2.12. Functional Measure in Fourier Space

The periodic path of a 1-D particle can be expanded in a Fourier series as

$$x(\tau) = \sum_{m=-\infty}^{\infty} x_m \, e^{-i\,\omega_m\,\tau} \qquad\qquad \omega_m = \frac{2\,\pi\,m}{\beta\,\hbar}$$

Since $x(\tau)$ is real, we have

$$x_m^* = x_{-m} \qquad \rightarrow \qquad x_0 = \text{real} \tag{2.441a}$$

$$x(\tau) = x_0 + \sum_{m=1}^{\infty} \left(x_m \, e^{-i\,\omega_m\,\tau} + c.c. \right) \tag{2.441}$$

$$= x_0 + \eta(\tau)$$

From the inverse transform

$$x_m = \frac{1}{\beta\,\hbar} \int_0^{\beta\,\hbar} d\,\tau \, e^{i\,\omega_m\,\tau} \, x(\tau) \tag{2.441b}$$

we have

$$x_0 = \frac{1}{\beta\,\hbar} \int_0^{\beta\,\hbar} d\,\tau \, x(\tau) \;= \overline{x} \tag{2.442}$$

so that x_0 is the temporal average of $x(\tau)$.

Furthermore

$$\overline{\eta} = \frac{1}{\beta\,\hbar} \int_0^{\beta\,\hbar} d\,\tau \, \eta(\tau)$$

$$= \frac{1}{\beta\,\hbar} \sum_{m=1}^{\infty} \int_0^{\beta\,\hbar} d\,\tau \left(x_m \, e^{-i\,\omega_m\,\tau} + c.c \right)$$

$$= \frac{1}{\beta\,\hbar} \sum_{m=1}^{\infty} \left[-\frac{x_m}{i\,\omega_m} \left(e^{-i\,\omega_m\,\beta\,\hbar} - 1 \right) + c.c. \right]$$

$$= 0 \qquad\qquad \text{since} \quad e^{i\,\omega_m\,\beta\,\hbar} = e^{i\,2\,\pi\,m} = 1 \tag{2.442a}$$

so that $\eta(\tau)$ can be identified as the temporal fluctuation about the average x_0 of the path.

For the linear oscillator, the Euclidean action

$$\mathcal{A}_e = \frac{1}{2} M \int_0^{\beta\,\hbar} d\,\tau \left(\dot{x}^2 + \omega^2\,x^2 \right)$$

can be written in terms of the Fourier components as

$$\mathcal{A}_e = \frac{1}{2} M \sum_{m,\,n=-\infty}^{\infty} \int_0^{\beta\,\hbar} d\,\tau \, e^{i\,(\,\omega_m - \omega_n\,)\,\tau} \left(\omega_m\,\omega_n + \omega^2 \right) x_m^*\,x_n$$

$$= \frac{1}{2} M\,\beta\,\hbar \sum_{m,\,n=-\infty}^{\infty} \delta_{m\,n} \left(\omega_m\,\omega_n + \omega^2 \right) x_m^*\,x_n$$

$$= \frac{1}{2} M\,\beta\,\hbar \sum_{m=-\infty}^{\infty} \left(\omega_m^2 + \omega^2 \right) |\,x_m\,|^2$$

$$= M\,\beta\,\hbar \left[\frac{1}{2}\,\omega^2\,x_0^2 + \sum_{m=1}^{\infty} \left(\omega_m^2 + \omega^2 \right) |\,x_m\,|^2 \right] \tag{2.443}$$

$$= M\,\beta\,\hbar \left\{ \frac{1}{2}\,\omega^2\,x_0^2 + \sum_{m=1}^{\infty} \left(\omega_m^2 + \omega^2 \right) \left[(\text{Re}\,x_m)^2 + (\text{Im}\,x_m)^2 \right] \right\}$$

The independent Fourier components can be chosen as

$$\{\, x_0, \;\text{Re}\,x_m, \;\text{Im}\,x_m \;;\; m = 1, \;\ldots\,\infty \,\}$$

The time-sliced path intgeration can therefore be written as

$$\prod_{n=0}^{N}\int_{-\infty}^{\infty} d\,x(\tau_n) = \int_{-\infty}^{\infty} d\,x_0 \prod_{m=1}^{N/2}\int_{-\infty}^{\infty} d\,\mathrm{Re}\,x_m \int_{-\infty}^{\infty} d\,\mathrm{Im}\,x_m\,J \tag{2.444}$$

where $J = \dfrac{\partial(\,x(\tau_0),\ \ldots,\ x(\tau_N)\,)}{\partial(\,x_0,\ \mathrm{Re}\,x_1,\ \mathrm{Im}\,x_1,\ \ldots,\ \mathrm{Re}\,x_{N/2},\ \mathrm{Im}\,x_{N/2}\,)}$ is the Jacobian and we've assumed for convenience N is even.

The rather tedious calculation of J can be side-stepped by calculating Z_ω directly by evaluating the x_m integrals and compare the previous result (2.408).

Thus, assuming J is a simple constant,

$$\begin{aligned} Z_\omega &= J\int_{-\infty}^{\infty} d\,x_0 \prod_{m=1}^{\infty}\int_{-\infty}^{\infty} d\,\mathrm{Re}\,x_m \int_{-\infty}^{\infty} d\,\mathrm{Im}\,x_m \exp\!\left(-\frac{1}{\hbar}\mathcal{A}_e\right) \\ &= J\sqrt{\frac{2\pi}{M\beta\,\omega^2}}\prod_{m=1}^{\infty}\frac{\pi}{M\beta\left(\omega_m^2+\omega^2\right)} \end{aligned}$$

Comparing with the continuum result of (2.408), we have

$$\begin{aligned} J &= \sqrt{\frac{M\beta}{2\pi\hbar^2}}\prod_{m=1}^{\infty}\frac{M\beta\,\omega_m^2}{\pi} \\ &= \frac{1}{l_e}\prod_{m=1}^{\infty}\frac{M\beta\,\omega_m^2}{\pi} \qquad \text{[(2.351) used.]} \end{aligned} \tag{2.444a}$$

so that the path integral measure in terms of the Fourier components is

$$\oint\mathcal{D}\,x = \int_{-\infty}^{\infty}\frac{d\,x_0}{l_e}\prod_{m=1}^{N/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\,\mathrm{Re}\,x_m\ d\,\mathrm{Im}\,x_m}{\pi/M\beta\,\omega_m^2} \tag{2.445}$$

$$\equiv \int_{-\infty}^{\infty}\frac{d\,x_0}{l_e}\oint\mathcal{D}'\,x \tag{2.446}$$

As an exercise,

$$\begin{aligned} Z_\omega^{x_0} &\equiv \oint\mathcal{D}'\,x\exp\!\left(-\frac{1}{\hbar}\mathcal{A}_e\right) \\ &= \prod_{m=1}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\,\mathrm{Re}\,x_m\ d\,\mathrm{Im}\,x_m}{\pi/M\beta\,\omega_m^2} \\ &\quad\times\exp\!\left(-M\beta\left\{\frac{1}{2}\omega^2 x_0^2+\sum_{m=1}^{\infty}\left(\omega_m^2+\omega^2\right)\left[(\mathrm{Re}\,x_m)^2+(\mathrm{Im}\,x_m)^2\right]\right\}\right) \\ &= \exp\!\left(-\frac{1}{2}M\beta\ \omega^2 x_0^2\right)\prod_{m=1}^{\infty}\left(\frac{\omega_m^2+\omega^2}{\omega_m^2}\right)^{-1} \end{aligned} \tag{2.447}$$

Then

$$\begin{aligned} Z_\omega &= \int_{-\infty}^{\infty}\frac{d\,x_0}{l_e}Z_\omega^{x_0} \\ &= \frac{1}{l_e}\sqrt{\frac{2\pi}{M\beta\ \omega^2}}\prod_{m=1}^{\infty}\left(\frac{\omega_m^2+\omega^2}{\omega_m^2}\right)^{-1} \\ &= \frac{1}{\beta\hbar\,\omega}\prod_{m=1}^{\infty}\left(\frac{\omega_m^2+\omega^2}{\omega_m^2}\right)^{-1} \end{aligned}$$

in agreement with (2.408).

We now proceed to find the integration measure for open end paths needed for the open-end partition function in (2.411). For reasons to be explained later, we start with the special class of open paths that satisfy the Neumann boundary conditions

$$\dot{x}(\tau_a) = v_a = 0 \qquad \dot{x}(\tau_b) = v_b = 0 \tag{2.449}$$

Starting with the Fourier series with real coefficients

$$x(\tau) = x_0 + \sum_{m=1}^{\infty} \Big(x_m \cos\nu_m (\tau-\tau_a) + x'_m \sin\nu_m(\tau-\tau_a) \Big) \tag{2.449a}$$

we have

$$\dot{x}(\tau) = \sum_{m=1}^{\infty} \nu_m \Big(-x_m \sin\nu_m(\tau-\tau_a) + x'_m \cos\nu_m(\tau-\tau_a) \Big) \tag{2.449b}$$

The B.C. at τ_a can be satisfied by setting

$$x'_m = 0 \qquad \forall\ m$$

so that

$$x(\tau) = x_0 + \sum_{m=1}^{\infty} x_m \cos\nu_m (\tau-\tau_a) \tag{2.450}$$

$$\dot{x}(\tau) = -\sum_{m=1}^{\infty} \nu_m\, x_m \sin\nu_m(\tau-\tau_a)$$

The B.C. at τ_b then requires

$$\nu_m = \frac{m\,\pi}{\tau_b-\tau_a} = \frac{m\,\pi}{\beta\,\hbar} \tag{2.450a}$$

which are the Euclidean version of the frequencies (3.64) for Dirichlet B.C.

Using

$$\int d\,\theta\, \cos m\,\theta\, \cos n\,\theta = \frac{\sin(m-n)\,\theta}{2\,(m-n)} + \frac{\sin(m+n)\,\theta}{2\,(m+n)}$$

$$\int d\,\theta\, \sin m\,\theta\, \sin n\,\theta = \frac{\sin(m-n)\,\theta}{2\,(m-n)} - \frac{\sin(m+n)\,\theta}{2\,(m+n)}$$

$$\int d\,\theta\, \sin m\,\theta\, \cos n\,\theta = -\frac{\cos(m-n)\,\theta}{2\,(m-n)} - \frac{\cos(m+n)\,\theta}{2\,(m+n)}$$

we get

$$\frac{\pi}{\beta\,\hbar}\int_{\tau_a}^{\tau_b} d\,\tau \cos\nu_m(\tau-\tau_a)\cos\nu_n(\tau-\tau_a) = \frac{\sin(m-n)\,\pi}{2\,(m-n)} + \frac{\sin(m+n)\,\pi}{2\,(m+n)}$$
$$= \frac{\pi}{2}\,(\,\delta_{m\,n} + \delta_{m\,0}\,\delta_{n\,0}\,) \tag{2.450b}$$

$$\frac{\pi}{\beta\,\hbar}\int_{\tau_a}^{\tau_b} d\,\tau \sin\nu_m(\tau-\tau_a)\sin\nu_n(\tau-\tau_a) = \frac{\pi}{2}\,(\,\delta_{m\,n} - \delta_{m\,0}\,\delta_{n\,0}\,) \tag{2.450c}$$

The Euclidean action

$$\mathcal{A}_e = \frac{1}{2}\,M\int_{\tau_a}^{\tau_b} d\,\tau\left(\dot{x}^2 + \omega^2\,x^2\right)$$

thus becomes

$$\mathcal{A}_e = \frac{1}{2} M \sum_{m,n=1}^{\infty} \int_{\tau_a}^{\tau_b} d\tau \; x_m x_n \Big\{ \nu_m \nu_n \sin\nu_m(\tau-\tau_a) \sin\nu_n(\tau-\tau_a)$$
$$+ \omega^2 \cos\nu_m(\tau-\tau_a) \cos\nu_n(\tau-\tau_a) \Big\}$$
$$+\frac{1}{2} M \omega^2 \int_{\tau_a}^{\tau_b} d\tau \Big[x_0^2 + 2 x_0 \sum_{m=1}^{\infty} x_m \cos\nu_m(\tau-\tau_a) \Big]$$
$$= \frac{1}{4} M \beta \hbar \sum_{m=1}^{\infty} x_m^2 \left(\nu_m^2 + \omega^2\right) + \frac{1}{2} \beta \hbar M \omega^2 x_0^2$$
$$= \frac{1}{2} \beta \hbar M \Big[\omega^2 x_0^2 + \frac{1}{2} \sum_{m=1}^{\infty} x_m^2 \left(\nu_m^2 + \omega^2\right) \Big] \qquad (2.451)$$

Writing (2.450) in the form of (2.441) gives

$$x(\tau) = y_0 + \sum_{m=1}^{\infty} \left(y_m e^{-i\omega_m \tau} + c.c. \right)$$
$$= y_0 + \sum_{m=1}^{\infty} \Big[(y_m + y_m^*) \cos\omega_m \tau + i(-y_m + y_m^*) \sin\omega_m \tau \Big]$$
$$= y_0 + 2 \sum_{m=1}^{\infty} \Big[\cos(\omega_m \tau) \,\mathrm{Re}\, y_m + \sin(\omega_m \tau) \,\mathrm{Im}\, y_m \Big] \qquad (2.451)$$

The corresponding path integral measure is [see (2.445-6)]

$$\oint \mathcal{D} y = \int_{-\infty}^{\infty} \frac{dy_0}{l_e} \prod_{m=1}^{N/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\,\mathrm{Re}\, y_m \; d\,\mathrm{Im}\, y_m}{\pi / M \beta \omega_m^2} \qquad (2.451a)$$
$$\equiv \int_{-\infty}^{\infty} \frac{dy_0}{l_e} \oint \mathcal{D}' y \qquad (2.451b)$$

Comparing with (2.449a), we see that

$$x_0 = y_0 \qquad x_m = 2\,\mathrm{Re}\, y_m \qquad x'_m = 2\,\mathrm{Im}\, y_m$$

so that the integral measure is

$$\oint \mathcal{D} x = \int_{-\infty}^{\infty} \frac{dx_0}{l_e} \prod_{m=1}^{N/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx_m \; dx'_m}{4\pi / M \beta \omega_m^2} \qquad (2.451c)$$

For (2.450),

$$x_0 = y_0 \qquad x_m = 2\,\mathrm{Re}\, y_m \qquad 0 = 2\,\mathrm{Im}\, y_m$$

The number of $m \neq 0$ variables is reduced by half. However

$$\nu_m = \frac{1}{2} \omega_m$$

so that the number of non-zero modes are twice as much, thus keeping the total number of independent Fourier components the same. The integral measure is therefore

$$\oint \mathcal{D} x = \int_{-\infty}^{\infty} \frac{dx_0}{l_e} \prod_{m=1}^{N} \int_{-\infty}^{\infty} \frac{dx_m}{\sqrt{4\pi / M \beta \nu_m^2}}$$
$$= \int_{-\infty}^{\infty} \frac{dx_0}{l_e} \left(\prod_{m=1}^{N/2} \int_{-\infty}^{\infty} \frac{dx_m}{\sqrt{4\pi / M \beta \nu_m^2}} \right)^2 \qquad (2.452a)$$
$$\equiv \int_{-\infty}^{\infty} \frac{dx_0}{l_e} \oint \mathcal{D}' x \qquad (2.452b)$$

where the peculiar form of (2.452a) is necessary to account for the double degeneracy of the $m \neq 0$

modes [see (2.447)].

Thus,

$$
\begin{aligned}
Z_\omega^{x_0} &\equiv \oint \mathcal{D}' x \exp\left(-\frac{1}{\hbar}\mathcal{A}_e\right) \\
&= \prod_{m=1}^{\infty} \int_{-\infty}^{\infty} \frac{d x_m}{\sqrt{4\pi / M\beta\nu_m^2}} \exp\left\{-\frac{1}{2}\beta M\left[\omega^2 x_0^2 + \frac{1}{2}\sum_{m=1}^{\infty} x_m^2\left(\nu_m^2+\omega^2\right)\right]\right\} \\
&= \exp\left(-\frac{1}{2}\beta M \omega^2 x_0^2\right)\left(\prod_{m=1}^{\infty} \frac{1}{\sqrt{4\pi / M\beta\nu_m^2}} \sqrt{\frac{4\pi}{\beta M\left(\nu_m^2+\omega^2\right)}}\right)^2 \\
&= \exp\left(-\frac{1}{2}\beta M \omega^2 x_0^2\right)\prod_{m=1}^{\infty}\frac{\nu_m^2}{\nu_m^2+\omega^2}
\end{aligned}
\tag{2.453}
$$

in agreement with (2.447).

Using the product formula (2.181), this becomes

$$
Z_\omega^{x_0} = \sqrt{\frac{\beta\hbar\omega}{\sinh\beta\hbar\omega}}\ \exp\left(-\frac{1}{2}\beta M\omega^2 x_0^2\right) \tag{2.454}
$$

The partition function is therefore

$$
\begin{aligned}
Z_\omega &= \int_{-\infty}^{\infty}\frac{d x_0}{l_e} Z_\omega^{x_0} \\
&= \frac{1}{l_e}\sqrt{\frac{\beta\hbar\omega}{\sinh\beta\hbar\omega}}\sqrt{\frac{2\pi}{\beta M\omega^2}} \\
&= \frac{1}{l_e}\sqrt{\frac{2\pi\hbar}{M\omega\sinh\beta\hbar\omega}}
\end{aligned}
\tag{2.455}
$$

which differs from Z_ω^{open} of (2.411) by the factor l_e^{-1}.

The reason that Z_ω differs only by a trivial factor from Z_ω^{open} is that the integral over the end points in (2.411) forces the end point momenta to vanish, i.e.,

$$
Z_\omega^{\text{open}} = (\,p_b,\ \beta\hbar\ |\ \ p_a\, 0\,)_{p_b=p_a=0}
$$

In case where $p = m\,v$, the paths in Z_ω^{open} are therefore those obeying the Dirichlet B.C.