dfrac{\partial}{i\,\partial t}$, its counterpart in the ω-space is $-i\nabla$. The eigenvalues of $-i\epsilon\nabla$ are

$$-i\epsilon\,\Omega_m = 1 - e^{i\omega_m\epsilon} = 1 - \cos(\omega_m\epsilon) - i\sin(\omega_m\epsilon) \tag{2.385a}$$

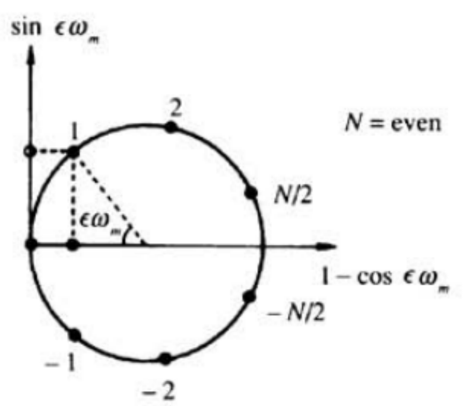
Those of $i\epsilon\overline{\nabla}$ are

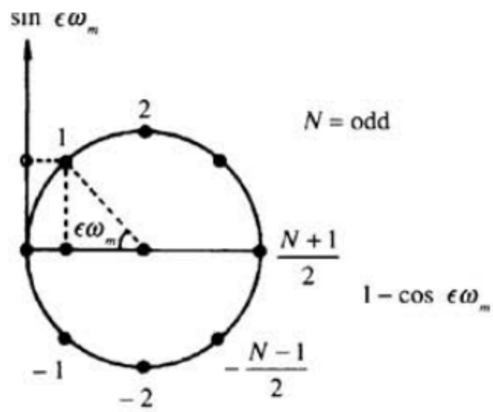
$$i\epsilon\,\overline{\Omega}_m = 1 - e^{-i\omega_m\epsilon} = 1 - \cos(\omega_m\epsilon) + i\sin(\omega_m\epsilon)$$

so that

$$\begin{aligned}
(-i\epsilon\,\Omega_m)\left(i\epsilon\,\overline{\Omega}_m\right) &= [1 - \cos(\omega_m\epsilon)]^2 + \sin^2(\omega_m\epsilon) \\
&= 2\left[1 - \cos(\omega_m\epsilon)\right] = \epsilon^2\,\Omega_m\overline{\Omega}_m
\end{aligned}$$

The eigenvalues $i\epsilon\overline{\Omega}_m$ are illustrated in the figures below for $N =$ even & odd.





By (2.384a), $\nabla\overline{\nabla}$ is diagonal in the basis $x_m \equiv x(\omega_m)$, which is the Fourier transform of the basis $x_n \equiv x(\tau_n)$.

Consider now the Fourier transform [see (2.380a)].

$$x_n = \frac{1}{\sqrt{N+1}} \sum_{m=-M_-}^{M_+} e^{-i\,\omega_m \tau_n}\, x_m \qquad \begin{array}{ll} M_\pm = \frac{1}{2}(N \pm 1) & \text{for } N \text{ odd} \\ M_\pm = \frac{1}{2} N & \text{for } N \text{ even} \end{array} \qquad (2.386a)$$

Since x_n real implies $x_m^* = x_{-m}$, we have $x_0^* = x_0$ so that $x_{m=0}$ is real. Also, since $x_{-(N+1)/2}$ is not included in the sum, $x_{m=(N+1)/2}$ for N odd must also real.

Using $\omega_m \tau_n = \dfrac{2\pi m n}{N+1}$, we have

$$e^{-i\,\omega_0 \tau_n} = 1, \qquad e^{-i\,\omega_{(N+1)/2} \tau_n} = e^{-i\pi n} = (-)^n \qquad \text{for } N \text{ odd}$$

(2.386b)

and

$$x_n = \begin{cases} \dfrac{1}{\sqrt{N+1}}\left[x_0 + \sum_{m=1}^{(N-1)/2}\left(e^{-i\,\omega_m \tau_n} x_m + e^{i\,\omega_m \tau_n} x_m^*\right) + (-)^n x_{(N+1)/2}\right] & N \text{ odd} \\ \dfrac{1}{\sqrt{N+1}}\left[x_0 + \sum_{m=1}^{N/2}\left(e^{-i\,\omega_m \tau_n} x_m + e^{i\,\omega_m \tau_n} x_m^*\right)\right] & N \text{ even} \end{cases}$$

$$= \begin{cases} \dfrac{1}{\sqrt{N+1}}\left\{x_0 + 2\sum_{m=1}^{(N-1)/2}\left[\cos(\omega_m \tau_n)\,\mathrm{Re}\,x_m + \sin(\omega_m \tau_n)\,\mathrm{Im}\,x_m\right] + (-)^n x_{(N+1)/2}\right\} & N \text{ odd} \\ \dfrac{1}{\sqrt{N+1}}\left\{x_0 + 2\sum_{m=1}^{N/2}\left[\cos(\omega_m \tau_n)\,\mathrm{Re}\,x_m + \sin(\omega_m \tau_n)\,\mathrm{Im}\,x_m\right]\right\} & N \text{ even} \end{cases} \qquad (2.386c)$$

Setting

$$\boldsymbol{X}_\tau = (x_0, x_1, \ldots, x_N)^T$$
$$\boldsymbol{X}_\omega = \left(x_{-M_-}, x_{-M_-+1}, \ldots, x_{M_+}\right)^T \qquad (2.386d)$$

the transform (2.386a) can be written as

$$\boldsymbol{X}_\tau = \mathbb{S}\,\boldsymbol{X}_\omega$$

The n^{th} row of $\mathbb{S}$ being

$$\mathbb{S}_n = \frac{1}{\sqrt{N+1}} \begin{cases} \left(e^{-i\,\omega_{-(N-1)/2} \tau_n}, e^{-i\,\omega_{-(N-3)/2} \tau_n}, \ldots, e^{-i\,\omega_{(N+1)/2} \tau_n}\right) & N \text{ odd} \\ \left(e^{-i\,\omega_{-N/2} \tau_n}, e^{-i\,\omega_{-(N-2)/2} \tau_n}, \ldots, e^{-i\,\omega_{N/2} \tau_n}\right) & N \text{ even} \end{cases} \qquad (2.386e)$$

The dot product of the n^{th} & k^{th} row is

$$\mathbb{S}_n^+ \mathbb{S}_k = \frac{1}{N+1} \begin{cases} \sum_{m=-(N-1)/2}^{(N+1)/2} e^{-i\,\omega_m (\tau_k - \tau_n)} & N \text{ odd} \\ \sum_{m=-N/2}^{N/2} e^{-i\,\omega_m (\tau_k - \tau_n)} & N \text{ even} \end{cases}$$

$$= \delta_{n k} \tag{2.386f}$$

where

$$\sum_{m=-(N-1)/2}^{(N+1)/2} e^{-i\,\omega_m (\tau_k-\tau_n)} = \sum_{m=-(N-1)/2}^{(N+1)/2} e^{-i\,2\,\pi\,m\,(k-n)/(N+1)}$$
$$= \frac{e^{i\,\pi\,(k-n)\,(N-1)/(N+1)} - e^{-i\,\pi\,(k-n)\,(N+3)/(N+1)}}{1 - e^{-i\,2\,\pi\,(k-n)/(N+1)}}$$
$$= e^{i\,\pi\,(k-n)\,(N-1)/(N+1)} \frac{1 - e^{-i\,2\,\pi\,(k-n)}}{1 - e^{-i\,2\,\pi\,(k-n)/(N+1)}}$$
$$= \begin{cases} 0 & \text{if } k \neq n \\ \lim\limits_{n\to k} \dfrac{i\,2\,\pi\,(k-n)}{i\,2\,\pi\,(k-n)\,/\,(N+1)} = N+1 & \text{if } k = n \end{cases}$$
$$= (N+1)\,\delta_{k n} \tag{2.386g}$$

$$\sum_{m=-N/2}^{N/2} e^{-i\,\omega_m (\tau_k-\tau_n)} = \frac{e^{i\,\pi\,(k-n)\,N/(N+1)} - e^{-i\,\pi\,(k-n)\,(N+2)/(N+1)}}{1 - e^{-i\,2\,\pi\,(k-n)/(N+1)}}$$
$$= e^{i\,\pi\,(k-n)\,N/(N+1)} \frac{1 - e^{-i\,2\,\pi\,(k-n)}}{1 - e^{-i\,2\,\pi\,(k-n)/(N+1)}}$$
$$= (N+1)\,\delta_{k n} \tag{2.386h}$$

Hence, the transform $\mathbb{S}$ is unitary,

$$\mathbb{S}^+ \,\mathbb{S} = \mathbb{I} \tag{2.386i}$$

Unitary transforms preserve lengths:

$$\boldsymbol{X}_\tau^+ \boldsymbol{X}_\tau = \sum_{n=0}^{N} x_n^* \, x_n$$
$$= \frac{1}{N+1} \sum_{n=0}^{N} \sum_{m,\,m'=-M_-}^{M_+} \exp\left(\frac{i\,2\,\pi\,n\,(m-m')}{N+1} \right) x_m^* \, x_{m'}$$
$$= \frac{1}{N+1} \sum_{m,\,m'=-M_-}^{M_+} \frac{1 - \exp[\,i\,2\,\pi\,(m-m')\,]}{1 - \exp\left(\dfrac{i\,2\,\pi\,(m-m')}{N+1} \right)} x_m^* \, x_{m'}$$
$$= \sum_{m,\,m'=-M_-}^{M_+} \delta_{m m'} \, x_m^* \, x_{m'}$$
$$= \sum_{m=-M_-}^{M_+} x_m^* \, x_m = \boldsymbol{X}_\omega^+ \boldsymbol{X}_\omega \tag{2.386j}$$

For the transform (2.386c), all quantities are real. Setting

$$\tilde{\boldsymbol{X}}_\omega = \begin{cases} (\,x_0\,,\ \text{Re}\,x_1,\ \text{Im}\,x_1\,,\ \ldots,\ \text{Re}\,x_{(N-1)/2},\ \text{Im}\,x_{(N-1)/2}\,,\ x_{(N+1)/2})^T & N = \text{odd} \\ (\,x_0\,,\ \text{Re}\,x_1,\ \text{Im}\,x_1\,,\ \ldots,\ \text{Re}\,x_{N/2},\ \text{Im}\,x_{N/2}\,)^T & N = \text{even} \end{cases}$$

we have

$$\boldsymbol{X}_\tau = \tilde{\mathbb{S}}\,\tilde{\boldsymbol{X}}_\omega$$

with the n^{th} row of $\tilde{\mathbb{S}}$ being

$$\begin{cases} \dfrac{2}{\sqrt{N+1}} \left(\dfrac{1}{2},\ \cos(\omega_1\,\tau_n),\ \sin(\omega_1\,\tau_n),\ \ldots,\ \cos(\omega_{(N-1)/2}\,\tau_n),\ \sin(\omega_{(N-1)/2}\,\tau_n),\ \dfrac{1}{2}(-)^n \right) & N\,\text{odd} \\ \dfrac{2}{\sqrt{N+1}} \left(\dfrac{1}{2},\ \cos(\omega_1\,\tau_n),\ \sin(\omega_1\,\tau_n),\ \ldots,\ \cos(\omega_{N/2}\,\tau_n),\ \sin(\omega_{N/2}\,\tau_n) \right) & N\,\text{even} \end{cases}$$

(2.386k)

The dot product of the n^{th} & k^{th} rows is therefore

$$\tilde{\mathbb{S}}_n^T \tilde{\mathbb{S}}_k = \frac{4}{N+1} \begin{cases} \frac{1}{4} + \sum_{m=1}^{(N-1)/2} \left[\cos(\omega_m \tau_n) \cos(\omega_m \tau_k) + \sin(\omega_m \tau_n) \sin(\omega_m \tau_k) \right] + \frac{1}{4}(-)^{n+k} & N \text{ odd} \\ \frac{1}{4} + \sum_{m=1}^{N/2} \left[\cos(\omega_m \tau_n) \cos(\omega_m \tau_k) + \sin(\omega_m \tau_n) \sin(\omega_m \tau_k) \right] & N \text{ even} \end{cases}$$

$$= \frac{4}{N+1} \begin{cases} \frac{1}{4} + \sum_{m=1}^{(N-1)/2} \cos[\omega_m(\tau_n - \tau_k)] + \frac{1}{4}(-)^{n+k} & N \text{ odd} \\ \frac{1}{4} + \sum_{m=1}^{N/2} \cos[\omega_m(\tau_n - \tau_k)] & N \text{ even} \end{cases}$$

Using

$$\begin{aligned} \sum_{m=1}^{(N-1)/2} \cos(\omega_m \tau_n) &= \frac{1}{2} \sum_{m=1}^{(N-1)/2} \left(e^{i 2\pi m n/(N+1)} + e^{-i 2\pi m n/(N+1)} \right) \\ &= -\frac{1}{2} - \frac{1}{2}(-)^n + \frac{1}{2} \sum_{m=-(N-1)/2}^{(N+1)/2} e^{i 2\pi m n/(N+1)} \\ &= -\frac{1}{2} - \frac{1}{2}(-)^n + \frac{1}{2}(N+1)\,\delta_{n0} \end{aligned} \tag{2.386l}$$

$$\begin{aligned} \sum_{m=1}^{N/2} \cos(\omega_m \tau_n) &= \frac{1}{2} \sum_{m=1}^{N/2} \left(e^{i 2\pi m n/(N+1)} + e^{-i 2\pi m n/(N+1)} \right) \\ &= -\frac{1}{2} + \frac{1}{2} \sum_{m=-N/2}^{N/2} e^{i 2\pi m n/(N+1)} \\ &= -\frac{1}{2} + \frac{1}{2}(N+1)\,\delta_{n0} \end{aligned} \tag{2.386m}$$

we have

$$\tilde{\mathbb{S}}_n^T \tilde{\mathbb{S}}_k = \frac{4}{N+1} \begin{cases} -\frac{1}{4} + \frac{1}{2}(N+1)\,\delta_{nk} - \frac{1}{4}(-)^{n+k} & N \text{ odd} \\ -\frac{1}{4} + \frac{1}{2}(N+1)\,\delta_{nk} & N \text{ even} \end{cases} \tag{2.386n}$$

so that $\tilde{\mathbb{S}}$ is not orthogonal and hence difficult to use in practice.

This can be remedied by introducing another Fourier vector

$$\boldsymbol{Y}_\omega = \begin{cases} \sqrt{2} \left(\frac{1}{\sqrt{2}} x_0, \operatorname{Re} x_1, \operatorname{Im} x_1, \ldots, \operatorname{Re} x_{(N-1)/2}, \operatorname{Im} x_{(N-1)/2}, \frac{1}{\sqrt{2}} x_{(N+1)/2} \right)^T & N = \text{odd} \\ \sqrt{2} \left(\frac{1}{\sqrt{2}} x_0, \operatorname{Re} x_1, \operatorname{Im} x_1, \ldots, \operatorname{Re} x_{N/2}, \operatorname{Im} x_{N/2} \right)^T & N = \text{even} \end{cases} \tag{2.386o}$$

so that

$$\boldsymbol{X}_\tau = \mathbb{T}\, \boldsymbol{Y}_\omega \tag{2.386p}$$

with the n^{th} row of $\mathbb{T}$ being

$$\begin{cases} \sqrt{\dfrac{2}{N+1}} \left(\dfrac{1}{\sqrt{2}}, \cos(\omega_1 \tau_n), \sin(\omega_1 \tau_n), \ \ldots, \ \cos(\omega_{(N-1)/2} \tau_n), \sin(\omega_{(N-1)/2} \tau_n), \ \dfrac{1}{\sqrt{2}} (-)^n \right) & N \text{ odd} \\ \sqrt{\dfrac{2}{N+1}} \left(\dfrac{1}{\sqrt{2}}, \cos(\omega_1 \tau_n), \sin(\omega_1 \tau_n), \ \ldots, \ \cos(\omega_{N/2} \tau_n), \sin(\omega_{N/2} \tau_n) \right) & N \text{ even} \end{cases}$$

(2.386q)

The dot product of the n^{th} & k^{th} rows is therefore

$$\mathbb{T}_n^T \mathbb{T}_k = \frac{2}{N+1} \begin{cases} \dfrac{1}{2} + \displaystyle\sum_{m=1}^{(N-1)/2} \left[\cos(\omega_m \tau_n) \cos(\omega_m \tau_k) + \sin(\omega_m \tau_n) \sin(\omega_m \tau_k) \right] + \dfrac{1}{2} (-)^{n+k} & N \text{ odd} \\ \dfrac{1}{2} + \displaystyle\sum_{m=1}^{N/2} \left[\cos(\omega_m \tau_n) \cos(\omega_m \tau_k) + \sin(\omega_m \tau_n) \sin(\omega_m \tau_k) \right] & N \text{ even} \end{cases}$$

$$= \delta_{nk} \qquad (2.386r)$$

so that $\mathbb{T}$ is orthogonal.

Since $-\nabla\overline{\nabla}+\omega^2$ is diagonal in the basis x_m, (2.383) can be written as

$$\mathcal{A}_e^N = \frac{1}{2} M \epsilon \sum_{m=-M_-}^{M_+} x_m^* \left(\Omega_m \overline{\Omega}_m + \omega^2 \right) x_m$$

$$= \frac{1}{2} M \epsilon \begin{cases} \omega^2 x_0^2 + 2 \displaystyle\sum_{m=1}^{(N-1)/2} \left(\Omega_m \overline{\Omega}_m + \omega^2 \right) |x_m|^2 + \left(\Omega_{(N+1)/2} \overline{\Omega}_{(N+1)/2} + \omega^2 \right) |x_{(N+1)/2}|^2 & N \text{ odd} \\ \omega^2 x_0^2 + 2 \displaystyle\sum_{m=1}^{N/2} \left(\Omega_m \overline{\Omega}_m + \omega^2 \right) |x_m|^2 & N \text{ even} \end{cases} \qquad (2.387)$$

where $x_m^* = x_{-m}$, (2.384a), (2.384c) & (2.386a) were used.

Owing to the unitarity of $\mathbb{S}$ and orthogonality of $\mathbb{T}$, we have [see (2.386o)],

$$\prod_{n=0}^{N} \int_{-\infty}^{\infty} d x_n = \prod_{m=-M_-}^{M_+} \int d x_m = \prod_{m=0}^{N} \int_{-\infty}^{\infty} d y_m$$

$$= \begin{cases} 2^{(N-1)/2} \displaystyle\int_{-\infty}^{\infty} d x_0 \left(\prod_{m=1}^{(N-1)/2} \int_{-\infty}^{\infty} d \,\text{Re}\, x_m \int_{-\infty}^{\infty} d \,\text{Im}\, x_m \right) \int_{-\infty}^{\infty} d x_{(N+1)/2} & N \text{ odd} \\ 2^{N/2} \displaystyle\int_{-\infty}^{\infty} d x_0 \left(\prod_{m=1}^{N/2} \int_{-\infty}^{\infty} d \,\text{Re}\, x_m \int_{-\infty}^{\infty} d \,\text{Im}\, x_m \right) & N \text{ even} \end{cases} \qquad (2.388)$$

Using (2.384c), (2.384) gives

$$Z_\omega^N = \frac{1}{\sqrt{\det_{N+1}\left(-\epsilon^2 \nabla\overline{\nabla} + \epsilon^2 \omega^2\right)}} = \prod_{m=0}^{N} \frac{1}{\sqrt{\epsilon^2 \left(\Omega_m \overline{\Omega}_m + \omega^2 \right)}}$$

$$= \prod_{m=0}^{N} \frac{1}{\sqrt{2 \left[1 - \cos(\omega_m \epsilon) \right] + \epsilon^2 \omega^2}} = \prod_{m=0}^{N} \frac{1}{\sqrt{4 \sin^2 \frac{\omega_m \epsilon}{2} + \epsilon^2 \omega^2}} \qquad (2.389)$$

which, unlike its real-time counterpart (2.160), has no phase subtleties.

Using

$$-\cos \frac{\omega_m \epsilon}{2} = \cos\left(\pi - \frac{\omega_m \epsilon}{2} \right) = \cos\left(\pi - \frac{\pi m}{N+1} \right) = \cos\left(\pi \frac{N+1-m}{N+1} \right)$$

$$\prod_{m=1}^{N} \cos\left(\pi \frac{N+1-m}{N+1}\right) = \prod_{m=N}^{1} \cos\left(\frac{\pi m}{N+1}\right) = \prod_{m=1}^{N} \cos\left(\frac{\pi m}{N+1}\right)$$

and

$$\sin^2 \frac{\omega_m \epsilon}{2} = 1 - \cos^2 \frac{\omega_m \epsilon}{2} = \left(1 + \cos \frac{\omega_m \epsilon}{2}\right)\left(1 - \cos \frac{\omega_m \epsilon}{2}\right)$$
$$= \left[1 - \cos\left(\pi \frac{N+1-m}{N+1}\right)\right]\left(1 - \cos \frac{\pi m}{N+1}\right)$$

we have

$$\prod_{m=1}^{N} \sin^2 \frac{\omega_m \epsilon}{2} = \prod_{m=1}^{N} \left(1 - \cos \frac{\pi m}{N+1}\right)^2 = \prod_{m=1}^{N} \left(1 - \cos \frac{\omega_m \epsilon}{2}\right)^2$$

(2.389) thus becomes

$$Z_\omega^N = \frac{1}{\epsilon\,\omega} \prod_{m=1}^{N} \frac{1}{\sqrt{4 \sin^2 \frac{\omega_m \epsilon}{2} + \epsilon^2 \omega^2}} = \frac{1}{\epsilon\,\omega} \prod_{m=1}^{N} \frac{1}{2\sqrt{\sin^2 \frac{\omega_m \epsilon}{2}}\sqrt{1 + \frac{\epsilon^2 \omega^2}{4 \sin^2 \frac{\omega_m \epsilon}{2}}}}$$

$$= \frac{1}{\epsilon\,\omega} \prod_{m=1}^{N} \frac{1}{2\left(1 - \cos \frac{\omega_m \epsilon}{2}\right)\sqrt{1 + \frac{\epsilon^2 \omega^2}{4 \sin^2 \frac{\omega_m \epsilon}{2}}}} \qquad (2.393)$$

$$= \frac{1}{(N+1)\,\epsilon\,\omega} \prod_{m=1}^{N} \frac{1}{\sqrt{1 + \frac{\epsilon^2 \omega^2}{4 \sin^2 \frac{\omega_m \epsilon}{2}}}} \qquad \text{[(2.123) used.]}$$

$$= \frac{1}{\beta \hbar \omega} \prod_{m=1}^{N} \left(1 + \frac{\epsilon^2 \omega^2}{4 \sin^2 \frac{\omega_m \epsilon}{2}}\right)^{-1/2} \qquad (2.394)$$

Consider

$$\sin^2 \frac{\omega_m \epsilon}{2} = \sin^2 \frac{\pi m}{N+1} = \sin^2\left(\pi - \frac{\pi m}{N+1}\right) = \sin^2\left(\pi \frac{N+1-m}{N+1}\right)$$

For N odd and $m = \frac{N-1}{2} + k$,

$$\sin^2 \frac{\omega_{k+(N-1)/2}\, \epsilon}{2} = \sin^2\left(\pi \frac{\frac{N-1}{2} - (k-2)}{N+1}\right) = \sin^2 \frac{\omega_{-k+2+(N-1)/2}\, \epsilon}{2}$$

$$k = 1 \quad \rightarrow \qquad \sin^2 \frac{\omega_{(N+1)/2}\, \epsilon}{2} = \sin^2 \frac{\pi}{2} = 1$$

Thus, for $k = 2, \ldots, (N-1)/2$, we have

$$\left\{\sin^2 \frac{\omega_{2+(N-1)/2}\, \epsilon}{2}, \ldots, \sin^2 \frac{\omega_N \epsilon}{2}\right\} = \left\{\sin^2 \frac{\omega_{(N-1)/2}\, \epsilon}{2}, \ldots, \sin^2 \frac{\omega_1 \epsilon}{2}\right\}$$

$$\rightarrow \qquad Z_\omega^N = \frac{1}{\beta \hbar \omega}\left(1 + \frac{\epsilon^2 \omega^2}{4 \sin^2 \frac{\omega_{(N+1)/2}\, \epsilon}{2}}\right)^{-1/2} \prod_{m=1}^{(N-1)/2} \left(1 + \frac{\epsilon^2 \omega^2}{4 \sin^2 \frac{\omega_m \epsilon}{2}}\right)^{-1}$$

$$= \frac{1}{\beta \hbar \omega}\left(1 + \frac{\epsilon^2 \omega^2}{4}\right)^{-1/2} \prod_{m=1}^{(N-1)/2} \left(1 + \frac{\epsilon^2 \omega^2}{4 \sin^2 \frac{\omega_m \epsilon}{2}}\right)^{-1} \quad \text{for } N \text{ odd} \qquad (2.396)$$

For N even and $m = \frac{N}{2} + k$,

$$\sin^2\frac{\omega_{k+N/2}\,\epsilon}{2}=\sin^2\left(\pi\,\frac{\frac{N}{2}-(k-1)}{N+1}\right)=\sin^2\frac{\omega_{-k+1+N/2}\,\epsilon}{2}$$

Thus, for $k=1,\ \dots,\ N/2$, we have

$$\left\{\sin^2\frac{\omega_{1+N/2}\,\epsilon}{2},\ \dots,\ \sin^2\frac{\omega_N\,\epsilon}{2}\right\}=\left\{\sin^2\frac{\omega_{N/2}\,\epsilon}{2},\ \dots,\ \sin^2\frac{\omega_1\,\epsilon}{2}\right\}$$

$$\rightarrow\qquad Z_\omega^N=\frac{1}{\beta\hbar\omega}\prod_{m=1}^{N/2}\left(1+\frac{\epsilon^2\,\omega^2}{4\sin^2\frac{\omega_m\epsilon}{2}}\right)^{-1}\qquad\text{for } N \text{ even}\qquad(2.395)$$

The Euclidean analog of (2.161) is

$$\sin\left(i\,\frac{\tilde{\omega}_e\,\epsilon}{2}\right)=i\sinh\left(\frac{\tilde{\omega}_e\,\epsilon}{2}\right)\equiv i\,\frac{\omega\,\epsilon}{2}\qquad(2.397)$$

For N odd, the formula [see Gradshteyn & Ryzhik, Formula 1.391.1]

$$\prod_{m=1}^{(N-1)/2}\left(1-\frac{\sin^2 x}{\sin^2\frac{m\pi}{N+1}}\right)=\frac{2}{\sin 2x}\,\frac{\sin[\,(N+1)\,x\,]}{N+1}\qquad(2.398)$$

turns (2.396) into

$$Z_\omega^N=\frac{1}{\beta\hbar\omega}\left[1-\sin^2\left(i\,\frac{\tilde{\omega}_e\,\epsilon}{2}\right)\right]^{-1/2}\prod_{m=1}^{(N-1)/2}\left(1-\frac{\sin^2\left(i\,\frac{\tilde{\omega}_e\,\epsilon}{2}\right)}{4\sin^2\frac{m\pi}{N+1}}\right)^{-1}$$

$$=\frac{1}{\beta\hbar\omega}\,\frac{1}{\cos\left(i\,\frac{\tilde{\omega}_e\,\epsilon}{2}\right)}\,\frac{\sin(\,i\,\tilde{\omega}_e\,\epsilon)}{2}\,\frac{N+1}{\sin\left[\frac{i\,\tilde{\omega}_e(N+1)\,\epsilon}{2}\right]}$$

$$=\frac{1}{\beta\hbar\omega}\sin\left(i\,\frac{\tilde{\omega}_e\,\epsilon}{2}\right)\frac{N+1}{\sin\left[\frac{i\,\tilde{\omega}_e(N+1)\,\epsilon}{2}\right]}$$

$$=\frac{1}{\beta\hbar\omega}\sinh\left(\frac{\tilde{\omega}_e\,\epsilon}{2}\right)\frac{N+1}{\sinh\left[\frac{\tilde{\omega}_e(N+1)\,\epsilon}{2}\right]}\qquad N\text{ odd}\qquad(2.399)$$

For N even, the formula [see Gradshteyn & Ryzhik, Formula 1.391.3]

$$\prod_{m=1}^{N/2}\left(1-\frac{\sin^2 x}{\sin^2\frac{m\pi}{N+1}}\right)=\frac{1}{\sin x}\,\frac{\sin[\,(N+1)\,x\,]}{N+1}\qquad(2.400)$$

turns (2.395) into

$$Z_\omega^N=\frac{1}{\beta\hbar\omega}\prod_{m=1}^{N/2}\left(1-\frac{\sin^2\left(i\,\frac{\tilde{\omega}_e\,\epsilon}{2}\right)}{4\sin^2\frac{m\pi}{N+1}}\right)^{-1}$$

$$=\frac{1}{\beta\hbar\omega}\,\sin(\,i\,\tilde{\omega}_e\,\epsilon)\,\frac{N+1}{\sin\left[\frac{i\,\tilde{\omega}_e(N+1)\,\epsilon}{2}\right]}$$

$$=\frac{1}{\beta\hbar\omega}\sinh\left(\frac{\tilde{\omega}_e\,\epsilon}{2}\right)\frac{N+1}{\sinh\left[\frac{\tilde{\omega}_e(N+1)\,\epsilon}{2}\right]}\qquad N\text{ even}$$

which is the same as (2.399).

Using (2.397), (2.399) becomes

$$Z_\omega^N=\frac{1}{\beta\hbar\omega}\,\frac{\omega\,\epsilon}{2}\,\frac{N+1}{\sinh\left[\frac{\tilde{\omega}_e(N+1)\,\epsilon}{2}\right]}$$

$$= \frac{1}{2\sinh\left(\frac{\tilde{\omega}_e \beta \hbar}{2}\right)} \tag{2.401}$$

$$= \frac{1}{e^{\tilde{\omega}_e \beta\hbar/2} - e^{-\tilde{\omega}_e \beta\hbar/2}} = e^{-\tilde{\omega}_e \beta\hbar/2}\left(1 - e^{-\tilde{\omega}_e \beta\hbar}\right)^{-1}$$

$$= e^{-\tilde{\omega}_e \beta\hbar/2} \sum_{n=0}^{\infty} e^{-n\,\tilde{\omega}_e \beta\hbar} = \sum_{n=0}^{\infty} e^{-(n+1/2)\,\tilde{\omega}_e \beta\hbar}$$

$$= e^{-\tilde{\omega}_e \beta\hbar/2}\left(1 + e^{-2\,\tilde{\omega}_e \beta\hbar} + e^{-3\,\tilde{\omega}_e \beta\hbar} + \ldots\right)$$

$$= e^{-\tilde{\omega}_e \beta\hbar/2} + e^{-3\,\tilde{\omega}_e \beta\hbar/2} + e^{-5\,\tilde{\omega}_e \beta\hbar/2} + \ldots$$

$$= e^{-\tilde{\omega}_e \hbar/2 k_B T} + e^{-3\,\tilde{\omega}_e \hbar/2 k_B T} + e^{-5\,\tilde{\omega}_e \hbar/2 k_B T} + \ldots \tag{2.402}$$

Comparing with the spectral expansion (2.326), we obtain the eigenvalues of the system

$$E_n = \left(n + \frac{1}{2}\right)\hbar\,\tilde{\omega}_e \tag{2.403}$$

Inverting (2.397) gives

$$\tilde{\omega}_e = \frac{2}{\epsilon}\sinh^{-1}\left(\frac{\omega\,\epsilon}{2}\right) \tag{2.404}$$

In the continuum limit $\epsilon \to 0$,

$$Z_\omega = \frac{1}{2\sinh\left(\frac{\omega\beta\hbar}{2}\right)} \tag{2.405}$$

in agreement of the usual statistical mechanics result.

Alternatively, the $\epsilon \to 0$ limit can be taken directly on (2.394). However, according to the analysis that led to (2.395-6), each ω_m, except for $m = 0$, occurs twice in the product. Therefore, the limit should be taken on either (2.395) or (2.396) instead. Thus

$$Z_\omega = \frac{1}{\beta\hbar\omega}\prod_{m=1}^{\infty}\left(1 + \frac{\omega^2}{\omega_m^2}\right)^{-1} \tag{2.406}$$

$$= \frac{1}{\beta\hbar\omega}\prod_{m=1}^{\infty}\left[1 + \left(\frac{\beta\hbar\omega}{2\,\pi\,m}\right)^2\right]^{-1}$$

$$= \frac{1}{\beta\hbar\omega}\,\frac{\beta\hbar\omega/2}{\sinh\left(\frac{\beta\hbar\omega}{2}\right)} = \frac{1}{2\sinh\left(\frac{\omega\beta\hbar}{2}\right)} \tag{2.407}$$

where we've used the $x \to i\,x$ version of (2.171),

$$\prod_{m=1}^{\infty}\left(1 + \frac{x^2}{m^2\,\pi^2}\right) = \frac{\sin(i\,x)}{i\,x} = \frac{\sinh x}{x} \tag{2.407a}$$

As in the QM case, the continuum result must be obtained from the ratio of operators. Thus, (2.384) becomes

$$Z_\omega^N = \left[\det_{N+1}\left(-\epsilon^2\,\nabla\overline{\nabla} + \epsilon^2\,\omega^2\right)\right]^{-1/2}$$

$$= \left[\det'_{N+1}\left(-\epsilon^2\,\nabla\overline{\nabla}\right)\right]^{-1/2}\left[\frac{\det_{N+1}\left(-\epsilon^2\,\nabla\overline{\nabla} + \epsilon^2\,\omega^2\right)}{\det'_{N+1}\left(-\epsilon^2\,\nabla\overline{\nabla}\right)}\right]^{-1/2} \tag{2.408a}$$

where the prime $'$ indicates the exclusion of the ω_0 term, which would render the entire determinant zero.

$$Z_\omega = \frac{1}{\beta\hbar}\left[\frac{\det\left(-\frac{\partial^2}{\partial\tau^2}+\omega^2\right)}{\det'\left(-\frac{\partial^2}{\partial\tau^2}\right)}\right]^{-1/2}$$

$$= \frac{1}{\beta\hbar}\left(\frac{\prod_{m=-\infty}^{\infty}\left(\omega_m^2 + \omega^2\right)}{\prod_{m=-\infty}^{\infty\,\prime}\omega_m^2}\right)^{-1/2}$$

$$= \frac{1}{\beta\hbar\omega}\prod_{m=1}^{\infty}\left(\frac{\omega_m^2 + \omega^2}{\omega_m^2}\right)^{-1} \qquad (2.408)$$

Finally, all these results could have been obtained from the QM amplitude (2.173) via the analytic continuation

$$t_b - t_a \;\rightarrow\; -i\,(\tau_b - \tau_a) = -i\,\beta\,\hbar$$

Hence,

$$(x_b\,\tau_b \mid x_a\,\tau_a) = \sqrt{\frac{M}{2\pi\hbar}}\,\sqrt{\frac{\omega}{\sinh\omega\,(\tau_b-\tau_a)}} \qquad (2.409)$$

$$\times\exp\left\{-\frac{1}{\hbar}\,\frac{M\omega}{2\sinh\omega\,(\tau_b-\tau_a)}\left[\left(x_b^2+x_a^2\right)\cosh\omega(\tau_b-\tau_a) - 2\,x_b\,x_a\right]\right\}$$

$$Z_\omega = \int_{-\infty}^{\infty} d\,x\,(x\,\tau_b \mid x\,\tau_a)$$

$$= \sqrt{\frac{M}{2\pi\hbar}}\,\sqrt{\frac{\omega}{\sinh\omega\,(\tau_b-\tau_a)}}\int_{-\infty}^{\infty} d\,x\,\exp\left\{-\frac{1}{\hbar}\,\frac{M\,\omega\left[\cosh\omega\,(\tau_b-\tau_a) - 1\right]}{\sinh\omega\,(\tau_b-\tau_a)}\,x^2\right\}$$

$$= \sqrt{\frac{M}{2\pi\hbar}}\,\sqrt{\frac{\omega}{\sinh\omega\,(\tau_b-\tau_a)}}\,\sqrt{\frac{\pi\hbar\sinh\omega\,(\tau_b-\tau_a)}{M\,\omega\left[\cosh\omega\,(\tau_b-\tau_a) - 1\right]}}$$

$$= \frac{1}{\sqrt{2\left[\cosh\omega\,(\tau_b-\tau_a) - 1\right]}}$$

$$= \frac{1}{2\sinh\frac{\omega\,(\tau_b-\tau_a)}{2}} \qquad (2.410)$$

$$= \frac{1}{2\sinh\frac{\omega\beta\hbar}{2}}$$

in agreement with (2.407). A similar treatment of the discrete-time version (2.197) would have led to (2.401).

For applications in polymer physics (see Chapter 15) one also needs the partition function of all path fluctuations with open ends

$$Z_\omega^{\text{open}} = \int_{-\infty}^{\infty} d\,x_b\int_{-\infty}^{\infty} d\,x_a\,(x_b\,\tau_b \mid x_a\,\tau_a)$$

$$= \sqrt{\frac{M}{2\pi\hbar}} \sqrt{\frac{\omega}{\sinh\omega(\tau_b-\tau_a)}} \int_{-\infty}^{\infty} d\,x_b \int_{-\infty}^{\infty} d\,x_a$$

$$\times \exp\left\{-\frac{1}{\hbar}\frac{M\omega}{2\sinh\omega(\tau_b-\tau_a)}\left[\left(x_b^2+x_a^2\right)\cosh\omega(\tau_b-\tau_a) - 2\,x_b\,x_a\right]\right\}$$

Using

$$\int_{-\infty}^{\infty} d\,x\, e^{-a x^2 + b x} = \sqrt{\frac{\pi}{a}}\; e^{b^2/4a}$$

we have

$$\int_{-\infty}^{\infty} d\,x_b \int_{-\infty}^{\infty} d\,x_a \exp\left[-A\left(x_a^2+x_b^2\right) + B\,x_a\,x_b\right]$$

$$= \sqrt{\frac{\pi}{A}} \int_{-\infty}^{\infty} d\,x_b \exp\left[-\left(A - \frac{B^2}{4A}\right)x_b^2\right]$$

$$= \sqrt{\frac{\pi}{A}} \sqrt{\frac{\pi}{A - \frac{B^2}{4A}}} = \frac{2\pi}{\sqrt{4A^2-B^2}}$$

With

$$A = \frac{1}{\hbar}\frac{M\omega}{2\sinh\omega(\tau_b-\tau_a)}\cosh\omega(\tau_b-\tau_a) \qquad B = \frac{1}{\hbar}\frac{M\omega}{\sinh\omega(\tau_b-\tau_a)}$$

we get

$$\sqrt{4A^2-B^2} = \frac{1}{\hbar}\frac{M\omega}{\sinh\omega(\tau_b-\tau_a)}\sqrt{\cosh^2\omega(\tau_b-\tau_a)-1}$$

$$= \frac{M\omega}{\hbar}$$

$$\rightarrow \qquad Z_\omega^{\text{open}} = \sqrt{\frac{M}{2\pi\hbar}} \sqrt{\frac{\omega}{\sinh\omega(\tau_b-\tau_a)}} \frac{2\pi\hbar}{M\omega}$$

$$= \sqrt{\frac{2\pi\hbar}{M\omega}} \frac{1}{\sqrt{\sinh\omega(\tau_b-\tau_a)}} \qquad (2.411)$$

$$= \frac{\sqrt{2\pi}\;\lambda_\omega}{\sqrt{\sinh\omega(\tau_b-\tau_a)}}$$ [(2.301) used.]