2.15. Field-Theoretic Definition of Harmonic Path Integrals by Analytic Regularization

Caution: Kleinert's text contains a number of minor errors.

The partition function of a 1-D harmonic oscillator

$$Z_\omega = \oint \mathcal{D}x \exp\left[-\int_0^{\hbar\beta} d\tau \frac{1}{2} M\left(\dot{x}^2 + \omega^2 x^2\right)\right]$$
$$= \oint \mathcal{D}x \exp\left[-\int_0^{\hbar\beta} d\tau \frac{1}{2} M x\left(-\frac{\partial^2}{\partial\tau^2} + \omega^2\right) x\right] \qquad (2.484)$$

can be formally evaluated as

$$Z_\omega = \frac{1}{\sqrt{\det\left(-\frac{\partial^2}{\partial\tau^2} + \omega^2\right)}} = \exp\left[-\frac{1}{2}\operatorname{Tr}\ln\left(-\frac{\partial^2}{\partial\tau^2} + \omega^2\right)\right] \qquad (2.485)$$

In terms of the eigenvalues ω'^2 of $-\dfrac{\partial^2}{\partial\tau^2}$,

$$Z_\omega = \prod_{\omega'} \frac{1}{\sqrt{\omega'^2 + \omega^2}} \qquad (2.486)$$

Since there is no upper bound to ω'^2, (2.486) is a divergent product, or an equivalent divergent sum:

$$Z_\omega \equiv e^{-\beta F_\omega} = \exp\left[-\frac{1}{2}\sum_{\omega'} \ln\left(\omega'^2 + \omega^2\right)\right] \qquad (2.487)$$

There are 2 features in (2.487) that require clarification.

First, since ω' can be a continuous spectrum, we need to turn $\sum_{\omega'}$ properly into an integration. This will be the subject of discussion in the next few sub-sections.

Secondly, since ω has the dimension of time^{-1}, its logarithm is ill-defined. This can be remedied by dealing instead with frequency ratios, i.e., $\ln\left(\dfrac{\omega'^2 + \omega^2}{\Omega^2}\right)$, provided $\sum_{\omega'} \ln\Omega^2 = 0$. Fortunately, it was found that [see (2.512)], with a proper definition of the sum, $\sum_{\omega'} 1 = 0$, so that we can set $\Omega = \omega$.

At finite temperatures, the periodic B.C. turns ω' into a discrete spectrum given by the Matsubara frequencies

$$\omega_m = \frac{2\pi m}{\hbar\beta} \qquad m = 0, \pm 1, \pm 2, \ \dots. \qquad (2.488a)$$

so that

$$Z_\omega = \exp\left[-\frac{1}{2}\sum_{m=-\infty}^{\infty} \ln\left(\omega_m^2 + \omega^2\right)\right] \qquad (2.488)$$

$$F_\omega = -\frac{1}{\beta}\ln Z_\omega$$
$$= \frac{1}{2\beta}\operatorname{Tr}\ln\left(-\frac{\partial^2}{\partial\tau^2} + \omega^2\right)_{\text{periodic } B.C.}$$

$$= \frac{1}{2\beta} \sum_{m=-\infty}^{\infty} \ln\left(\omega_m^2 + \omega^2\right) \tag{2.489}$$

2.15.1. Zero-Temperature Evaluation of the Frequency Sum

In the $T \to 0$ limit, $\beta \to \infty$ so that (2.489) becomes

$$F_\omega = \frac{1}{2\beta} \operatorname{Tr} \ln\left(-\frac{\partial^2}{\partial \tau^2} + \omega^2 \right)_{x=0 \text{ at } \tau=\pm\infty} \tag{2.490a}$$

The pushing of the boundary to $\pm\infty$ squeezes the eigenvalues into a continuum. Using

$$d\omega' \approx \Delta\omega_m = \frac{2\pi}{\hbar\beta} \Delta m \to \qquad \int d\omega' \approx \frac{2\pi}{\hbar\beta} \sum_m \tag{2.491}$$

(2.490a) becomes

$$F_\omega = \frac{\hbar}{2} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \ln\left(\omega'^2 + \omega^2\right) \qquad \text{for } \beta \to \infty \tag{2.490}$$

(2.491) can also be obtained from Planck's phase space rule which states that the quantum state for each DoF occupies an area of h in phase space. Choosing the conjugate variables as $E = \hbar\omega$ and τ,

$$\int\int \frac{dE\, d\tau}{h} = \int \frac{d\omega}{2\pi} \int d\tau = \beta\hbar \int \frac{d\omega}{2\pi}$$
$$= \text{number of states in phase space} = \sum_m$$

Hence, (2.491).

Note that

$$\int d\omega' \ln\left(\omega'^2 + \omega^2\right) = \omega' \ln\left(\omega'^2 + \omega^2\right) - 2\omega' + 2\omega \tan^{-1}\frac{\omega'}{\omega}$$

$$\to \qquad \int_{-\omega_M}^{\omega_M} d\omega' \ln\left(\omega'^2 + \omega^2\right) = 2\omega_M \ln\left(\omega_M^2 + \omega^2\right) - 4\omega_M + 4\omega \tan^{-1}\frac{\omega_M}{\omega}$$

$$\xrightarrow[\omega_M \to \infty]{} 4\omega_M(\ln\omega_M - 1) + 2\pi\omega$$

This is called ultraviolet divergence (UV-divergence) since it involves the high frequency part of the spectrum.

The important observation now is that the divergent integral (2.490) can be made finite by a mathematical technique called analytic regularization [see G. 't Hooft & M. Veltman, Nucl. Phys. B44, 189 (1972)]. Analytic regularization is at present the only method that allows renormalization of non-abelian gauge theories without destroying
gauge invariance. See also the review by G. Leibbrandt, Rev. Mod. Phys. 74 ,843 (1975).

To begin, we use

$$\frac{d a^x}{dx} = \frac{d e^{x \ln a}}{dx} = (\ln a)\, e^{x \ln a} = a^x \ln a \tag{2.492a}$$

to get

$$-\frac{d}{d\epsilon}\left(\omega'^2 + \omega^2\right)^{-\epsilon} = \left(\omega'^2 + \omega^2\right)^{-\epsilon} \ln\left(\omega'^2 + \omega^2\right)$$

$$\to \qquad \ln\left(\omega'^2 + \omega^2\right) = -\frac{d}{d\epsilon}\left(\omega'^2 + \omega^2\right)^{-\epsilon}\Big|_{\epsilon=0} \tag{2.492}$$

$$= -\lim_{\epsilon \to 0} \frac{\left(\omega'^2 + \omega^2\right)^{-\epsilon} - 1}{\epsilon}$$

The subtraction of the pole term $\frac{1}{\epsilon}$ is called a minimal subtraction. Thus,

$$\ln\left(\omega'^2+\omega^2\right)_{MS} \equiv -\lim_{\epsilon\to 0}\frac{\left(\omega'^2+\omega^2\right)^{-\epsilon}}{\epsilon} \tag{2.494}$$

(2.490) thus becomes

$$\begin{aligned}\frac{2}{\hbar}F_\omega &= \frac{1}{\beta\hbar}\operatorname{Tr}\ln\left(-\frac{\partial^2}{\partial\tau^2}+\omega^2\right) \qquad \text{for} \qquad T\to 0\\ &= \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\ln\left(\omega'^2+\omega^2\right)\\ &= -\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\frac{d}{d\epsilon}\left(\omega'^2+\omega^2\right)^{-\epsilon}\bigg|_{\epsilon=0} \qquad \text{[(2.492) used.]}\end{aligned} \tag{2.495}$$

From the definition of the gamma function

$$\begin{aligned}\Gamma(\mu) &= \int_0^\infty d\tau\,\tau^{\mu-1}e^{-\tau}\\ &= \int_0^\infty d(a\tau)\,(a\tau)^{\mu-1}e^{-a\tau}\end{aligned}$$

we have

$$\int_0^\infty d\tau\,\tau^{\mu-1}e^{-a\tau} = a^{-\mu}\,\Gamma(\mu) \tag{2.496a}$$

$$\int_0^\infty d\tau\,\tau^{\mu-1}e^{-\omega^2\tau} = \omega^{-2\mu}\,\Gamma(\mu) \tag{2.496}$$

Setting $\mu=\epsilon$, (2.496a) becomes

$$a^{-\epsilon} = \frac{1}{\Gamma(\epsilon)}\int_0^\infty\frac{d\tau}{\tau}\,\tau^\epsilon e^{-a\tau} \tag{2.497}$$

which allows (2.495) to be written as

$$\frac{1}{\beta\hbar}\operatorname{Tr}\ln\left(-\frac{\partial^2}{\partial\tau^2}+\omega^2\right) = -\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\frac{d}{d\epsilon}\frac{1}{\Gamma(\epsilon)}\int_0^\infty\frac{d\tau}{\tau}\,\tau^\epsilon e^{-\left(\omega'^2+\omega^2\right)\tau}\bigg|_{\epsilon=0} \tag{2.498}$$

Assuming we may interchange the order of derivatives & integrations

$$\frac{1}{\beta\hbar}\operatorname{Tr}\ln\left(-\frac{\partial^2}{\partial\tau^2}+\omega^2\right) = -\frac{d}{d\epsilon}\frac{1}{\Gamma(\epsilon)}\int_0^\infty\frac{d\tau}{\tau}\,\tau^\epsilon\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}e^{-\left(\omega'^2+\omega^2\right)\tau}\bigg|_{\epsilon=0} \tag{2.499}$$

$$= -\frac{d}{d\epsilon}\frac{1}{\Gamma(\epsilon)}\int_0^\infty\frac{d\tau}{\tau}\,\tau^\epsilon\frac{e^{-\omega^2\tau}}{2\sqrt{\pi\tau}}\bigg|_{\epsilon=0} \tag{2.500}$$

$$= -\frac{1}{2\sqrt{\pi}}\frac{d}{d\epsilon}\frac{1}{\Gamma(\epsilon)}\int_0^\infty\frac{d\tau}{\tau}\,\tau^{\epsilon-\frac{1}{2}}e^{-\omega^2\tau}\bigg|_{\epsilon=0}$$

$$= -\frac{1}{2\sqrt{\pi}}\frac{d}{d\epsilon}\frac{1}{\Gamma(\epsilon)}\,\omega^{-2\epsilon+1}\,\Gamma\left(\epsilon-\frac{1}{2}\right)\bigg|_{\epsilon=0} \tag{2.501}$$

Using

$$x\,\Gamma(x) = \Gamma(x+1)$$

we have

$$\epsilon\,\Gamma(\epsilon) = \Gamma(\epsilon+1) \to \qquad \frac{1}{\Gamma(\epsilon)} = \frac{\epsilon}{\Gamma(1+\epsilon)} = \epsilon + O\left(\epsilon^2\right) \tag{2.501a}$$

Hence,

$$\frac{1}{\Gamma(\epsilon)}\,\omega^{-2\epsilon+1}\,\Gamma\left(\epsilon-\frac{1}{2}\right) = \epsilon\,\omega\,\Gamma\left(-\frac{1}{2}\right) + O\left(\epsilon^2\right)$$

$$\rightarrow \qquad \frac{d}{d\epsilon}\frac{1}{\Gamma(\epsilon)}\omega^{-2\epsilon+1}\Gamma\left(\epsilon-\frac{1}{2}\right)\Bigg|_{\epsilon=0} = \omega\,\Gamma\left(-\frac{1}{2}\right)+O(\epsilon)\Bigg|_{\epsilon=0} = \omega\,\Gamma\left(-\frac{1}{2}\right)$$

With

$$\Gamma\left(-\frac{1}{2}\right) = -2\,\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

(2.501) becomes

$$\frac{1}{\beta\hbar}\operatorname{Tr}\ln\left(-\frac{\partial^2}{\partial\tau^2}+\omega^2\right) = \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\ln\left(\omega'^2+\omega^2\right) = \omega \qquad (2.502)$$

Using (2.495), we have

$$F_\omega = \frac{1}{2}\hbar\omega \qquad \text{for} \qquad T\rightarrow 0 \qquad (2.503)$$

as expected.

In summary, the recipe (2.491) for turning the sum into integral gave us the formula

$$F_\omega = \frac{\hbar}{2}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\ln\left(\omega'^2+\omega^2\right)$$

which diverges if evaluated directly. However, using [see (2.492)]

$$\ln(\omega'^2+\omega^2) = -\frac{d}{d\epsilon}(\omega'^2+\omega^2)^{-\epsilon}\Big|_{\epsilon=0}$$
$$= -\lim_{\epsilon\rightarrow 0}\left[\frac{1}{\epsilon}(\omega'^2+\omega^2)^{-\epsilon}-\frac{1}{\epsilon}\right] \qquad (2.503a)$$

we were able to obtain the correct result by allowing the interchange of integration order [see (2.499)]. This mathematically illegal action thus compensates for the inadequacy of (2.491). The whole thing is then dignified by the name of analytic regularization.

Since

$$\int_{-\infty}^{\infty}d\omega'\,(\omega'^2+\omega^2)^{-\epsilon} = \frac{\sqrt{\pi}\,\omega^{1-2\epsilon}}{\Gamma(\epsilon)}\Gamma\left(-\frac{1}{2}+\epsilon\right)$$
$$\rightarrow \qquad -\lim_{\epsilon\rightarrow 0}\frac{1}{\epsilon}\int_{-\infty}^{\infty}d\omega'\,(\omega'^2+\omega^2)^{-\epsilon} = 2\pi\omega$$

the procedure is equivalent to dropping the $\frac{1}{\epsilon}$ term in (2.503a), i.e., applying the minimal subtraction defined in (2.494):

$$\ln\left(\omega'^2+\omega^2\right)_{MS} = -\lim_{\epsilon\rightarrow 0}\frac{1}{\epsilon}(\omega'^2+\omega^2)^{-\epsilon} \qquad (2.505a)$$
$$= -\lim_{\epsilon\rightarrow 0}\int_0^{\infty}\frac{d\tau}{\tau}\tau^{\epsilon}e^{-(\omega'^2+\omega^2)\tau} \qquad \text{[(2.496a) used.]} \qquad (2.504a)$$
$$= -\int_0^{\infty}\frac{d\tau}{\tau}e^{-(\omega'^2+\omega^2)\tau} \qquad (2.504)$$

where (2.504) is meant to be a short-hand for (2.504a) since it is not defined as it stands.

(2.495) therefore should be written as

$$\frac{1}{\beta\hbar}\operatorname{Tr}\ln\left(-\frac{\partial^2}{\partial\tau^2}+\omega^2\right) = \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\ln\left(\omega'^2+\omega^2\right)_{MS}$$
$$= -\lim_{\epsilon\rightarrow 0}\frac{1}{\epsilon}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}(\omega'^2+\omega^2)^{-\epsilon} \qquad (2.505)$$

$$= -\int_0^\infty \frac{d\tau}{\tau} \int_{-\infty}^\infty \frac{d\omega'}{2\pi} e^{-(\omega'^2+\omega^2)\tau} \qquad (2.505a)$$

$$= -\int_0^\infty \frac{d\tau}{\tau} \frac{1}{2\sqrt{\pi\tau}} e^{-\omega^2\tau}$$

$$= -\frac{1}{2\sqrt{\pi}} \omega \, \Gamma\left(-\frac{1}{2}\right) = \omega$$

Note that the ω' integral must be evaluated first in (2.505a) since the τ integral is ill-defined at this point.

Consider now the slightly more general integral

$$\mathcal{I} = \int_{-\infty}^\infty \frac{d\omega'}{2\pi} (\omega'^2)^\gamma \ln\left(\omega'^2 + \omega^2\right)_{MS} \qquad (2.507a)$$

$$= -\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{-\infty}^\infty \frac{d\omega'}{2\pi} (\omega'^2)^\gamma (\omega'^2 + \omega^2)^{-\epsilon}$$

According to the MS rule (2.504), we have

$$\mathcal{I} = -\int_0^\infty \frac{d\tau}{\tau} \int_{-\infty}^\infty \frac{d\omega'}{2\pi} (\omega'^2)^\gamma e^{-(\omega'^2+\omega^2)\tau} \qquad (2.508)$$

$$= -\frac{1}{2\pi} \Gamma\left(\gamma + \frac{1}{2}\right) \int_0^\infty \frac{d\tau}{\tau} \tau^{-\gamma-\frac{1}{2}} e^{-\omega^2\tau} \qquad (2.509)$$

$$= -\frac{1}{2\pi} \Gamma\left(\gamma + \frac{1}{2}\right) \Gamma\left(-\gamma - \frac{1}{2}\right) \omega^{2\gamma+1} \qquad \text{[(2.496) used.]} \ (2.510)$$

According to (2.503a), the MS rule in (2.507a) corresponds to setting

$$\int_{-\infty}^\infty \frac{d\omega'}{2\pi} (\omega'^2)^\gamma = 0 \qquad \forall \ \gamma \qquad (2.506)$$

which is called the Veltman's rule.

Alternatively, if we multiply (2.510) by ϵ, we get

$$\lim_{\epsilon\to 0} \int_{-\infty}^\infty \frac{d\omega'}{2\pi} (\omega'^2)^\gamma (\omega'^2 + \omega^2)^{-\epsilon} = \epsilon \frac{1}{2\pi} \Gamma\left(\gamma + \frac{1}{2}\right) \Gamma\left(-\gamma - \frac{1}{2}\right) \omega^{2\gamma+1} \qquad (2.507)$$

$$\to \quad \int_{-\infty}^\infty \frac{d\omega'}{2\pi} (\omega'^2)^\gamma = 0$$

thus verifying the Veltman's rule.

Setting $\gamma = 0$, we have

$$\int_{-\infty}^\infty \frac{d\omega'}{2\pi} = 0 \qquad (2.512)$$

which was used in the discussion of (2.487) to turn the arguments of the logarithm of frequencies dimensionless.

2.15.2. Finite-Temperature Evaluation of the Frequency Sum

Let the difference between the frequency sum & integral be denoted by

$$\Delta F_\omega = \frac{1}{2\beta} \sum_{m=-\infty}^\infty \ln\left(\frac{\omega_m^2}{\omega^2} + 1\right) - \frac{1}{2}\hbar \int_{-\infty}^\infty \frac{d\omega_m}{2\pi} \ln\left(\frac{\omega_m^2}{\omega^2} + 1\right) \qquad (2.513)$$

$$= \frac{1}{2\beta} \sum_{m=-\infty}^\infty \ln\left(\frac{\omega_m^2}{\omega^2} + 1\right) - \frac{1}{2}\hbar\omega \qquad (2.513a)$$

Let

$$\frac{1}{2\beta}\sum_{m=-\infty}^{\infty}\ln\left(\frac{\omega_m^2}{\omega^2}+1\right)=\frac{1}{\beta}\sum_{m=1}^{\infty}\ln\left(\frac{\omega_m^2}{\omega^2}+1\right) \qquad \omega_m=\frac{2\pi m}{\beta\hbar}$$

$$=\Delta_1 F_\omega+\Delta_2 F_\omega \qquad (2.513b)$$

where

$$\Delta_1 F_\omega=\frac{1}{\beta}\sum_{m=1}^{\infty}\left[\ln\left(\frac{\omega_m^2}{\omega^2}+1\right)-\ln\frac{\omega_m^2}{\omega^2}\right]=\frac{1}{\beta}\sum_{m=1}^{\infty}\ln\left(1+\frac{\omega^2}{\omega_m^2}\right) \qquad (2.514)$$

$$\Delta_2 F_\omega=\frac{1}{\beta}\sum_{m=1}^{\infty}\ln\frac{\omega_m^2}{\omega^2} \qquad (2.515)$$

Using [see (2.406)]

$$\prod_{m=1}^{\infty}\left(1+\frac{\omega^2}{\omega_m^2}\right)=\frac{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)}{\frac{1}{2}\beta\hbar\omega} \qquad (2.516)$$

we have

$$\Delta_1 F_\omega=\frac{1}{\beta}\ln\left[\prod_{m=1}^{\infty}\left(1+\frac{\omega^2}{\omega_m^2}\right)\right]=\frac{1}{\beta}\ln\frac{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)}{\frac{1}{2}\beta\hbar\omega} \qquad (2.517)$$

$\Delta_2 F_\omega$ is divergent. Using (2.492), we have

$$\Delta_2 F_\omega=\frac{2}{\beta}\sum_{m=1}^{\infty}\ln\frac{\omega_m}{\omega}$$

$$=-\frac{2}{\beta}\lim_{\epsilon\to 0}\frac{d}{d\epsilon}\sum_{m=1}^{\infty}\left(\frac{\omega_m}{\omega}\right)^{-\epsilon}$$

$$=-\frac{2}{\beta}\lim_{\epsilon\to 0}\frac{d}{d\epsilon}\left[\left(\frac{2\pi}{\beta\hbar\omega}\right)^{-\epsilon}\sum_{m=1}^{\infty}m^{-\epsilon}\right] \qquad (2.518)$$

$$=-\frac{2}{\beta}\lim_{\epsilon\to 0}\frac{d}{d\epsilon}\left[\left(\frac{2\pi}{\beta\hbar\omega}\right)^{-\epsilon}\zeta(\epsilon)\right] \qquad (2.518a)$$

where

$$\zeta(x)=\sum_{m=1}^{\infty}m^{-x} \qquad x>1 \qquad (2.519)$$

is the Riemann zeta function.

ζ can be analytic continued onto the complex plane via the integral representation [Gradshteyn & Ryzhik, Formula 9.513.1]

$$\zeta(z)=\frac{1}{\left(1-2^{1-z}\right)\Gamma(z)}\int_0^\infty dt\,\frac{t^{z-1}}{e^t+1} \qquad (2.519a)$$

$$=\frac{1}{\Gamma(z)}\int_0^\infty dt\,\frac{t^{z-1}}{e^t-1} \qquad (2.519b)$$

Defined this way, $\zeta(z)$ has only 1 pole at $z=1$. Furthermore [Gradshteyn & Ryzhik, Formula 9.542.1-4]

$$\zeta(2n)=\frac{(-)^{n+1}}{2(2n)!}(2\pi)^{2n}B_{2n} \qquad n=0,1,2,3,\ \ldots \qquad (2.520a)$$

$$=\frac{1}{2(2m)!}(2\pi)^{2m}\left|B_{2m}\right| \qquad m=1,2,3,\ \ldots$$

$$\zeta(-2m) = 0 \tag{2.520b}$$

$$\zeta(1-2m) = -\frac{B_{2m}}{2m} \tag{2.520c}$$

$$\zeta'(0) = -\frac{1}{2}\ln 2\pi \qquad \zeta(0) = -\frac{1}{2} \tag{2.520}$$

where B_{2n} are the Bernoulli numbers with

$$\{B_0, B_1, \ldots, B_{10}, \ldots\} = \left\{1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, \ldots\right\} \tag{2.520d}$$

Using (2.492a), we have

$$\zeta(z) \approx -\frac{1}{2}(2\pi)^z \qquad \text{for} \qquad z \approx 0 \tag{2.521}$$

(2.518a) thus becomes

$$\begin{aligned}\Delta_2 F_\omega &= -\frac{2}{\beta}\lim_{\epsilon\to 0}\left[\left(\frac{2\pi}{\beta\hbar\omega}\right)^{-\epsilon}\ln\left(\frac{2\pi}{\beta\hbar\omega}\right)\zeta(\epsilon) + \left(\frac{2\pi}{\beta\hbar\omega}\right)^{-\epsilon}\zeta'(\epsilon)\right] \\ &= -\frac{2}{\beta}\left[\frac{1}{2}\ln\left(\frac{2\pi}{\beta\hbar\omega}\right) - \frac{1}{2}\ln 2\pi\right] \\ &= \frac{1}{\beta}\ln(\beta\hbar\omega)\end{aligned} \tag{2.522}$$

Using (2.517) & (2.522) , (2.513) becomes

$$\begin{aligned}\Delta F_\omega &= \Delta_1 F_\omega + \Delta_2 F_\omega - \frac{1}{2}\hbar\omega \\ &= \frac{1}{\beta}\ln\frac{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)}{\frac{1}{2}\beta\hbar\omega} + \frac{1}{\beta}\ln(\beta\hbar\omega) - \frac{1}{2}\hbar\omega \\ &= \frac{1}{\beta}\ln\frac{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}}{\beta\hbar\omega} + \frac{1}{\beta}\ln(\beta\hbar\omega) - \frac{1}{\beta}\ln e^{\beta\hbar\omega/2} \\ &= \frac{1}{\beta}\ln\left(1 - e^{-\beta\hbar\omega}\right)\end{aligned} \tag{2.523}$$

$$\begin{aligned}\rightarrow \qquad F_\omega &= \frac{1}{2\beta}\operatorname{Tr}\ln\left(-\frac{\partial^2}{\partial\tau^2} + \omega^2\right) = \frac{1}{2\beta}\sum_{m=-\infty}^{\infty}\ln\left(\omega_m^2 + \omega^2\right) \\ &= \frac{1}{2\beta}\sum_{m=-\infty}^{\infty}\ln\left(\frac{\omega_m^2}{\omega^2} + 1\right) = \frac{1}{2}\hbar\omega + \Delta F_\omega \\ &= \frac{1}{2}\hbar\omega + \frac{1}{\beta}\ln\left(1 - e^{-\beta\hbar\omega}\right) = \frac{1}{\beta}\ln\left(e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}\right) \\ &= \frac{1}{\beta}\ln\left[2\sinh\left(\frac{1}{2}\beta\hbar\omega\right)\right]\end{aligned} \tag{2.524}$$

in agreement with (2.483).

Note that using (2.520):

$$\zeta(0) = \sum_{m=1}^{\infty} 1 = \sum_{m=-1}^{-\infty} 1 = -\frac{1}{2} \tag{2.526}$$

we have

$$\sum_{m=-\infty}^{\infty} 1 = \sum_{m=1}^{\infty} 1 + 1 + \sum_{m=-1}^{-\infty} 1 = 0 \tag{2.525}$$

thus proving (2.512), which allows us to write

$$\sum_{m=-\infty}^{\infty} \ln\left(\omega_m^2+\omega^2\right) = \sum_{m=-\infty}^{\infty} \ln\left(\omega_m^2+\omega^2\right) - \sum_{m=-\infty}^{\infty} \ln\omega^2$$
$$= \sum_{m=-\infty}^{\infty} \ln\left(\frac{\omega_m^2}{\omega^2}+1\right)$$
$$= 2\sum_{m=1}^{\infty} \ln\left(\frac{\omega_m^2}{\omega^2}+1\right) \qquad \omega_0 = 0 \qquad (2.527)$$

2.15.3. Quantum-Mechanical Harmonic Oscillator

Consider the quantum fluctuation factor (2.86) of a free particle

$$F_0(\Delta t) = \int \mathcal{D}\,\delta x \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} d t \frac{1}{2} M\,\delta x\left(-\frac{\partial^2}{\partial t^2}\right)\delta x\right]$$
$$= \sqrt{\frac{M}{2\pi\hbar i\,\Delta t}} \qquad [(2.125)\text{ used}] \qquad (2.528)$$

where $\Delta t = t_b - t_a$.

For the harmonic oscillator [see (2.186)],

$$F_\omega(\Delta t) = F_0(\Delta t)\left[\frac{\det\left(-\partial_t^2-\omega^2\right)}{\det\left(-\partial_t^2\right)}\right]^{-1/2} \qquad (2.529)$$

In terms of the Fourier components [see (2.186)],

$$\frac{\det\left(-\partial_t^2-\omega^2\right)}{\det\left(-\partial_t^2\right)} = \exp\left\{\sum_{n=1}^{\infty}\left[\ln\left(\nu_n^2-\omega^2\right)-\ln\nu_n^2\right]\right\} \qquad \nu_n = \frac{n\pi}{\Delta t} \qquad (2.530)$$
$$= \frac{\sin(\omega\,\Delta t)}{\omega\,\Delta t} \qquad (2.531)$$

This result can be obtained using (2.527) & (2.524) as follows.

$$\det\left(-\partial_t^2-\omega^2\right) = \prod_{n=1}^{\infty}\left(\nu_n^2-\omega^2\right) = \exp\left[\sum_{n=1}^{\infty}\ln\left(\nu_n^2-\omega^2\right)\right]$$
$$= \exp\left[\sum_{n=1}^{\infty}\ln\left(\nu_n^2+\omega^2\right)\right]_{\omega\to i\omega}$$
$$= \exp\left\{\sum_{n=1}^{\infty}\left[\ln\left(\frac{\nu_n^2}{\omega^2}+1\right)+\ln\omega^2\right]\right\}_{\omega\to i\omega}$$
$$= \exp\left\{-\frac{1}{2}\ln\omega^2+\sum_{n=1}^{\infty}\ln\left(\frac{\nu_n^2}{\omega^2}+1\right)\right\}_{\omega\to i\omega} \qquad [(2.526)\text{ used.}]$$

In order to use the results of the last section, we set

$$\omega_m = \frac{2\pi m}{\beta\hbar} = \frac{m\pi}{\Delta t} = \nu_m \qquad \rightarrow \qquad 2\,\Delta t = \beta\hbar$$

From (2.513b) & (2.523), we have

$$\sum_{m=1}^{\infty}\ln\left(\frac{\omega_m^2}{\omega^2}+1\right) = \ln\frac{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)}{\frac{1}{2}\beta\hbar\omega}+\ln(\beta\hbar\omega)$$
$$= \ln\left[2\sinh\left(\frac{1}{2}\beta\hbar\omega\right)\right]$$

Hence,

$$\det\left(-\partial_t^2 - \omega^2\right) = \exp\left\{-\frac{1}{2}\ln\omega^2 + \ln\left[2\sinh(\omega\,\Delta\,t)\right]\right\}_{\omega\to i\omega}$$

$$= \exp\left\{\ln\left[\frac{2\sinh(\omega\,\Delta\,t)}{\omega}\right]\right\}_{\omega\to i\omega}$$

$$= \frac{2\sin(\omega\,\Delta\,t)}{\omega} \qquad (2.532)$$

$$\to \qquad \det\left(-\partial_t^2\right) = \lim_{\omega\to 0}\frac{2\sin(\omega\,\Delta\,t)}{\omega} = 2\,\Delta\,t$$

$$\therefore \qquad \frac{\det\left(-\partial_t^2 - \omega^2\right)}{\det\left(-\partial_t^2\right)} = \frac{\sin(\omega\,\Delta\,t)}{\omega\,\Delta\,t}$$

thus reproducing (2.531).

The amplitude is therefore

$$(x_b\,t_b \mid x_a\,t_a) = F_\omega(\Delta\,t)\,e^{i\mathcal{A}_{cl}/\hbar}$$

$$= F_0(\Delta\,t)\left[\frac{\det\left(-\partial_t^2 - \omega^2\right)}{\det\left(-\partial_t^2\right)}\right]^{-1/2} e^{i\mathcal{A}_{cl}/\hbar}$$

$$= \sqrt{\frac{M}{2\,\pi\,\hbar\,i\,\Delta\,t}}\left[\frac{\sin(\omega\,\Delta\,t)}{\omega\,\Delta\,t}\right]^{-1/2} e^{i\mathcal{A}_{cl}/\hbar}$$

$$= \sqrt{\frac{M\,\omega}{2\,\pi\,\hbar\,i\sin(\omega\,\Delta\,t)}}\; e^{i\mathcal{A}_{cl}/\hbar} \qquad (2.533)$$

in agreement with (2.173).

2.15.4. Tracelog of the First-Order Differential Operator

Consider

$$\operatorname{Tr}\ln\left(-\partial_\tau^2+\omega^2\right) = \operatorname{Tr}\ln\left[(-\partial_\tau+\omega)\,(-\partial_\tau+\omega)\right]$$

$$= \operatorname{Tr}\ln(\partial_\tau+\omega) + \operatorname{Tr}\ln(-\partial_\tau+\omega) \qquad (2.534)$$

which is the sum of the tracelog of two 1st order differential operators.

In terms of the continuous eigenvalues $-i\,\omega'$ of ∂_r, we have [see (2.491)],

$$\operatorname{Tr}\ln(\partial_\tau+\omega) = \beta\,\hbar\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\,\pi}\ln(-i\,\omega' + \omega)$$

$$= -\beta\,\hbar\int_{\infty}^{-\infty}\frac{d\,\omega'}{2\,\pi}\ln(i\,\omega' + \omega) \qquad (\omega' \to -\omega')$$

$$= \beta\,\hbar\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\,\pi}\ln(i\,\omega' + \omega)$$

$$= \operatorname{Tr}\ln(-\partial_\tau+\omega)$$

$$= \frac{1}{2}\operatorname{Tr}\ln\left(-\partial_\tau^2+\omega^2\right) \qquad (2.535a)$$

In the low temperature limit, we get from (2.502)

$$\operatorname{Tr}\ln(\pm\partial_\tau+\omega) = \frac{1}{2}\beta\,\hbar\,\omega \qquad (2.535)$$

For finite temperatures, we get from (2.524)

$$\text{Tr}\ln(\pm\partial_\tau+\omega)=\ln\left(2\sinh\frac{\beta\hbar\omega}{2}\right) \qquad (2.536)$$

Setting $\omega'\to -i\,\omega'$ in (2.502) gives

$$\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\,\pi}\ln\left(-\omega'^2+\omega^2\right)=i\,\omega \qquad \omega\geq 0$$

The integral is singular due to the poles at $\omega'=\pm\omega$. A finite value can be obtained by specifying a limiting contour path for the integration. If we choose

$$\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\,\pi}\ln\left(-\omega'^2+\omega^2-i\,\eta\right)=i\,\omega \qquad \eta=0^+ \qquad (2.537)$$

then we've moved the poles to

$$\omega'=\pm\omega\,(1-i\,\eta)=\begin{cases}\omega-i\,\eta\\ -\omega+i\,\eta\end{cases}$$

where

$\alpha\,\eta=\eta$ for $\quad \eta=0^+,\ \ \alpha\geq 0$

If the contour is to be closed in the upper complex plane, then (2.537) is equivalent to bending the contour below (above) the pole at $-\omega$ ($+\omega$). An implicit assumption is that the integral along the parts of the contour outside the real line has zero value.

Obviously, the integral value depends on the path chosen. However, the principal value of the integral, which exclude all portions of the contour around poles, is independent of the path.

Using

$$-\omega'^2+\omega^2-i\,\eta=-(\omega'-\omega+i\,\eta)\,(\omega'+\omega-i\,\eta)$$

(2.537) can be written as

$$\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\,\pi}\ln\left(-\omega'^2+\omega^2-i\,\eta\right)$$

$$=\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\,\pi}\left[i\,\pi+\ln(\omega'-\omega+i\,\eta)+\ln(\omega'+\omega-i\,\eta)\right] \qquad \ln(-1)=i\,\pi$$

$$=\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\,\pi}\ln(\omega'-\omega+i\,\eta)+\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\,\pi}\ln(\omega'+\omega-i\,\eta) \qquad [(2.512)\text{ used.}]$$

Since the two integrals are equal, we have

$$\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\,\pi}\ln\left[\omega'\mp(\omega\pm i\,\eta)\right]=\frac{1}{2}\,i\,\omega \qquad \omega\geq 0 \qquad (2.538)$$

Taking the complex conjugate gives

$$\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\,\pi}\ln\left[\omega'\mp(\omega\mp i\,\eta)\right]=-\frac{1}{2}\,i\,\omega \qquad \omega\geq 0 \qquad (2.538a)$$

Generalizing to a complex $\omega=\omega_R+i\,\omega_I$, we have

For $\omega_I>0$,

$$\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\,\pi}\ln\left(\omega'-\omega\right)=\frac{1}{2}\,i\,\omega \qquad [\text{ From upper-sign in (2.538) }]$$

$$\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\,\pi}\ln\left(\omega'+\omega\right)=-\frac{1}{2}\,i\,\omega \qquad [\text{ From lower-sign in (2.538a) }]$$

For $\omega_I<0$,

$$\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\,\pi}\ln\left(\omega'-\omega\right)=-\frac{1}{2}\,i\,\omega \qquad [\text{ From upper-sign in (2.538a) }]$$

$$\int_{-\infty}^{\infty} \frac{d\,\omega'}{2\,\pi} \ln\Big(\,\omega' + \omega\Big) = \frac{1}{2}\,i\,\omega \qquad \text{[From lower-sign in (2.538)]}$$

Combining both cases gives

$$\int_{-\infty}^{\infty} \frac{d\,\omega'}{2\,\pi} \ln\Big(\,\omega' \pm \,\omega\Big) = -\epsilon\,(\omega_I)\,\frac{1}{2}\,i\,\omega \qquad (2.540)$$

where

$$\epsilon(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Using $i\,\omega = i\,\omega_R - \omega_I$, the above analysis becomes

For $\omega_R > 0$,

$$\int_{-\infty}^{\infty} \frac{d\,\omega'}{2\,\pi} \ln\Big(\,\omega' - i\,\omega\Big) = -\frac{1}{2}\,\omega \qquad \text{[From upper-sign in (2.538)]}$$

$$\int_{-\infty}^{\infty} \frac{d\,\omega'}{2\,\pi} \ln\Big(\,\omega' + i\,\omega\Big) = \frac{1}{2}\,\omega \qquad \text{[From lower-sign in (2.538a)]}$$

For $\omega_R < 0$,

$$\int_{-\infty}^{\infty} \frac{d\,\omega'}{2\,\pi} \ln\Big(\,\omega' - i\,\omega\Big) = \frac{1}{2}\,\omega \qquad \text{[From upper-sign in (2.538a)]}$$

$$\int_{-\infty}^{\infty} \frac{d\,\omega'}{2\,\pi} \ln\Big(\,\omega' + i\,\omega\Big) = -\frac{1}{2}\,\omega \qquad \text{[From lower-sign in (2.538)]}$$

Combining both cases gives

$$\int_{-\infty}^{\infty} \frac{d\,\omega'}{2\,\pi} \ln\Big(\,\omega' \pm \,i\,\omega\Big) = \pm\,\epsilon(\omega_R)\,\frac{1}{2}\,\omega \qquad (2.539)$$

which has the opposite sign to Kleinert’s result.

Using

$$\int_{-\infty}^{\infty} \frac{d\,\omega'}{2\,\pi} \ln\Big(\,\omega' \pm \,\omega\Big) = \frac{1}{\beta\,\hbar} \sum_{m=-\infty}^{\infty} \ln(\omega_m \pm \omega)$$

we can extrapolate to finite temperatures according to (2.535-6):

$$\begin{aligned} \frac{1}{\beta\,\hbar} \sum_{m=-\infty}^{\infty} \ln(\omega_m \pm i\,\omega) &= \frac{1}{\beta\,\hbar} \ln\left[2\,\sinh\left(\pm\,\epsilon(\omega_R)\,\frac{\beta\,\hbar\,\omega}{2}\right)\right] \\ &= \frac{1}{\beta\,\hbar} \ln\left[\pm\,2\,\epsilon(\omega_R)\,\sinh\left(\frac{1}{2}\,\omega\,\beta\,\hbar\right)\right] \end{aligned} \qquad (2.541)$$

$$\begin{aligned} \frac{1}{\beta\,\hbar} \sum_{m=-\infty}^{\infty} \ln(\omega_m \pm \omega) &= \frac{1}{\beta\,\hbar} \ln\left[2\,\sinh\left(-i\,\epsilon(\omega_I)\,\frac{\beta\,\hbar\,\omega}{2}\right)\right]\Big] \\ &= \frac{1}{\beta\,\hbar} \ln\left[2\,\epsilon(\omega_I)\,\sin\left(\frac{1}{2}\,\omega\,\beta\,\hbar\right)\right] \end{aligned} \qquad (2.542)$$

Using

$$\begin{aligned} \sinh[\,a\,(\,\omega_R + i\,\omega_I\,)\,] &= \sinh(a\,\omega_R)\,\cosh(i\,a\,\omega_I) + \cosh(a\,\omega_R)\,\sinh(i\,a\,\omega_I) \\ &= \sinh(a\,\omega_R)\,\cos(a\,\omega_I) + i\cosh(a\,\omega_R)\,\sin(a\,\omega_I) \end{aligned}$$

we see that (2.541) is periodic in ω_I with period $\frac{4\,\pi}{\beta\,\hbar}$. Similarly, (2.542) is periodic in ω_R with the same period. For large times ($\beta\,\hbar \to \infty$, $T \to 0$), the period becomes vanishingly small and the sum fails to converge. Thus, a meaningful sum is possible only if the periodic part of ω vanishes. In many applications, both types of sum in (2.541) & (2.542) may be present, albeit with different ω.

The same argument then applies to each. In which case, using (2.439) & (2.540) directly may be preferable. [See §18.9.2]

In §3.3.2, (2.536) will be generalized to arbitrary positive time-dependent frequencies $\Omega(t)$ to read [see (3.133)]

$$\begin{aligned}\operatorname{Tr}\ln\left[\pm\partial_\tau+\Omega(\tau)\right] &= \ln\left\{2\sinh\left[\frac{1}{2}\int_0^{\beta\hbar} d\tau\,\Omega(\tau)\right]\right\} \\ &= \frac{1}{2}\int_0^{\beta\hbar} d\tau\,\Omega(\tau) + \ln\left\{1-\exp\left[-\int_0^{\beta\hbar} d\tau\,\Omega(\tau)\right]\right\} \qquad (2.543)\end{aligned}$$

2.15.5. Gradient Expansion of the 1-D Tracelog

Consider

$$\begin{aligned}\left[-\partial_\tau-\overline{\Omega}(\tau)\right]\left[\partial_\tau-\overline{\Omega}(\tau)\right]x(\tau) &= -\partial_\tau^2 x+\overline{\Omega}^2 x+\partial_\tau\left(\overline{\Omega}\,x\right)-\overline{\Omega}\,\partial_\tau x \\ &= \left(-\partial_\tau^2+\overline{\Omega}^2+\frac{\partial\,\overline{\Omega}}{\partial\,\tau}\right)x \qquad (2.544a)\end{aligned}$$

Thus, by setting

$$\Omega^2 = \overline{\Omega}^2+\frac{\partial\,\overline{\Omega}}{\partial\,\tau} \qquad (2.544b)$$

the 2nd order differential operator $-\partial_\tau^2+\Omega(\tau)^2$ can be factorized as $\left[-\partial_\tau-\overline{\Omega}(\tau)\right]\left[\partial_\tau-\overline{\Omega}(\tau)\right]$.

This can be used with (2.543) to obtain a semi-classical expansion for the tracelog of $-\partial_\tau^2+\Omega(\tau)^2$. To begin, we introduce the expansion parameter $\hbar$ by setting

$$w(\tau)\equiv\hbar\,\Omega(\tau) \qquad (2.544c)$$

(2.544a) thus implies

$$\det\left[-\hbar^2\,\partial_\tau^2+w^2(\tau)\right] = \det\left[-\hbar\,\partial_\tau-\overline{w}(\tau)\right]\det\left[\hbar\,\partial_\tau-\overline{w}(\tau)\right] \qquad (2.544)$$

while(2.544b) becomes

$$\hbar\,\frac{\partial\,\overline{w}}{\partial\,\tau}+\overline{w}^2 = w^2 \qquad (2.545)$$

which is a special case of the Riccati differential eq.

$$y' = f(x)\,y+g(x)\,y^2+h(x)$$

Consider

$$\begin{aligned}\operatorname{Tr}\ln\left[-\hbar^2\,\partial_\tau^2+w^2(\tau)\right] &= \operatorname{Tr}\left\{\ln\left[-\hbar\,\partial_\tau-\overline{w}(\tau)\right]+\ln\left[\hbar\,\partial_\tau-\overline{w}(\tau)\right]\right\} && \ln(A\,B)=\ln A+\ln B \\ &= \operatorname{Tr}\ln\left[-\hbar\,\partial_\tau-\overline{w}(\tau)\right]+\operatorname{Tr}\ln\left[\hbar\,\partial_\tau-\overline{w}(\tau)\right] && \operatorname{Tr}(A+B)=\operatorname{Tr}A+\operatorname{Tr}B \\ &= 2\operatorname{Tr}\ln\left[-\hbar\,\partial_\tau-\overline{w}(\tau)\right] && \text{[(2.543) used.]}\end{aligned}$$

Using the Veltman's rule $\operatorname{Tr}\ln c = 0$, we can write (2.543) as

$$\begin{aligned}\operatorname{Tr}\ln\left[\pm\hbar\,\partial_\tau-\overline{w}(\tau)\right] &= \operatorname{Tr}\ln\left[\pm\partial_\tau-\frac{\overline{w}(\tau)}{\hbar}\right] \\ &= \ln\left\{-2\sinh\left[\frac{1}{2\,\hbar}\int_0^{\beta\hbar} d\tau\,\overline{w}(\tau)\right]\right\}\end{aligned}$$

$$\rightarrow \qquad \operatorname{Tr}\ln\left[-\hbar^2\,\partial_\tau^2+w^2(\tau)\right] = 2\ln\left\{-2\sinh\left[\frac{1}{2\,\hbar}\int_0^{\beta\hbar} d\tau\,\overline{w}(\tau)\right]\right\}$$

$$= \ln\left\{4\sinh^2\left[\frac{1}{2\hbar}\int_0^{\beta\hbar} d\tau\,\overline{w}(\tau)\right]\right\} \tag{2.546}$$

Therefore, solving (2.545) tantamounts to calculating $\operatorname{Tr}\ln\left[-\hbar^2\partial_\tau^2+w^2(\tau)\right]$ and hence

$$\det\left[-\hbar^2\partial_\tau^2+w^2(\tau)\right] = \exp\left\{\operatorname{Tr}\ln\left[-\hbar^2\partial_\tau^2+w^2(\tau)\right]\right\} \tag{2.546a}$$

For a time-independent frequency $w(\tau)=\hbar\,\omega$, the solution to (2.545) is $\overline{w}(\tau)=\hbar\,\omega$ and (2.546) gives

$$\det\left[-\hbar^2\partial_\tau^2+w^2(\tau)\right] = 4\sinh^2\frac{\beta\hbar\omega}{2}$$

in agreement with the Gelfand-Yaglom result (2.436) for a periodic B.C.

This agreement is no coincidence. For, if we know the solution $D_a(\tau)$ of [see (2.226)]

$$\left[-\hbar^2\partial_\tau^2+w^2(\tau)\right] D_a(\tau) = 0 \tag{2.546b}$$

with periodic B.C., we've found $D_{\text{ren}} = D_a(t_b)$ and hence [see (2.432)]

$$\det\left[-\hbar^2\partial_\tau^2+w^2(\tau)\right] = 2\left[\dot{D}_{\text{ren}}(\hbar\beta)-1\right] \tag{2.546c}$$

The exponential of (2.546) thus becomes, with the integration limit $\beta\hbar$ replaced by τ,

$$2\left[\dot{D}_{\text{ren}}(\tau)-1\right] = 4\sinh^2\left[\frac{1}{2\hbar}\int_0^{\tau} d\tau\,\overline{w}(\tau)\right]$$

$$\to \qquad \frac{1}{2\hbar}\int_0^{\tau} d\tau\,\overline{w}(\tau) = \sinh^{-1}\sqrt{\frac{1}{2}\left[\dot{D}_{\text{ren}}(\tau)-1\right]}$$

$$\overline{w}(\tau) = 2\hbar\frac{\partial}{\partial\tau}\sinh^{-1}\sqrt{\frac{1}{2}\left[\dot{D}_{\text{ren}}(\tau)-1\right]} \tag{2.547}$$

For the harmonic oscillator with fixed frequency ω, $D_{\text{ren}}(\tau)$ is given by (2.435) and hence leads to $\overline{w}(\tau)=\hbar\,\omega$, as expected.

If we can't solve (2.546b), a solution to the Riccati eq. (2.545) can still be found in terms of a power series

$$\overline{w}(\tau) = \sum_{n=0}^{\infty} \overline{w}_n(\tau)\,\hbar^n \tag{2.548}$$

which leads to the so-called gradient expansion of the tracelog.

Putting (2.548) into (2.545) gives

$$\sum_{n=0}^{\infty}\left(\hbar^{n+1}\dot{\overline{w}}_n + \sum_{k=0}^{\infty} h^{n+k}\,\overline{w}_n\,\overline{w}_k\right) = w^2$$

Using

$$\sum_{n=0}^{\infty}\hbar^{n+1}\dot{\overline{w}}_n = \sum_{n=1}^{\infty}\hbar^n\,\dot{\overline{w}}_{n-1}$$

$$\sum_{n,k=0}^{\infty} h^{n+k}\,\overline{w}_n\,\overline{w}_k = \sum_{m=0}^{\infty}\sum_{k=0}^{m} h^m\,\overline{w}_{m-k}\,\overline{w}_k = \overline{w}_0^2 + \sum_{n=1}^{\infty}\sum_{k=0}^{n} h^n\,\overline{w}_{n-k}\,\overline{w}_k$$

we have

$$\overline{w}_0^2 + \sum_{n=1}^{\infty}\hbar^n\left(\dot{\overline{w}}_{n-1} + \sum_{k=0}^{n}\overline{w}_{n-k}\,\overline{w}_k\right) = w^2$$

$\to$

$$\overline{w}_0 = \pm w$$

$$0 = \dot{\overline{w}}_{n-1} + \sum_{k=0}^{n} \overline{w}_{n-k}\,\overline{w}_k \qquad \forall\; n = 1, 2, \ldots$$

$$= \dot{\overline{w}}_{n-1} + 2\,\overline{w}_0\,\overline{w}_n + \sum_{k=1}^{n-1} \overline{w}_{n-k}\,\overline{w}_k$$

$$\rightarrow \qquad \overline{w}_n = -\frac{1}{2\,\overline{w}_0}\left(\dot{\overline{w}}_{n-1} + \sum_{k=1}^{n-1} \overline{w}_{n-k}\,\overline{w}_k\right) \qquad \forall\; n = 1, 2, \ldots$$

$$= \mp\frac{1}{2\,w}\left(\dot{\overline{w}}_{n-1} + \sum_{k=1}^{n-1} \overline{w}_{n-k}\,\overline{w}_k\right) \tag{2.549}$$

Choosing

$$\overline{w}_0 = w = \sqrt{v} \qquad \text{where} \qquad v = w^2$$

we have

$$\overline{w}_1 = -\frac{1}{2\,w}\,\dot{w} = -\frac{1}{2}\,\frac{d\ln w}{d\tau} = -\frac{1}{4}\,\frac{d\ln v}{d\tau} = -\frac{\dot{v}}{4\,v}$$

$$\dot{\overline{w}}_1 = -\frac{\ddot{v}}{4\,v} + \frac{\dot{v}^2}{4\,v^2}$$

$$\overline{w}_2 = -\frac{1}{2\,w}\left(\dot{\overline{w}}_1 + \overline{w}_1^2\right) = -\frac{1}{2\sqrt{v}}\left(-\frac{\ddot{v}}{4\,v} + \frac{\dot{v}^2}{4\,v^2} + \frac{\dot{v}^2}{16\,v^2}\right) = \frac{\ddot{v}}{8\,v^{3/2}} - \frac{5\,\dot{v}^2}{32\,v^{5/2}} \tag{2.550}$$

Higher n terms can be obtained from the *Mathematica* file “2.15._Code.nb”.

2.15.6. Duality Transformation and Low-Temperature Expansion

(2.513) can be written as

$$\Delta F_\omega = \frac{1}{2\,\beta}\left(\sum_{m=-\infty}^{\infty} - \beta\,\hbar\int_{-\infty}^{\infty}\frac{d\,\omega_m}{2\,\pi}\right)\ln\!\left(\omega_m^2 + \omega^2\right) \tag{2.551}$$

where we have withdrawn the effort to make the argument of the logarithm dimensionless.

Using $\omega_m = \dfrac{2\,\pi\,m}{\beta\,\hbar}$, we have

$$\Delta F_\omega = \frac{1}{2\,\beta}\left(\sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} d\,m\right)\ln\!\left[\left(\frac{2\,\pi}{\beta\,\hbar}\right)^2 m^2 + \omega^2\right] \tag{2.552}$$

where m can be discrete or continuous as the situation demands.

With the minimal subtraction formula (2.504), (2.552) becomes

$$\Delta F_\omega = -\frac{1}{2\,\beta}\int_0^\infty \frac{d\,\tau}{\tau}\left(\sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} d\,m\right)\exp\left\{-\tau\left[\left(\frac{2\,\pi}{\beta\,\hbar}\right)^2 m^2 + \omega^2\right]\right\} \tag{2.553}$$

Using Poisson’s formula (1.205), we have

$$\Delta F_\omega = -\frac{1}{2\,\beta}\int_0^\infty \frac{d\,\tau}{\tau}\int_{-\infty}^{\infty} d\,\mu\left(\sum_{n=-\infty}^{\infty} e^{2\,\pi\,\mu\,n\,i} - 1\right)\exp\left\{-\tau\left[\left(\frac{2\,\pi}{\beta\,\hbar}\right)^2 \mu^2 + \omega^2\right]\right\} \tag{2.554}$$

$$= -\frac{1}{2\,\beta}\int_0^\infty \frac{d\,\tau}{\tau}\int_{-\infty}^{\infty} d\,\mu\sum_{n=1}^{\infty}\left(e^{2\,\pi\,\mu\,n\,i} + e^{-2\,\pi\,\mu\,n\,i}\right)\exp\left\{-\tau\left[\left(\frac{2\,\pi}{\beta\,\hbar}\right)^2 \mu^2 + \omega^2\right]\right\}$$

where μ is continuous. Going from (2.553) to (2.554) is called the duality transformation.

Using

$$\int_{-\infty}^{\infty} d\,x\,e^{-a\,x^2 + b\,x} = \int_{-\infty}^{\infty} d\,x\,e^{-a\left(x - \frac{b}{2\,a}\right)^2 + \frac{b^2}{4\,a}}$$

$$= \sqrt{\frac{\pi}{a}}\, e^{b^2/4a} \tag{2.555a}$$

to evaluate the μ integral, we have

$$\sqrt{\frac{\pi}{a}} = \sqrt{\frac{\pi}{\left(\frac{2\pi}{\beta\hbar}\right)^2 \tau}} = \frac{\beta\hbar}{2\sqrt{\pi\tau}} \qquad \frac{b^2}{4a} = \frac{(\pm 2\pi n i)^2}{4\left(\frac{2\pi}{\beta\hbar}\right)^2 \tau} = -\frac{(n\beta\hbar)^2}{4\tau}$$

$$\rightarrow \qquad \Delta F_\omega = -\frac{\hbar}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \int_0^\infty \frac{d\tau}{\tau} \tau^{-1/2} \exp\left[-\frac{(n\beta\hbar)^2}{4\tau} - \tau\omega^2\right] \tag{2.556}$$

Using [Gradshteyn & Ryzhik, Formulas 3.471.9 & 8.486.16]

$$\int_0^\infty d\tau\, \tau^{\nu-1} \exp\left(-\frac{a}{\tau} - b\tau\right) = 2\left(\frac{a}{b}\right)^{\nu/2} K_\nu\left(2\sqrt{ab}\right) \qquad K_{-\nu}(z) = K_\nu(z) \tag{2.557}$$

we have

$$\Delta F_\omega = -\frac{\hbar}{\sqrt{\pi}} \sum_{n=1}^{\infty} \left(\frac{n\beta\hbar}{2\omega}\right)^{-1/2} K_{1/2}(n\beta\hbar\omega) \tag{2.558}$$

Since

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z} \tag{2.559}$$

(2.558) becomes

$$\begin{aligned} \Delta F_\omega &= -\frac{\hbar}{\sqrt{\pi}} \sum_{n=1}^{\infty} \left(\frac{n\beta\hbar}{2\omega}\right)^{-1/2} \sqrt{\frac{\pi}{2n\beta\hbar\omega}}\, e^{-n\beta\hbar\omega} \\ &= -\frac{1}{\beta} \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\beta\hbar\omega} \\ &= \frac{1}{\beta} \ln\left(1 - e^{-\beta\hbar\omega}\right) \end{aligned} \tag{2.560}$$

in agreement with (2.523).

From (2.513-4) & (2.522), we have

$$\Delta F_\omega = \frac{1}{\beta} \sum_{m=1}^{\infty} \ln\left(1 + \frac{\omega^2}{\omega_m^2}\right) + \frac{1}{\beta} \ln(\beta\hbar\omega) - \frac{1}{2}\hbar\omega \tag{2,560a}$$

Comparing with (2.560), we see that the effect of the duality tranformation is to convert the sum over Matsubara frequencies $\omega_m = \dfrac{2\pi m}{\beta\hbar}$

$$S = \sum_{m=1}^{\infty} \ln\left(1 + \frac{\omega^2}{\omega_m^2}\right) \tag{2.561}$$

into a sum over the quantum numbers n of the harmonic oscillator:

$$S = \frac{1}{2}\beta\hbar\omega - \ln(\beta\hbar\omega) - \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\beta\hbar\omega} \tag{2.562}$$

For high T, or small β, $\dfrac{\omega^2}{\omega_m^2} << 1$ for all m's and drops as $\dfrac{1}{m^2}$. The sum in (2.561) therefore converges rapidly. Using

$$\ln(1+x) = \sum_{n=1}^{\infty} (-)^{n+1} \frac{x^n}{n}$$

(2.561) becomes

$$S = -\sum_{k,m=1}^{\infty} \frac{(-)^k}{k} \left(\frac{\omega^2}{\omega_m^2}\right)^k = -\sum_{k,m=1}^{\infty} \frac{(-)^k}{k} \left(\frac{\beta\hbar\omega}{2\pi m}\right)^{2k}$$

$$= -\sum_{k=1}^{\infty} \frac{(-)^k}{k} \left(\sum_{m=1}^{\infty} \frac{1}{m^{2k}}\right)\left(\frac{\beta\hbar\omega}{2\pi}\right)^{2k} \qquad (2.563)$$

$$= -\sum_{k=1}^{\infty} \frac{(-)^k}{k} \zeta(2k) \left(\frac{\beta\hbar\omega}{2\pi}\right)^{2k} \qquad (2.564)$$

where $\zeta(z)$ is the Reimann zeta function with [see (2.520a)]

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} \left| B_{2n} \right| \qquad (2.565)$$

The Bernoulli numbers are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad (2.566)$$

with the lowest few already listed in (2.520d). Following Kleinert, we copy (2.520c) here

$$\zeta(1-2n) = -\frac{B_{2n}}{2n} \qquad n = 1, 2, \ldots \qquad (2.567)$$

which is a consequence of the general identity [Gradshteyn & Ryzhik, Formula 9.535.2-3]

$$2^z \,\Gamma(1-z)\,\zeta(1-z) \sin\frac{z\pi}{2} = \pi^{1-z} \zeta(z) \qquad (2.568a)$$

$$2^{1-z} \,\Gamma(z)\,\zeta(z) \cos\frac{z\pi}{2} = \pi^z \zeta(1-z) \qquad (2.568)$$

Thus, setting $z = 2n$ in (2.568) gives

$$2^{1-2n}\,\Gamma(2n)\,\zeta(2n)\cos n\pi = \pi^{2n}\,\zeta(1-2n)$$

$$\rightarrow \quad \zeta(1-2n) = (-)^n \pi^{-2n} 2^{1-2n} (2n-1)!\,\zeta(2n)$$

$$= (-)^n \pi^{-2n} 2^{1-2n} (2n-1)! \frac{(2\pi)^{2n}}{2(2n)!} \left| B_{2n} \right| \qquad \text{[(2.565) used.]}$$

$$= (-)^n \frac{1}{2n} \left| B_{2n} \right|$$

$$= -\frac{1}{2n} B_{2n}$$

Some values of $\zeta(z)$ of use here are [see *Mathematica* file “2.15._Code.nb”]

$$\left\{\zeta(0) = -\tfrac{1}{2},\ \zeta(2) = \tfrac{\pi^2}{6},\ \zeta(3) = 1.20206,\ \zeta(4) = \tfrac{\pi^4}{90},\ \zeta(5) = 1.03693\right\} \qquad (2.569)$$

In contrast, the duality transformed sum (2.562) converges rapidly for the opposite limit, namely, small β, or high T. In which case, there exists a large number N such that

$$e^{-\beta\hbar\omega n} \approx 1 \qquad \forall\ n < N \qquad (2.570a)$$

with acceptable error. Hence

$$\sum_{n=1}^{\infty} \frac{1}{n} e^{-n\beta\hbar\omega} \approx \sum_{n=1}^{N-1} \frac{1}{n} + \sum_{n=N}^{\infty} \frac{1}{n} e^{-n\beta\hbar\omega} \qquad (2.570)$$

Since N is large,

$$\sum_{n=N}^{\infty} \frac{1}{n} e^{-n\beta\hbar\omega} \approx \int_N^{\infty} \frac{dn}{n} e^{-n\beta\hbar\omega} = \int_{N\beta\hbar\omega}^{\infty} \frac{dx}{x} e^{-x} \qquad (2.570b)$$

$$= E_1(N\beta\hbar\omega)$$

where E_1 is the exponential integral [see (2.468)] and, by (2.570a), $N\beta\hbar\omega << 1$.

The 1st sum in (2.570) can be calculated using the Digamma function [see Arfken]

$$\psi(z) \equiv \frac{\Gamma'(z)}{\Gamma(z)} \tag{2.571}$$

which has the expansion [see Gradshteyn & Ryzhik, Formula 8.362.1]

$$\psi(x) = -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{x+k} - \frac{1}{k+1}\right) \tag{2.572}$$

Using

$$\sum_{k=0}^{\infty}\frac{1}{N+k} = \sum_{k=N}^{\infty}\frac{1}{k}$$

we have

$$\psi(N) = -\gamma + \sum_{k=1}^{N-1}\frac{1}{k} \tag{2.573}$$

For large z [see Abramowitz & Stegun, Formula 6.3.18],

$$\psi(z) \approx \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty}\frac{B_{2n}}{2n z^{2n}} \tag{2.574}$$

Combining this with (2.469) for small x :

$$E_1(x) = -\gamma - \ln x - \sum_{k=1}^{\infty}\frac{(-x)^k}{k\,k!}$$

and (2.573-4), the sum (2.570) becomes

$$\sum_{n=1}^{\infty}\frac{1}{n}e^{-n\beta\hbar\omega} \approx \psi(N) + \gamma + E_1(N\beta\hbar\omega)$$

$$\approx \ln N - \frac{1}{2N} - \sum_{n=1}^{\infty}\frac{B_{2n}}{2n N^{2n}} + \gamma - \gamma - \ln(N\beta\hbar\omega) - \sum_{k=1}^{\infty}\frac{(-N\beta\hbar\omega)^k}{k\,k!}$$

$$\approx -\ln(\beta\hbar\omega) - \frac{1}{2N} - \sum_{n=1}^{\infty}\left[\frac{B_{2n}}{2n N^{2n}} - \frac{(-N\beta\hbar\omega)^n}{n\,n!}\right]$$

Since we've asssumed that $\beta << 1$ with $N >> 1$ while $N\beta\hbar\omega << 1$, we have

$$\sum_{n=1}^{\infty}\frac{1}{n}e^{-n\beta\hbar\omega} \approx -\ln(\beta\hbar\omega) + O(\beta) \tag{2.575}$$

The sum in (2.562) thus approches $\frac{1}{2}\beta\hbar\omega$ for high T.

The low T series (2.562) can be used to illustrate the power of analytic regularization (AR). Suppose we want to extract from it the large-T behavior, where the sum

$$g(\beta\hbar\omega) \equiv \sum_{n=1}^{\infty}\frac{1}{n}e^{-n\beta\hbar\omega} \tag{2.576}$$

converges slowly. If we expand $e^{-n\beta\hbar\omega}$ as a power series, then g is a sum of n^k with $k \ge 0$.

In order to apply the technique of AR, we generalize (2.576) to

$$\zeta_\nu\left(e^{\beta\hbar\omega}\right) \equiv \sum_{n=1}^{\infty}\frac{1}{n^\nu}e^{-n\beta\hbar\omega} \tag{2.577}$$

with

$$\zeta_1\left(e^{\beta\hbar\omega}\right) = g(\beta\hbar\omega) \tag{2.577a}$$

Since

$$\int_0^\infty dn\,\frac{1}{n^\nu}\,e^{-n\beta\hbar\omega} = (\beta\hbar\omega)^{\nu-1}\,\Gamma(1-\nu) \tag{2.578a}$$

is well defined except for $\nu = 1, 2, 3, \ldots$, we re-write (2.577) as

$$\zeta_\nu\left(e^{\beta\hbar\omega}\right) = \int_0^\infty dn\,\frac{1}{n^\nu}\,e^{-n\beta\hbar\omega} + \left(\sum_{n=1}^\infty - \int_0^\infty dn\right)\frac{1}{n^\nu}\,e^{-n\beta\hbar\omega} \tag{2.578}$$

$$= \int_0^\infty dn\,\frac{1}{n^\nu}\,e^{-n\beta\hbar\omega} + \left(\sum_{n=1}^\infty - \int_0^\infty dn\right)\frac{1}{n^\nu}\sum_{k=0}^\infty \frac{(-n\beta\hbar\omega)^k}{k!}$$

$$= \int_0^\infty dn\,\frac{1}{n^\nu}\,e^{-n\beta\hbar\omega} + \left(\sum_{n=1}^\infty - \int_0^\infty dn\right)\frac{1}{n^\nu} \tag{2.579}$$

$$+\sum_{k=1}^\infty\left(\sum_{n=1}^\infty - \int_0^\infty dn\right) n^{k-\nu}\,\frac{(-\beta\hbar\omega)^k}{k!}$$

Applying the Veltman's rule (2.506), and hence the AR, we have

$$\int_0^\infty dn\,\frac{1}{n^\gamma} = 0 \qquad \to \qquad \left(\sum_{n=1}^\infty - \int_0^\infty dn\right)\frac{1}{n^\gamma} = \sum_{n=1}^\infty \frac{1}{n^\gamma} = \zeta(\gamma) \tag{2.580}$$

so that (2.579) becomes, with (2.578a),

$$\zeta_\nu\left(e^{\beta\hbar\omega}\right) = (\beta\hbar\omega)^{\nu-1}\,\Gamma(1-\nu) + \zeta(\nu) + \sum_{k=1}^\infty \zeta(\nu-k)\,\frac{(-\beta\hbar\omega)^k}{k!} \tag{2.581}$$

which is known as the Robinson expansion and will be used in the discussion of the Einstein condensation [see (7.38)].

It is also useful to define the auxiliary function

$$\bar{\zeta}_\nu\left(e^{\beta\hbar\omega}\right) \equiv \sum_{k=1}^\infty \zeta(\nu-k)\,\frac{(-\beta\hbar\omega)^k}{k!}$$

$$= \sum_{n=1}^\infty \frac{1}{n^\nu}\,e^{-n\beta\hbar\omega} - \int_0^\infty dn\,\frac{1}{n^\nu}\,e^{-n\beta\hbar\omega} - \left(\sum_{n=1}^\infty - \int_0^\infty dn\right)\frac{1}{n^\nu}$$

$$= \left(\sum_{n=1}^\infty - \int_0^\infty dn\right)\frac{1}{n^\nu}\,e^{-n\beta\hbar\omega} - \zeta(\nu) \tag{2.582}$$

Using

$$(1-\nu)\,\Gamma(1-\nu) = \Gamma(2-\nu)$$

we have

$$\Gamma(1-\nu) = \frac{1}{1-\nu}\,\Gamma(2-\nu) \xrightarrow[\nu\to 1]{} \frac{1}{1-\nu}\,\Gamma(1) = \frac{1}{1-\nu}$$

which is singular. From (2.568a)

$$2^z\,\Gamma(1-z)\,\zeta(1-z)\sin\frac{z\pi}{2} = \pi^{1-z}\,\zeta(z)$$

we get

$$\zeta(\nu) = \pi^{-1+\nu}\,2^\nu\,\Gamma(1-\nu)\,\zeta(1-\nu)\sin\frac{\nu\pi}{2}$$

$$\xrightarrow[\nu\to 1]{} 2\,\Gamma(1-\nu)\left(-\frac{1}{2}\right) = -\Gamma(1-\nu) \qquad \text{[(2.520) used.]} \tag{2.584}$$

Hence, as $\nu \to 1$, the first 2 terms in (2.581) become

$$(\beta\hbar\omega)^{\nu-1}\,\Gamma(1-\nu) + \zeta(\nu) \approx \frac{1}{1-\nu}\left[(\beta\hbar\omega)^{\nu-1} - 1\right]$$

$$= -\lim_{\epsilon\to 0} \frac{(\beta\hbar\omega)^\epsilon - 1}{\epsilon}$$
$$= -\frac{d}{d\epsilon}(\beta\hbar\omega)^\epsilon \Big|_{\epsilon=0}$$
$$= -(\beta\hbar\omega)^\epsilon \ln(\beta\hbar\omega) \Big|_{\epsilon=0}$$
$$= -\ln(\beta\hbar\omega)$$

$$\therefore \qquad g\left(e^{\beta\hbar\omega}\right) = \zeta_1\left(e^{\beta\hbar\omega}\right) = -\ln(\beta\hbar\omega) + \sum_{k=1}^{\infty} \zeta(1-k)\frac{(-\beta\hbar\omega)^k}{k!} \qquad (2.584a)$$

Since [see (2.520 & b)]

$$\zeta(0) = -\frac{1}{2} \qquad \text{and} \qquad \zeta(-2m) = 0 \qquad m = 1, 2, 3, \ldots$$

(2.584a) simplifies to

$$g\left(e^{\beta\hbar\omega}\right) = -\ln(\beta\hbar\omega) + \frac{1}{2}\beta\hbar\omega + \sum_{n=1}^{\infty} \zeta(1-2n)\frac{(\beta\hbar\omega)^{2n}}{(2n)!} \qquad (2.584b)$$

From [see (2.520a & c)]

$$\zeta(2n) = \frac{(-)^{n+1}}{2(2n)!}(2\pi)^{2n} B_{2n} \qquad n = 1, 2, 3, \ldots$$
$$\zeta(1-2n) = -\frac{B_{2n}}{2n}$$

we have

$$\zeta(1-2n) = (-)^n \frac{(2n)!}{n(2\pi)^{2n}}\zeta(2n) \qquad n = 1, 2, 3, \ldots \qquad (2.585)$$

(2.584b) becomes

$$g\left(e^{\beta\hbar\omega}\right) = -\ln(\beta\hbar\omega) + \frac{1}{2}\beta\hbar\omega + \sum_{n=1}^{\infty}(-)^n \frac{\zeta(2n)}{n}\left(\frac{\beta\hbar\omega}{2\pi}\right)^{2n} \qquad (2.586)$$

Inserting this AR result into (2.562) recovers (2.564), which was derived by the duality transformation. Thus proving our remarks stated just before (2.576).

Note that an essential part of the AR procedure is the separation (2.578) of an integral from the sum, thus allowing the application of the Veltman's rule in (2.581). If we omit (2.578) and expand (2.576) directly, we get

$$\zeta_\nu\left(e^{\beta\hbar\omega}\right) = \sum_{n=1}^{\infty}\frac{1}{n^\nu}\sum_{k=0}^{\infty}\frac{(-n\beta\hbar\omega)^k}{k!}$$
$$= \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} n^{k-\nu}\frac{(-\beta\hbar\omega)^k}{k!}$$
$$= \sum_{k=0}^{\infty}\zeta(\nu-k)\frac{(-\beta\hbar\omega)^k}{k!}$$

$$\to \qquad \zeta_1\left(e^{\beta\hbar\omega}\right) = \zeta(1) + \sum_{k=1}^{\infty}\zeta(1-k)\frac{(-\beta\hbar\omega)^k}{k!} \qquad (2.587)$$

which is singular due to $\zeta(1)$. Comparing with (2.586), we see that the AR acts to replace $\zeta(1)$ with $-\ln(\beta\hbar\omega)$, which may be formalized as

$$\zeta(1) \to \zeta_{\text{reg}}(1) = -\ln(\beta\hbar\omega) \qquad (2.588)$$

The Robinson expansion (2.581) can be supplemented by a dual version as follows. Using the Poisson formula (1.197 or 1.205), we can write (2.577) as

$$\zeta_\nu\left(e^{\beta\hbar\omega}\right) = \sum_{m=-\infty}^{\infty} \int_0^\infty dn\, \frac{1}{n^\nu}\, e^{-(\beta\hbar\omega - 2\pi i m)n} \qquad (2.589a)$$

$$= \sum_{m=-\infty}^{\infty} (\beta\hbar\omega - 2\pi i m)^{\nu-1}\,\Gamma(1-\nu) \qquad \text{[(2.496a) used.]}$$

$$= (\beta\hbar\omega)^{\nu-1}\,\Gamma(1-\nu) + \sum_{m=1}^{\infty}\left[(\beta\hbar\omega - 2\pi i m)^{\nu-1} + (\beta\hbar\omega + 2\pi i m)^{\nu-1}\right]\Gamma(1-\nu)$$

$$= (\beta\hbar\omega)^{\nu-1}\,\Gamma(1-\nu) + \Gamma(1-\nu)\sum_{m=1}^{\infty} 2\,\mathrm{Re}\,(\beta\hbar\omega - 2\pi i m)^{\nu-1} \qquad (2.589)$$

The sum can be expanded in powers of ω :

$$\sum_{m=1}^{\infty} 2\,\mathrm{Re}\,(\beta\hbar\omega - 2\pi i m)^{\nu-1} = 2\,\mathrm{Re}\sum_{m=1}^{\infty}(-2\pi i m)^{\nu-1}\left(1 + i\,\frac{\beta\hbar\omega}{2\pi m}\right)^{\nu-1}$$

$$= 2\,\mathrm{Re}\sum_{m=1}^{\infty}(-2\pi i m)^{\nu-1}\sum_{k=0}^{\infty} C_k^{\nu-1}\left(i\,\frac{\beta\hbar\omega}{2\pi m}\right)^k$$

$$= 2\,\mathrm{Re}\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}(2\pi m)^{\nu-1-k}\, e^{i(1-\nu+k)\pi/2}\, C_k^{\nu-1}\,(\beta\hbar\omega)^k$$

$$= 2\,\mathrm{Re}\sum_{k=0}^{\infty}\zeta(k+1-\nu)\,(2\pi)^{\nu-1-k}\, e^{i(1-\nu+k)\pi/2}\, C_k^{\nu-1}\,(\beta\hbar\omega)^k$$

$$= 2\sum_{k=0}^{\infty}\zeta(k+1-\nu)\,(2\pi)^{\nu-1-k}\cos\left[(k+1-\nu)\,\frac{\pi}{2}\right] C_k^{\nu-1}\,(\beta\hbar\omega)^k \qquad (2.590)$$

$$= 2\sum_{k=0}^{\infty}\zeta(k+1-\nu)\,(2\pi)^{\nu-1-k}\sin\left[(\nu-k)\,\frac{\pi}{2}\right] C_k^{\nu-1}\,(\beta\hbar\omega)^k$$

$$= \sum_{k=0}^{\infty}\frac{\zeta(\nu-k)}{\Gamma(k+1-\nu)}\, C_k^{\nu-1}\,(\beta\hbar\omega)^k \qquad \text{[(2.583) used.]}$$

$$= \sum_{k=0}^{\infty}\frac{\zeta(\nu-k)\,\Gamma(\nu)}{k!\,\Gamma(k+1-\nu)\,\Gamma(\nu-k)}\,(\beta\hbar\omega)^k$$

Using [Gradshteyn & Ryzhik, Formula 8.334.3]

$$\Gamma(1-x)\,\Gamma(x) = \frac{\pi}{\sin\pi x}$$

(2.589) becomes

$$\zeta_\nu\left(e^{\beta\hbar\omega}\right) = (\beta\hbar\omega)^{\nu-1}\,\Gamma(1-\nu) + \Gamma(1-\nu)\sum_{k=0}^{\infty}\frac{\zeta(\nu-k)\,\Gamma(\nu)}{k!\,\Gamma(k+1-\nu)\,\Gamma(\nu-k)}\,(\beta\hbar\omega)^k$$

$$= (\beta\hbar\omega)^{\nu-1}\,\Gamma(1-\nu) + \sum_{k=0}^{\infty}\frac{\zeta(\nu-k)\,\sin\pi(\nu-k)}{k!\,\sin\pi\nu}\,(\beta\hbar\omega)^k$$

$$= (\beta\hbar\omega)^{\nu-1}\,\Gamma(1-\nu) + \sum_{k=0}^{\infty}\frac{\zeta(\nu-k)}{k!}\,(-\beta\hbar\omega)^k$$

which agrees with (2.581).

Using

$$(\beta\hbar\omega - 2\pi i m)\,n = (\omega - i\,\omega_m)\,\beta\hbar n \qquad \omega_m = \frac{2\pi m}{\beta\hbar}$$

(2.589a) can be written as

$$\zeta_\nu\left(e^{\beta\hbar\omega}\right)=\sum_{m=-\infty}^{\infty}\int_0^\infty dn\,\frac{1}{n^\nu}\,e^{(i\omega_m-\omega)\beta\hbar n}$$

$$=\int_0^\infty dn\,\frac{1}{n^\nu}\,e^{-\omega\beta\hbar n}+\sum_{m=1}^{\infty}\int_0^\infty dn\,\frac{1}{n^\nu}\left[e^{(i\omega_m-\omega)\beta\hbar n}+e^{(-i\omega_m-\omega)\beta\hbar n}\right]$$

$$=\Gamma(1-\nu)\,(\beta\hbar\omega)^{\nu-1}+2\,\mathrm{Re}\sum_{m=1}^{\infty}\int_0^\infty dn\,\frac{1}{n^\nu}\,e^{(i\omega_m-\omega)\beta\hbar n}\qquad\text{[(2.496a) used.]}$$

$$=\Gamma(1-\nu)\,(\beta\hbar\omega)^{\nu-1}+2\,\mathrm{Re}\sum_{m=1}^{\infty}\left[(\omega-i\omega_m)\,\beta\hbar\right]^{\nu-1}\Gamma(1-\nu)$$

$$=\Gamma(1-\nu)\,(\beta\hbar\omega)^{\nu-1}\left[1+2\,\mathrm{Re}\sum_{m=1}^{\infty}\left(1-\frac{i\omega_m}{\omega}\right)^{\nu-1}\right]\qquad(2.591)$$

in agreement with (2.589). The first term in (2.591) represents the high T or classical limit of the expansion. The rest therefore contains quantum fluctuations.

For an analytic function $F(t)$ on an interval (a, b), two of the Euler-Maclaurin formulas [Abramowitz & Stegun, Formulas 23.1.30 & 23.1.32 with $n=\infty$] are

$$\sum_{k=0}^{M}F(a+k\Delta)=\frac{1}{\Delta}\int_a^b F(t)\,dt+\frac{1}{2}[F(b)+F(a)]\qquad\Delta=\frac{b-a}{M}\qquad(2.592)$$

$$+\sum_{j=1}^{\infty}\frac{\Delta^{2j-1}}{(2j)!}B_{2j}\left[F^{(2j-1)}(b)-F^{(2j-1)}(a)\right]$$

where

$$F^{(j)}(b)\equiv\partial_t^j F(t)\Big|_{t=b}$$

and

$$\sum_{k=0}^{M-1}F[a+(k+\theta)\Delta]=\frac{1}{\Delta}\int_a^b F(t)\,dt+\sum_{j=1}^{\infty}\frac{\Delta^{j-1}}{j!}B_j(\theta)\left[F^{(j-1)}(b)-F^{(j-1)}(a)\right]\qquad(2.593)$$

where $1\geq\theta\geq0$ and $B_j(\theta)$ are the Bernoulli functions defined by a generalization of the expansion (2 .566) :

$$\frac{t\,e^{\theta t}}{e^t-1}=\sum_{n=0}^{\infty}B_n(\theta)\,\frac{t^n}{n!}\qquad(2.594)$$

so that

$$B_n(0)=B_n\qquad\text{and}\qquad B_0(\theta)=1\qquad(2.594a)$$

Using

$$F^{(j-1)}(b)-F^{(j-1)}(a)=\int_a^b dt\,\partial_t^j F(t)$$

we can write (2.593) as

$$\sum_{k=0}^{M-1}F[a+(k+\theta)\Delta]=\frac{1}{\Delta}\int_a^b dt\left[1+\sum_{j=1}^{\infty}\frac{\Delta^j}{j!}B_j(\theta)\,\partial_t^j\right]F(t)\qquad(2.595)$$

$$=\frac{1}{\Delta}\int_a^b dt\sum_{j=0}^{\infty}\frac{\Delta^j}{j!}B_j(\theta)\,\partial_t^j F(t)\qquad(2.595a)$$

with the understanding that $\partial_t^0 F(t)=F(t)$.

Thus, a sum over discrete values of $F(t)$ can be replaced by an integral over the gradient expansion of $F(t)$.

The sum (2.561) can be evaluated by setting

$$F(\omega_m)=\ln\left(1+\frac{\omega^2}{\omega_m^2}\right)=\ln\left(\omega_m^2+\omega^2\right)-\ln\omega_m^2\qquad\omega_m=\frac{2\pi m}{\beta\hbar}=m\,\omega_1$$

with

$$a = \omega_1 \quad b = \omega_M \qquad \& \qquad \Delta = \omega_1$$

we have

$$a + k\,\Delta = \omega_1 + k\,\omega_1 = \omega_{k+1}$$

so that the 1st formula (2.592) becomes

$$S = \lim_{M\to\infty} \sum_{m=1}^{M} \ln\left(1 + \frac{\omega^2}{\omega_m^2}\right) = \sum_{k=0}^{M-1} \ln\left(1 + \frac{\omega^2}{\omega_{k+1}^2}\right)$$

$$= \lim_{M\to\infty} \left\{ \frac{1}{\omega_1} \int_{\omega_1}^{\omega_M} \ln\left(1 + \frac{\omega^2}{t^2}\right) d\,t + \frac{1}{2}\left[\ln\left(1 + \frac{\omega^2}{\omega_M^2}\right) + \ln\left(1 + \frac{\omega^2}{\omega_1^2}\right)\right] + \sum_{j=1}^{\infty} \frac{\omega_1^{2j-1}}{(2\,j)!} B_{2j}\left[F^{(2j-1)}(\omega_M) - F^{(2j-1)}(\omega_1)\right]\right\} \tag{2.597a}$$

Consider the integral

$$\mathcal{I} = \int_{\omega_1}^{\omega_M} \ln\left(1 + \frac{\omega^2}{t^2}\right) d\,t = \left[2\,\omega \tan^{-1}\frac{t}{\omega} + t\ln\left(1 + \frac{\omega^2}{t^2}\right)\right] \Bigg|_{\omega_1}^{\omega_M}$$

Using

$$\lim_{t\to\infty} t\ln\left(1 + \frac{\omega^2}{t^2}\right) = \lim_{t\to\infty} t\,\frac{\omega^2}{t^2} = 0 \qquad\qquad \tan\frac{\pi}{2} = \infty$$

we have

$$\lim_{\omega_M\to\infty} \mathcal{I} = \omega\,\pi - 2\,\omega\tan^{-1}\frac{\omega_1}{\omega} - \omega_1 \ln\left(1 + \frac{\omega^2}{\omega_1^2}\right)$$

and (2.597a) becomes

$$S = \frac{\omega}{\omega_1}\pi - 2\,\frac{\omega}{\omega_1}\tan^{-1}\frac{\omega_1}{\omega} - \frac{1}{2}\ln\left(1 + \frac{\omega^2}{\omega_1^2}\right) + O(\omega_1) \tag{2.597b}$$

For low T, or large β, $\omega_1 = \dfrac{2\,\pi}{\beta\,\hbar}$ is small and (2.597b) becomes

$$\frac{\omega}{\omega_1}\pi - 2 - \frac{1}{2}\ln\left(\frac{\omega^2}{\omega_1^2}\right) + O(\omega_1)$$

the 1st two leading terms of which are

$$\frac{\omega}{\omega_1}\pi - \frac{1}{2}\ln\left(\frac{\omega^2}{\omega_1^2}\right) = \frac{1}{2}\beta\,\hbar\,\omega - \ln(\beta\,\hbar\,\omega) \tag{2.597}$$

where a term $\ln(2\,\pi) << \beta\,\hbar\,\omega$ was dropped. This is the same as the low T series (2.562). However, the Euler-Maclaurin formula is unable to recover the exponentially small terms in (2.562), since they are not expandable in powers of T.