2.17. Time Evolution Amplitude of Freely Falling Particle

The gravitational potential of a particle on the surface of the earth is

$$V(\boldsymbol{x}) = V_0 + M\,\boldsymbol{g}\cdot\boldsymbol{x} \tag{2.609}$$

where $-\boldsymbol{g}$ is Earth's acceleration vector pointing towards the ground and V_0 is the potential at the ground where $\boldsymbol{x}=0$. The eq. of motion is simply

$$\ddot{\boldsymbol{x}} = -\boldsymbol{g} \tag{2.610}$$

For the I.C.

$$\boldsymbol{x}(t_a) = \boldsymbol{x}_a \qquad \dot{\boldsymbol{x}}(t_a) = \boldsymbol{v}_a$$

the solution is

$$\dot{\boldsymbol{x}} = \boldsymbol{v}_a - \boldsymbol{g}\,(t-t_a) \tag{2.611a}$$

$$\boldsymbol{x} = \boldsymbol{x}_a + \boldsymbol{v}_a(t-t_a) - \frac{1}{2}\boldsymbol{g}\,(t-t_a)^2 \tag{2.611}$$

To simplify the notations, we set

$$\Delta\,\boldsymbol{x} = \boldsymbol{x}_b - \boldsymbol{x}_a \qquad \tau = t_b - t_a \tag{2.611b}$$

$$\overline{\boldsymbol{x}} = \frac{1}{2}(\boldsymbol{x}_b + \boldsymbol{x}_a) \qquad \overline{t} = \frac{1}{2}(t_b + t_a)$$

Hence,

$$\boldsymbol{x}_b = \boldsymbol{x}_a + \boldsymbol{v}_a\,\tau - \frac{1}{2}\boldsymbol{g}\,\tau^2$$

$$\rightarrow \qquad \boldsymbol{v}_a = \frac{\Delta\,\boldsymbol{x}}{\tau} + \frac{1}{2}\boldsymbol{g}\,\tau \tag{2.612}$$

(2.611) & (2.611a) thus become

$$\boldsymbol{x} = \boldsymbol{x}_a + \frac{\Delta\,\boldsymbol{x}}{\tau}(t-t_a) + \frac{1}{2}\boldsymbol{g}\,(t_b-t)\,(t-t_a) \tag{2.612a}$$

$$\dot{\boldsymbol{x}} = \frac{\Delta\,\boldsymbol{x}}{\tau} - \boldsymbol{g}\,(t-\overline{t}) \tag{2.612b}$$

Note that (2.612a) satisfies the B.C.

$$\boldsymbol{x}(t_a) = \boldsymbol{x}_a \qquad \boldsymbol{x}(t_b) = \boldsymbol{x}_b \tag{2.612c}$$

Putting these into

$$E^g = \frac{1}{2}M\,\dot{\boldsymbol{x}}^2 + V_0 + M\,\boldsymbol{g}\cdot\boldsymbol{x}$$

$$\mathcal{A}^g = \int_{t_a}^{t_b} d\,t\left(\frac{1}{2}M\,\dot{\boldsymbol{x}}^2 - V_0 - M\,\boldsymbol{g}\cdot\boldsymbol{x}\right) \tag{2.613}$$

gives the energy & the classical action [see *Mathematica* file "2.17._Code.nb"]

$$E^g = M\left[\frac{(\Delta\boldsymbol{x})^2}{2\,\tau^2} + \frac{1}{8}\boldsymbol{g}^2\,\tau^2 + \boldsymbol{g}\cdot\overline{\boldsymbol{x}}\right] + V_0 \tag{2.614a}$$

$$\mathcal{A}^g_{\text{cl}} = M\left[\frac{(\Delta\boldsymbol{x})^2}{2\,\tau} - \boldsymbol{g}\cdot\overline{\boldsymbol{x}}\,\tau - \frac{1}{24}\boldsymbol{g}^2\,\tau^3\right] - V_0\,\tau \tag{2.614}$$

Since the velocity dependent part in $\mathcal{A}$ is the same as that for the free particle, they share the same fluctuation factor (2.125). The evolution amplitude is therefore

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \left(\frac{M}{2\pi i \hbar \tau} \right)^{D/2} \exp\left(-\frac{i}{\hbar} V_0\, \tau \right) \times \exp\left\{ \frac{i}{\hbar} M \left[\frac{(\Delta \boldsymbol{x})^2}{2\tau} - \boldsymbol{g}\cdot\overline{\boldsymbol{x}}\,\tau - \frac{1}{24} \boldsymbol{g}^2 \tau^3 \right] \right\} \quad (2.615)$$

Let us emulate the potential (2.609) by a harmonic potential

$$v(\boldsymbol{x}) = v_0 + \frac{1}{2} M \omega^2 (\boldsymbol{x} - \boldsymbol{x}_0)^2 = v_0 + \frac{1}{2} M \omega^2 \left(\boldsymbol{x}^2 - 2\,\boldsymbol{x}\cdot\boldsymbol{x}_0 + \boldsymbol{x}_0^2 \right) \quad (2.616)$$

The energy & classical action satisfying the B.C. (2.612c) are [see “2.17._Code.nb”]

$$E^\omega = v_0 + M \left\{ \frac{(\Delta \boldsymbol{x})^2}{2\tau^2} + \frac{1}{24}\omega^2 \left[(\Delta \boldsymbol{x})^2 + 12\,(\overline{\boldsymbol{x}} - \boldsymbol{x}_0)^2 \right] + \frac{1}{480} \tau^2 \omega^4 \left[(\Delta \boldsymbol{x})^2 + 60\,(\overline{\boldsymbol{x}} - \boldsymbol{x}_0)^2 \right] \right\} + O\left[\omega^6\right] \quad (2.616a)$$

$$\mathcal{A}^\omega = -v_0\,\tau + M\tau \left\{ \frac{(\Delta \boldsymbol{x})^2}{2\tau^2} - \frac{1}{24}\omega^2 \left[(\Delta \boldsymbol{x})^2 + 12\,(\overline{\boldsymbol{x}} - \boldsymbol{x}_0)^2 \right] - \frac{1}{1440} \tau^2 \omega^4 \left[(\Delta \boldsymbol{x})^2 + 60\,(\overline{\boldsymbol{x}} - \boldsymbol{x}_0)^2 \right] \right\} + O\left[\omega^6\right] \quad (2.616b)$$

These two systems emulates each other if

$$E^g = E^\omega = E \qquad \& \qquad \mathcal{A}^g = \mathcal{A}^\omega \quad (2.616c)$$

$(\Delta \boldsymbol{x})^2$ in (2.616c) can be eliminated using (2.614a). Since our emulation is expected to work only if $\omega\,\tau << 1$, it is sufficient to set the coefficients of τ^n up to $n = 3$ in (2.616c). If the system is 1-D, we get three equations which can be solved to give [see “2.17._Code.nb”]

$$g = \omega^2 (x_0 - \overline{x}) = \omega \sqrt{\frac{2}{M}(E - v_0)} \quad (2.618)$$

$$\overline{x} = \frac{E - V_0}{M g} \qquad x_0 = -\frac{E - 2\,v_0 + V_0}{M g} \quad (2.619)$$

$$\rightarrow \qquad x_0 - \overline{x} = \frac{2\,(E - v_0)}{M g} \quad (2.619a)$$

The wave function for a particle in a 1-D harmonic potential (2.616) is

$$\phi_n(x) = C_n\, e^{-a^2 (x - x_0)^2/2} H_n[a\,(x - x_0)] \quad (2.619i)$$

where H_n are the Hermite polynomials and

$$a = \sqrt{\frac{M\omega}{\hbar}} \qquad C_n = \frac{1}{\sqrt{2^n\, n!\, \sqrt{\pi a}}} \quad (2.619j)$$

ϕ_n is an eigenstate of the Hamiltonian with eigen-energy

$$E_n = \left(n + \frac{1}{2} \right) \hbar\,\omega + v_0 \quad (2.619k)$$

Using (2.618), these can be expressed as

$$a = \left(\frac{M^3 g^2}{2\,\hbar^2 (E - v_0)} \right)^{1/4} \qquad n = \sqrt{\frac{2\,(E - v_0)^3}{\hbar^2 g^2 M}} - \frac{1}{2} \quad (2.619m)$$

Consider now the 1-D Schrodinger eq. for the potential (2.609):

$$-\frac{\hbar^2}{2M}\psi''(x) + (V_0 + M g x)\,\psi(x) = E\,\psi(x) \qquad (2.620a)$$

Since the Airy function of the 1st kind satisfies the eq.

$$\mathrm{Ai}''(z) - z\,\mathrm{Ai}(z) = 0 \qquad (2.622)$$

we set

$$z = \alpha\,(x - x_0') \qquad \to \qquad x = \frac{z}{\alpha} + x_0'$$

so that (2.620a) becomes

$$\frac{\hbar^2\alpha^2}{2M}\partial_z^2\psi(z) - \frac{M g}{\alpha}\left[z + \alpha x_0' + \frac{\alpha}{M g}(V_0 - E)\right]\psi(z) = 0 \qquad (2.622a)$$

Setting

$$\frac{\hbar^2\alpha^3}{2M^2 g} = 1 \qquad \& \qquad \alpha x_0' + \frac{\alpha}{M g}(V_0 - E) = 0 \qquad (2.622b)$$

turns (2.622a) into (2.622) &

$$\psi(z) = C\,\mathrm{Ai}(z) = C\,\mathrm{Ai}\,[\alpha\,(x - x_0')] \qquad (2.622c)$$

where C is a normalization constant.

Solving (2.622b) gives

$$\alpha = \left(\frac{2M^2 g}{\hbar^2}\right)^{1/3} = \frac{1}{l} \qquad\qquad x_0' = \frac{1}{M g}(E - V_0)$$

where

$$l \equiv \left(\frac{\hbar^2}{2M^2 g}\right)^{1/3}$$

$$\to \qquad \alpha x_0' = \frac{E - V_0}{M g\, l} = \left(\frac{2}{M g^2\hbar^2}\right)^{1/3}(E - V_0) = \frac{E - V_0}{\epsilon}$$

with

$$\epsilon = \left(\frac{1}{2}M g^2\hbar^2\right)^{1/3}$$

and (2.622c) becomes

$$\psi(x) = C\,\mathrm{Ai}\left(\frac{x}{l} - \frac{E - V_0}{\epsilon}\right) \qquad (2.621a)$$

Using

$$\int_{-\infty}^{\infty} d t\,\mathrm{Ai}\,(t + x)\,\mathrm{Ai}\,(t + y) = \delta(x - y)$$

and the normalization

$$\int_{-\infty}^{\infty} d x\,\psi_E^*(x)\,\psi_{E'}(x) = \delta(E - E') \qquad (2.621b)$$

we have

$$C^2\int_{-\infty}^{\infty} d x\,\mathrm{Ai}\left(\frac{x}{l} - \frac{E - V_0}{\epsilon}\right)\mathrm{Ai}\left(\frac{x}{l} - \frac{E' - V_0}{\epsilon}\right) = C^2\, l\,\delta\left(\frac{E}{\epsilon} - \frac{E'}{\epsilon}\right) = \delta(E - E')$$

$$\to \qquad C = \frac{1}{\sqrt{l\,\epsilon}} = \left(\frac{4M}{\hbar^4 g}\right)^{1/6}$$

$$\psi_E(x) = \frac{1}{\sqrt{l\,\epsilon}}\,\text{Ai}\left(\frac{x}{l} - \frac{E - V_0}{\epsilon}\right) \qquad (2.621)$$

Note that for real arguments [see "2.17._Code.nb"]

$$\text{Ai}\,(x)\text{ is }\begin{cases}\text{oscillatory} & \text{for } x < 0\\ \text{exponential} & \text{for } x > 0\end{cases}$$

Using the asymptotic relation

$$e^{-x^2/2}\,H_n(x) \underset{n\to\infty}{\longrightarrow} \pi^{1/4}\,2^{\frac{n}{2}+\frac{1}{4}}\sqrt{n!}\;n^{-1/12}\,\text{Ai}\left(-3^{-1/3}\,t\right)$$

where

$$t = 2^{1/2}\,3^{1/3}\,n^{1/6}\left(\sqrt{2\,n+1} - x\right)$$

(2.619i) becomes

$$\begin{aligned}\phi_n(x) &\underset{n\to\infty}{\longrightarrow} a^{-1/4}\,\text{Ai}\left(-2^{1/2}\,n^{1/6}\left[\sqrt{2\,n} - a\,(x - x_0)\right]\right)\\ &= a^{-1/4}\,\text{Ai}\left(\frac{x}{l} - \frac{E - V_0}{\epsilon}\right) \qquad (2.621a)\\ &\propto \psi_E(x)\end{aligned}$$

For $z > 0$, we have [see Abramowitz & Stegun, Formula 10.4.14]

$$\begin{aligned}\text{Ai}(z) &= \frac{\sqrt{z}}{2}\,[\,I_{-1/3}(\zeta) - I_{1/3}(\zeta)\,] \qquad \zeta = \frac{2}{3}\,z^{3/2} \qquad (2.623)\\ &= \frac{1}{\pi}\sqrt{\frac{z}{3}}\;K_{1/3}(\zeta)\end{aligned}$$

where I_ν & K_ν are the modified Bessel functions.

For large $z > 0$,

$$\text{Ai}(z) \underset{z\to\infty}{\longrightarrow} \frac{1}{2\sqrt{\pi}\;z^{1/4}}\,e^{-\zeta} \qquad (2.624)$$

For $z < 0$, an analytic continuation [see Gradshteyn & Ryzhik, Formulas 8.406]

$$I_\nu(z) = \begin{cases} e^{-i\,\pi\,\nu/2}\,J_\nu\left(e^{i\,\pi/2}\,z\right) & \text{for } -\pi < \arg z \le \frac{\pi}{2}\\ e^{3\,i\,\pi\,\nu/2}\,J_\nu\left(e^{-3\,i\,\pi/2}\,z\right) & \text{for } \frac{\pi}{2} < \arg z \le \pi\end{cases} \qquad (2.625)$$

gives [see Abramowitz & Stegun, Formula 10.4.15]

$$\text{Ai}(z) = \frac{1}{3}\sqrt{-z}\,\left[J_{1/3}(\overline{\zeta}) + J_{-1/3}(\overline{\zeta})\right] \qquad \overline{\zeta} = \frac{2}{3}\,(-z)^{3/2} \qquad (2.626)$$

where J_ν are the Bessel functions with asymptotic form

$$J_\nu(x) \underset{x\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi\,x}}\;\cos\left(x - \frac{\pi\,\nu}{2} - \frac{\pi}{4}\right) + O\left(x^{-1}\right) \qquad (2.627)$$

from which we obtain the oscillatory part of the Airy function

$$\text{Ai}(z) \underset{z\to-\infty}{\longrightarrow} \frac{1}{\sqrt{\pi}\;(-z)^{1/4}}\,\sin\left(\overline{\zeta} + \frac{\pi}{4}\right) \qquad (2.628)$$

Ai(z) has the simple Fourier representation

$$\text{Ai}(x) = \int_{-\infty}^{\infty}\frac{d\,k}{2\,\pi}\,e^{i\,k\,x + i\,k^3/3} \qquad \text{or} \qquad \text{Ai}(k) = e^{i\,k^3/3} \qquad (2.629)$$

Taking the Fourier transform of (2.621a), we get

$$\psi_E(p) = \langle p \mid E \rangle = \int_{-\infty}^{\infty} dx\, e^{i p x/\hbar} \frac{1}{\sqrt{l\,\epsilon}} \mathrm{Ai}\left(\frac{x}{l} - \frac{E - V_0}{\epsilon}\right)$$

$$= \sqrt{\frac{l}{\epsilon}} \exp\left(\frac{i}{\hbar} p\, l \frac{E - V_0}{\epsilon}\right) \exp\left[\frac{i}{3}\left(\frac{p\, l}{\hbar}\right)^3\right] \qquad (2.630)$$

Hence,

$$\langle E' \mid E \rangle = \int \frac{dp}{2\pi\hbar} \langle E' \mid p \rangle \langle p \mid E \rangle$$

$$= \frac{l}{\epsilon} \int \frac{dp}{2\pi\hbar} \exp\left[\frac{i}{\hbar} \frac{p\, l}{\epsilon} (E - E')\right]$$

$$= \delta(E - E') \qquad (2.631)$$

$$\int dE\, \psi_E(p)\, \psi_E^*(p') = \frac{l}{\epsilon} \int dE \exp\left[\frac{i}{\hbar} (p - p')\, l \frac{E}{\epsilon}\right]$$

$$\times \exp\left[-\frac{i}{\hbar} (p - p')\, l \frac{V_0}{\epsilon} + \frac{i}{3}\left(\frac{l}{\hbar}\right)^3 \left(p^3 - p'^3\right)\right]$$

$$= 2\pi\hbar\, \delta(p - p') \exp\left[-\frac{i}{\hbar} (p - p')\, l \frac{V_0}{\epsilon} + \frac{i}{3}\left(\frac{l}{\hbar}\right)^3 \left(p^3 - p'^3\right)\right]$$

$$= 2\pi\hbar\, \delta(p - p') \qquad (2.631a)$$