2.11. Time-Dependent Harmonic Potential

For a 1-D harmonic oscillator with (imaginary) time-dependent frequency $\Omega(\tau)$ of period $\beta\hbar$, the evolution amplitude is

$$(x_b t_a \mid x_a t_a) = \oint \mathcal{D}' x \int \frac{\mathcal{D} p}{2\pi\hbar} \exp\left\{-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} d\tau \left[-i p \dot{x} + \frac{p^2}{2M} + \frac{1}{2} M \Omega^2(\tau)\, x^2\right]\right\}$$
$$= \oint \mathcal{D} x \exp\left\{-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} d\tau\, \frac{1}{2} M\left[\dot{x}^2 + \Omega^2(\tau)\, x^2\right]\right\} \tag{2.412}$$

The time-sliced fluctuation factor [c.f. (2.200)] is

$$F^N(\tau_b, \tau_a) = \left\{\det_{N+1}\left[-\epsilon^2 \nabla\overline{\nabla} + \epsilon^2 \Omega^2(\tau)\right]\right\}^{-1/2} \tag{2.413}$$

with the continuum limit

$$F(\tau_b, \tau_a) = \left[\det{}'\left(-\frac{\partial^2}{\partial\tau^2}\right)\right]^{-1/2} \left\{\frac{\det\left[-\frac{\partial^2}{\partial\tau^2} + \epsilon^2\Omega^2(\tau)\right]}{\det{}'\left(-\frac{\partial^2}{\partial\tau^2}\right)}\right\}^{-1/2}$$
$$= \frac{1}{\beta\hbar}\left\{\frac{\det\left[-\frac{\partial^2}{\partial\tau^2} + \epsilon^2\Omega^2(\tau)\right]}{\det{}'\left(-\frac{\partial^2}{\partial\tau^2}\right)}\right\}^{-1/2} \tag{2.414}$$

To avoid the special treatment required for det′, one can use the harmonic oscillator in the operator ratio instead:

$$F(\tau_b, \tau_a) = \left\{\det\left(-\frac{\partial^2}{\partial\tau^2} + \epsilon^2\omega^2\right)\right\}^{-1/2} \left\{\frac{\det\left[-\frac{\partial^2}{\partial\tau^2} + \epsilon^2\Omega^2(\tau)\right]}{\det\left(-\frac{\partial^2}{\partial\tau^2} + \epsilon^2\omega^2\right)}\right\}^{-1/2}$$
$$= \frac{1}{2\sinh\frac{\beta\hbar\omega}{2}}\left\{\frac{\det\left[-\frac{\partial^2}{\partial\tau^2} + \epsilon^2\Omega^2(\tau)\right]}{\det\left(-\frac{\partial^2}{\partial\tau^2} + \epsilon^2\omega^2\right)}\right\}^{-1/2} \tag{2.415}$$

Owing to the periodic B.C.,

$$\epsilon^2 \nabla\overline{\nabla} f_1 = \epsilon\nabla(f_1 - f_0) = f_2 - 2 f_1 + f_0 = f_2 - 2 f_1 + f_{N+1}$$
$$\epsilon^2 \nabla\overline{\nabla} f_{N+1} = \epsilon\nabla(f_{N+1} - f_N) = f_{N+2} - 2 f_{N+1} + f_N = f_1 - 2 f_{N+1} + f_N$$

so that the fluctuation matrix becomes

$$-\epsilon^2\nabla\overline{\nabla} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \tag{2.416}$$

which differs from the vanishing B.C. result (2.102) by the −1 elements at the upper-right & lower-left corners. In order to compare the effects of the B.C.'s, we set

$$\epsilon^2\left(-\nabla\overline{\nabla} + \Omega^2\right) = \begin{pmatrix} 2+\epsilon^2\Omega_{N+1}^2 & -1 & 0 & \cdots & 0 & 0 & -\alpha \\ -1 & 2+\epsilon^2\Omega_N^2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2+\epsilon^2\Omega_{N-1}^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2+\epsilon^2\Omega_3^2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2+\epsilon^2\Omega_2^2 & -1 \\ -\alpha & 0 & 0 & \cdots & 0 & -1 & 2+\epsilon^2\Omega_1^2 \end{pmatrix}$$

(2.417)

where $\alpha = 0$ or 1 for the vanishing or periodic B.C., respectively.

Laplace expansion of the 1st row gives

$$\tilde{D}_{N+1} \equiv \det_{N+1}\left[\epsilon^2\left(-\nabla\overline{\nabla}+\Omega^2\right)\right]$$
$$= \left(2+\epsilon^2\,\Omega_{N+1}^2\right)D_N + A_N + (-)^{N+1}\,\alpha\,B_N \quad (2.418)$$

where

$$D_N = \tilde{D}_N\big|_{\alpha=0} \quad (2.418a)$$

$$A_N \equiv \det_N\begin{pmatrix} -1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2+\epsilon^2\,\Omega_{N-1}^2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2+\epsilon^2\,\Omega_{N-2}^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2+\epsilon^2\,\Omega_3^2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2+\epsilon^2\,\Omega_2^2 & -1 \\ -\alpha & 0 & 0 & \cdots & 0 & -1 & 2+\epsilon^2\,\Omega_1^2 \end{pmatrix} \quad (2.418b)$$

$$B_N = \det_N\begin{pmatrix} -1 & 2+\epsilon^2\,\Omega_N^2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2+\epsilon^2\,\Omega_{N-1}^2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2+\epsilon^2\,\Omega_3^2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2+\epsilon^2\,\Omega_2^2 \\ -\alpha & 0 & 0 & \cdots & 0 & 0 & -1 \end{pmatrix} \quad (2.418c)$$

Analogous to (2.205), we have

$$\left(-\nabla\overline{\nabla}+\Omega_{N+1}^2\right)D_N = 0 \quad (2.419)$$

with the initial conditions

$$D_1 = 2+\epsilon^2\,\Omega_1^2$$
$$D_2 = \left(2+\epsilon^2\,\Omega_2^2\right)\left(2+\epsilon^2\,\Omega_1^2\right) - 1 \quad (2.420)$$

Laplace expansion of the 1st row of (2.418b) gives

$$A_N = -D_{N-1} + C_{N-1} \quad (2.420a)$$

where

$$C_{N-1} = \det_{N-1}\begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2+\epsilon^2\,\Omega_{N-2}^2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2+\epsilon^2\,\Omega_{N-3}^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2+\epsilon^2\,\Omega_3^2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2+\epsilon^2\,\Omega_2^2 & -1 \\ -\alpha & 0 & 0 & \cdots & 0 & -1 & 2+\epsilon^2\,\Omega_1^2 \end{pmatrix} \quad (2.420b)$$

Expanding the 1st row gives

$$C_{N-1} = \det_{N-2}\begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2+\epsilon^2\,\Omega_{N-3}^2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2+\epsilon^2\,\Omega_{N-4}^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2+\epsilon^2\,\Omega_3^2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2+\epsilon^2\,\Omega_2^2 & -1 \\ -\alpha & 0 & 0 & \cdots & 0 & -1 & 2+\epsilon^2\,\Omega_1^2 \end{pmatrix}$$

$$= C_{N-2}$$

Since

$$C_2 = \begin{pmatrix} 0 & -1 \\ -\alpha & 2+\epsilon^2\,\Omega_1^2 \end{pmatrix} = -\alpha$$

(2.420a) becomes

$$A_N = -D_{N-1} - \alpha \qquad (2.421)$$

Laplace expansion of the 1st row of (2.418c) gives

$$B_N = -G_{N-1} - \left(2+\epsilon^2\,\Omega_N^2\right) h_{N-1} - K_{N-1} \qquad (2.421a)$$

where

$$G_{N-1} = \det_{N-1}\begin{pmatrix} -1 & 2+\epsilon^2\,\Omega_{N-1}^2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2+\epsilon^2\,\Omega_{N-2}^2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2+\epsilon^2\,\Omega_3^2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2+\epsilon^2\,\Omega_2^2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{pmatrix} \qquad (2.421b)$$

$$h_{N-1} = \det_{N-1}\begin{pmatrix} 0 & 2+\epsilon^2\,\Omega_{N-1}^2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2+\epsilon^2\,\Omega_{N-2}^2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2+\epsilon^2\,\Omega_3^2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2+\epsilon^2\,\Omega_2^2 \\ -\alpha & 0 & 0 & \cdots & 0 & 0 & -1 \end{pmatrix} \qquad (2.421c)$$

$$K_{N-1} = \det_{N-1}\begin{pmatrix} 0 & -1 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2+\epsilon^2\,\Omega_{N-2}^2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2+\epsilon^2\,\Omega_3^2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2+\epsilon^2\,\Omega_2^2 \\ -\alpha & 0 & 0 & \cdots & 0 & 0 & -1 \end{pmatrix} \qquad (2.421d)$$

Expansion of the 1st column of (2.421b) gives

$$G_{N-1} = -G_{N-2} = (-)^{N-2}\,G_1 = (-)^{N-1} \qquad (2.421e)$$

Expansion of the 2nd column of (2.421d) gives

$$K_{N-1} = h_{N-2} \qquad (2.421e)$$

(2.421a) thus becomes

$$B_N = (-)^N - \left(2+\epsilon^2\,\Omega_N^2\right) h_{N-1} - h_{N-2} \qquad (2.422a)$$

$$= (-)^N \left[\, 1 + \left(2+\epsilon^2\,\Omega_N^2\right) H_{N-1} - H_{N-2} \,\right] \qquad (2.422)$$

where

$$H_{N-1} = (-)^{N-1}\,h_{N-1} \qquad (2.423)$$

Expansion of the 1st row of (2.421c) gives

$$h_{N-1} = -\left(2+\epsilon^2\,\Omega_{N-1}^2\right) h_{N-2} - K_{N-2}$$

$$= -\left(2+\epsilon^2\,\Omega_{N-1}^2\right) h_{N-2} - h_{N-3} \qquad (2.423a)$$

$$\rightarrow \quad H_{N-1} = \left(2+\epsilon^2\,\Omega_{N-1}^2\right) H_{N-2} - H_{N-3} \qquad (2.423b)$$

which is just the recurrence relation for the difference eq.

$$\epsilon^2 \left(-\nabla\overline{\nabla}+\Omega_{N+1}^2\right) H_N = 0 \tag{2.424}$$

with the I.C.

$$h_2 = \det_2\begin{pmatrix} 0 & 2+\epsilon^2\,\Omega_2^2 \\ -\alpha & -1 \end{pmatrix} = \alpha\left(2+\epsilon^2\,\Omega_2^2\right) = H_2 \tag{2.425}$$

$$h_3 = \det_3\begin{pmatrix} 0 & 2+\epsilon^2\,\Omega_3^2 & -1 \\ 0 & -1 & 2+\epsilon^2\,\Omega_2^2 \\ -\alpha & 0 & -1 \end{pmatrix}$$

$$= -\alpha\left[\left(2+\epsilon^2\,\Omega_3^2\right)\left(2+\epsilon^2\,\Omega_2^2\right)-1\right] = -H_3 \tag{2.426}$$

Comparing with the difference eq. for D_N

$$\epsilon^2 \left(-\nabla\overline{\nabla}+\Omega_{N+1}^2\right) D_N = 0$$

with I.C. [see (2.417) with $\alpha = 0$]

$$D_1 = 2+\epsilon^2\,\Omega_1^2$$

$$D_2 = \det_2\begin{pmatrix} 2+\epsilon^2\,\Omega_2^2 & -1 \\ -1 & 2+\epsilon^2\,\Omega_1^2 \end{pmatrix} = \left(2+\epsilon^2\,\Omega_2^2\right)\left(2+\epsilon^2\,\Omega_1^2\right)-1$$

we conclude

$$H_N = \alpha\, D_{N-1}^{(+)} \tag{2.427}$$

where

$$D_N^{(+)} \equiv D_N \Big|_{\{\,\Omega_m \to \Omega_{m+1}\ \forall\, m\,\}} \tag{2.427a}$$

Using (2.423b) on (2.422) gives

$$B_N = (-)^N\,(1+\,H_N)$$

$$= (-)^N\left(1+\,\alpha\, D_{N-1}^{(+)}\right) \tag{2.427b}$$

Putting (2.421) & (2.427b) into (2.418) gives

$$\tilde{D}_{N+1} = \left(2+\epsilon^2\,\Omega_{N+1}^2\right) D_N - D_{N-1} - \alpha - \alpha\left(1+\,\alpha\, D_{N-1}^{(+)}\right) \tag{2.428}$$

$$= D_{N+1} - 2\,\alpha - \,\alpha^2\, D_{N-1}^{(+)} \tag{2.429}$$

For periodic fluctuation, $\alpha = 1$,

$$\tilde{D}_{N+1} = D_{N+1} - D_{N-1}^{(+)} - 2 \tag{2.430}$$

The imaginary time version of the renormalized function (2.210) is

$$D_{\text{ren}}(\tau_N) = \epsilon\, D_N$$

satisfying [see (2.226) with $D_{\text{ren}} = D_a$]

$$\left(-\frac{\partial^2}{\partial\tau^2}+\Omega^2(\tau)\right) D_{\text{ren}}(\tau) = 0 \qquad \text{with I.C.} \qquad D_{\text{ren}}(0) = 0 \qquad \dot{D}_{\text{ren}}(0) = 1 \tag{2.431}$$

For $\epsilon \to 0$,

$$\dot{D}_{\text{ren}}(\tau_N) \approx \epsilon\,\frac{D_{N+1}-D_N}{\epsilon} = \epsilon\,\frac{D_N-D_{N-1}}{\epsilon} = \frac{1}{2}\left(D_{N+1}-D_{N-1}\right) \approx \frac{1}{2}\left(D_{N+1}-D_{N-1}^{(+)}\right) \tag{2.430a}$$

The continuum limit of (2.430) is therefore

$$\det\left[-\frac{\partial^2}{\partial\tau^2}+\Omega^2(\tau)\right]_{\hbar\beta} = 2\left[\dot{D}_{\text{ren}}(\hbar\beta)-1\right] \tag{2.432}$$

By (2.384), the partition function is

$$Z_\Omega = \frac{1}{\sqrt{2\left[\dot{D}_{\text{ren}}(\hbar\beta)-1\right]}} \tag{2.433}$$

This result can also be obtained by analytic continuation from the real-time amplitude [see (2.260)]

$$(\,x_b\,\tau_b \mid x_a\,\tau_a\,) = F_\Omega(\tau_b,\ \tau_a)\exp\left(-\frac{1}{\hbar}\mathcal{A}_{\text{cl}}\right)$$

where

$$F_\Omega(\tau_b,\ \tau_a)\Big|_{\tau_b-\tau_a=\hbar\beta} = \sqrt{\frac{M}{2\pi\hbar}}\ \frac{1}{\sqrt{D_{\rm ren}(\hbar\beta)}} \qquad \text{[see (2.261)]}$$

$$\mathcal{A}_{\rm cl}\Big|_{x_b=x_a=x} = \frac{M\left(\dot{D}_{\rm ren}-1\right)}{D_{\rm ren}(\hbar\beta)}\,x^2 \qquad \text{[see (2.266)]}$$

Hence,

$$\begin{aligned}
Z_\Omega &= \int d\,x\,(\,x\,\tau_b\mid x\,\tau_a\,)\Big|_{\tau_b-\tau_a=\hbar\beta} \\
&= \sqrt{\frac{M}{2\pi\hbar}}\ \frac{1}{\sqrt{D_{\rm ren}(\hbar\beta)}}\int_{-\infty}^{\infty} d\,x\exp\left[-\frac{M\left(\dot{D}_{\rm ren}-1\right)}{\hbar D_{\rm ren}(\hbar\beta)}\,x^2\right] \\
&= \sqrt{\frac{M}{2\pi\hbar}}\ \frac{1}{\sqrt{D_{\rm ren}(\hbar\beta)}}\sqrt{\frac{\pi\hbar D_{\rm ren}(\hbar\beta)}{M\left[\dot{D}_{\rm ren}(\hbar\beta)-1\right]}} \\
&= \frac{1}{\sqrt{2\left[\dot{D}_{\rm ren}(\hbar\beta)-1\right]}}
\end{aligned}$$

in agreement with (2.433).

For the harmonic oscillator of frequency ω, the analytic continuation of (2.214) gives

$$D_{\rm ren}(\tau) = \frac{1}{\omega}\sinh\omega\tau \tag{2.435}$$

$$\rightarrow \qquad \dot{D}_{\rm ren}(\tau) = \cosh\omega\tau$$

$$2\left[\dot{D}_{\rm ren}(\hbar\beta)-1\right] = 2\left[\cosh(\beta\hbar\omega)-1\right] = 4\sinh^2\frac{\beta\hbar\omega}{2} \tag{2.436}$$

So that

$$Z_\omega = \frac{1}{2\sinh\frac{\beta\hbar\omega}{2}} \tag{2.437}$$

as expected.

On a sliced imaginary-time axis, the case $\Omega^2 = \omega^2$ is solved as follows. The analytic continuation of (2.208) gives

$$D_N = \frac{\sinh[(N+1)\,\tilde{\omega}_e\,\epsilon]}{\sinh\tilde{\omega}_e\,\epsilon} \tag{2.438}$$

Using $D_{N-1}^{(+)} = D_{N-1}$, (2.430) simplifies to

$$\begin{aligned}
\tilde{D}_{N+1} &= D_{N+1} - D_{N-1} - 2 \\
&= \frac{\sinh[(N+2)\,\tilde{\omega}_e\,\epsilon] - \sinh(N\,\tilde{\omega}_e\,\epsilon)}{\sinh\tilde{\omega}_e\,\epsilon} - 2 \\
&= \frac{2}{\sinh\tilde{\omega}_e\,\epsilon}\sinh(\tilde{\omega}_e\,\epsilon)\cosh[(N+1)\,\tilde{\omega}_e\,\epsilon] - 2 \\
&= 2\left\{\cosh[(N+1)\,\tilde{\omega}_e\,\epsilon] - 1\right\} \\
&= 4\sinh^2\frac{(N+1)\,\tilde{\omega}_e\,\epsilon}{2} \qquad (2.439) \\
&= 4\sinh^2\frac{\beta\hbar\tilde{\omega}_e}{2}
\end{aligned}$$

Therefore

$$Z_\omega = \frac{1}{\sqrt{\tilde{D}_{N+1}}} = \frac{1}{2\sinh\frac{\beta\hbar\tilde{\omega}_e}{2}} \tag{2.440}$$

in agreement with (2.401).