2.18. Charged Particle in Magnetic Field

2.18.1. Action

The action of a particle of charge e in a magnetic field $\boldsymbol{B}$ is given by

$$\mathcal{A}[\boldsymbol{x}] = \mathcal{A}_0 + \mathcal{A}_{mag} \qquad (2.633)$$

where

$$\mathcal{A}_0 = \int_{t_a}^{t_b} dt\, \frac{1}{2} M \dot{\boldsymbol{x}}(t)^2 \qquad (2.632a)$$

$$\mathcal{A}_{mag} = \int_{t_a}^{t_b} dt\, \frac{e}{c} \dot{\boldsymbol{x}}(t) \cdot \boldsymbol{A}[\boldsymbol{x}(t)] \qquad (2.632)$$

and $\boldsymbol{A}(\boldsymbol{x})$ is the vector potential so that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad (2.635)$$

Note that if $\boldsymbol{A}$ satisfies (2.635), so does

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla \Lambda(\boldsymbol{x}) \qquad (2.636)$$

for any differentiable function Λ since

$$\nabla \times \nabla \Lambda = 0 \qquad (2.637a)$$

which is easily proved as follows

$$\epsilon_{ijk} \partial_j \partial_k \Lambda = \epsilon_{ikj} \partial_k \partial_j \Lambda = -\epsilon_{ijk} \partial_k \partial_j \Lambda = -\epsilon_{ijk} \partial_j \partial_k \Lambda = 0$$

where the last but one step invoked the Schwarz integrability condition

$$\partial_k \partial_j \Lambda = \partial_j \partial_k \Lambda \qquad (2.637)$$

The transformation $\boldsymbol{A} \to \boldsymbol{A}'$ is called a gauge transformation.

For

$$\boldsymbol{B} = B \hat{\boldsymbol{z}} = (0, 0, B) \qquad (2.634a)$$

we can set

$$\boldsymbol{A}(\boldsymbol{x}) = B x \hat{\boldsymbol{y}} = (0, B x, 0) \qquad (2.634)$$

Another choice is the axially symmetric

$$\tilde{\boldsymbol{A}}(\boldsymbol{x}) = \frac{1}{2} \boldsymbol{B} \times \boldsymbol{x} = \frac{1}{2} B(-y, x, 0) \qquad (2.638)$$

so that

$$\tilde{\boldsymbol{A}}(\boldsymbol{x}) = \boldsymbol{A} + \nabla \Lambda(\boldsymbol{x}) \qquad (2.639)$$

with

$$\Lambda = -\frac{1}{2} B x y \qquad (2.640)$$

$$\rightarrow \qquad \nabla \Lambda = -\frac{1}{2} B(y, x, 0)$$

The Lagrangian is

$$L = \frac{1}{2} M \dot{\boldsymbol{x}}^2 + \frac{e}{c} \dot{\boldsymbol{x}} \cdot \boldsymbol{A} \qquad (2.640a)$$

$$\rightarrow \qquad \boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = M \dot{\boldsymbol{x}} + \frac{e}{c} \boldsymbol{A} \qquad \dot{\boldsymbol{x}} = \frac{1}{M}\left(\boldsymbol{p} - \frac{e}{c} \boldsymbol{A}\right)$$

$$\therefore \qquad \dot{\boldsymbol{x}} \cdot \boldsymbol{p} = M \dot{\boldsymbol{x}}^2 + \frac{e}{c} \dot{\boldsymbol{x}} \cdot \boldsymbol{A}$$

$$\rightarrow \quad L = \dot{\boldsymbol{x}} \cdot \boldsymbol{p} - \frac{1}{2} M \dot{\boldsymbol{x}}^2$$

$$= \dot{\boldsymbol{x}} \cdot \boldsymbol{p} - \frac{1}{2M} \left(\boldsymbol{p} - \frac{e}{c} \boldsymbol{A} \right)^2 \qquad (2.641a)$$

(2.633) thus takes the canonical form

$$\mathcal{A}[\boldsymbol{x}, \boldsymbol{p}] = \int_{t_a}^{t_b} dt \left[\dot{\boldsymbol{x}} \cdot \boldsymbol{p} - \frac{1}{2M} \left(\boldsymbol{p} - \frac{e}{c} \boldsymbol{A} \right)^2 \right] \qquad (2.641)$$

The Hamiltonian is

$$H = \dot{\boldsymbol{x}} \cdot \boldsymbol{p} - L = \frac{1}{2M} \left(\boldsymbol{p} - \frac{e}{c} \boldsymbol{A} \right)^2 \qquad (2.641a)$$

which can be obtained from the free particle Hamiltonian by the minimal substitution

$$\boldsymbol{p} \rightarrow \boldsymbol{P} = \boldsymbol{p} - \frac{e}{c} \boldsymbol{A} \qquad (2.642)$$

(2.641) thus takes the canonical form

$$\mathcal{A}[\boldsymbol{x}, \boldsymbol{p}] = \int_{t_a}^{t_b} dt \left(\dot{\boldsymbol{x}} \cdot \boldsymbol{p} - H \right) \qquad (2.643)$$

For the vector potential (2.638), we have

$$H = \frac{1}{2M} \left[\boldsymbol{p} - \frac{eB}{2c} \left(-y \hat{\boldsymbol{x}} + x \hat{\boldsymbol{y}} \right) \right]^2$$

$$= \frac{\boldsymbol{p}^2}{2M} - \frac{eB}{2Mc} (x p_y - y p_x) + \frac{1}{8M} \left(\frac{eB}{c} \right)^2 (x^2 + y^2)$$

$$= \frac{\boldsymbol{p}_{//}^2}{2M} + \frac{1}{8} M \omega_L^2 \boldsymbol{\rho}^2 + \frac{\boldsymbol{p}_z^2}{2M} - \frac{1}{2} \frac{e}{|e|} \omega_L l_z \qquad (2.644)$$

where

$$\boldsymbol{p}_{//} = (p_x, p_y, 0) \qquad \boldsymbol{\rho} = (x, y, 0) \qquad \boldsymbol{l} = \boldsymbol{x} \times \boldsymbol{p}$$

$$l_z = x p_y - y p_x \qquad (2.645)$$

and

$$\omega_L = \frac{|e| B}{M c} \qquad (2.646)$$

is the cyclotron, or Landau, frequency.   For electrons, we can use the Bohr magneton

$$\mu_B = \frac{|e| \hbar}{M c} \qquad \text{(for electrons)} \qquad (2.647)$$

to write

$$\omega_L = \frac{\mu_B B}{\hbar} \qquad \text{(for electrons)} \qquad (2.648)$$

Note that the first two terms in (2.644) describe a harmonic oscillator in the xy-plane of (magnetic) frequency

$$\omega_B = \frac{1}{2} \omega_L \qquad (2.649)$$

On the other hand, if (2.634) was used, we would have

$$H = \frac{1}{2M} \left[\boldsymbol{p} - \frac{eB}{c} x \hat{\boldsymbol{y}} \right]^2$$

$$= \frac{\boldsymbol{p}^2}{2M} - \frac{eB}{Mc} x p_y + \frac{1}{2M} \left(\frac{eB}{c} \right)^2 x^2$$

$$= \frac{\boldsymbol{p}^2}{2M} + \frac{1}{2} M \omega_L^2 x^2 - \omega_L x p_y \tag{2.650}$$

which consists of an oscillator of frequency ω_L along the x-axis and free motions in the yz-plane.

The time-sliced form of the canonical action (2.641) then becomes

$$\mathcal{A}_e^N = \sum_{n=1}^{N+1} \left\{ \boldsymbol{p}_n \cdot (\boldsymbol{x}_n - \boldsymbol{x}_{n-1}) - \frac{\epsilon}{2M} \left[p_{x\,n}^2 + \left(p_{y\,n} - \frac{e}{c} B x_n \right)^2 + p_{z\,n}^2 \right] \right\} \tag{2.651}$$

with the corresponding evolution amplitude

$$(\boldsymbol{x}_b t_b \mid \boldsymbol{x}_a t_a) = \left(\prod_{n=1}^{N} \int d\boldsymbol{x}_n \right) \left(\prod_{k=1}^{N+1} \int \frac{d\boldsymbol{p}_k}{(2\pi\hbar)^3} \right) \exp\left(\frac{i}{\hbar} \mathcal{A}_e^N \right) \tag{2.652}$$

2.18.2. Gauge Properties

The evolution amplitude is not gauge invariant. If we switch to another gauge with vector potential

$$\boldsymbol{A}'(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}) + \nabla \Lambda \tag{2.654}$$

then the change in the action is given by (2.633) as

$$\begin{aligned} \Delta \mathcal{A} = \Delta \mathcal{A}_{\text{mag}} &= \int_{t_a}^{t_b} dt \, \frac{e}{c} \dot{\boldsymbol{x}} \cdot \nabla \Lambda \\ &= \frac{e}{c} \int_{\boldsymbol{x}_a}^{\boldsymbol{x}_b} d\boldsymbol{x} \cdot \nabla \Lambda \; = \frac{e}{c} \int_{\Lambda(\boldsymbol{x}_a)}^{\Lambda(\boldsymbol{x}_b)} d\Lambda \\ &= \frac{e}{c} \left[\Lambda(\boldsymbol{x}_b) - \Lambda(\boldsymbol{x}_a) \right] \end{aligned} \tag{2.655}$$

The evolution amplitude thus gains a phase factor

$$\begin{aligned} (\boldsymbol{x}_b t_b \mid \boldsymbol{x}_a t_a)_{\boldsymbol{A}'} &= e^{i \Delta \mathcal{A} / \hbar} (\boldsymbol{x}_b t_b \mid \boldsymbol{x}_a t_a)_{\boldsymbol{A}} \\ &= \exp\left\{ i \frac{e}{\hbar c} \left[\Lambda(\boldsymbol{x}_b) - \Lambda(\boldsymbol{x}_a) \right] \right\} (\boldsymbol{x}_b t_b \mid \boldsymbol{x}_a t_a)_{\boldsymbol{A}} \\ &= \exp\left[i \frac{e}{\hbar c} \Lambda(\boldsymbol{x}_b) \right] (\boldsymbol{x}_b t_b \mid \boldsymbol{x}_a t_a)_{\boldsymbol{A}} \exp\left[-i \frac{e}{\hbar c} \Lambda(\boldsymbol{x}_a) \right] \end{aligned} \tag{2.656}$$

This leaves all observables unchanged since they are given by the evolution probability, which equals to the absolute square of the evolution amplitude.

2.18.3. Time-Sliced Path Integration

Since the action $\mathcal{A}_e^N$ in (2.651) contains the variables y_n & z_n only in the term $\boldsymbol{p}_n \cdot \boldsymbol{x}_n$, we can carry out their integrations in (2.652) immediately & get for each n

$$\begin{aligned} \mathcal{P}_n &= \int dy_n \int dz_n \, \exp\left\{ \frac{i}{\hbar} [(p_{y\,n} - p_{y\,n+1}) y_n + (p_{z\,n} - p_{z\,n+1}) z_n) \right] \\ &= (2\pi\hbar)^2 \, \delta(p_{y\,n} - p_{y\,n+1}) \, \delta(p_{z\,n} - p_{z\,n+1}) \\ &= (2\pi\hbar)^2 \, \delta(\boldsymbol{p}'_{n+1} - \boldsymbol{p}'_n) \end{aligned}$$

where

$$\boldsymbol{p}'_n = (0, p_{y\,n}, p_{z\,n}) \tag{2.657a}$$

Since

$$\sum_{n=1}^{N+1} \left\{ \boldsymbol{p}'_n \cdot (\boldsymbol{x}_n - \boldsymbol{x}_{n-1}) = \boldsymbol{p}'_{N+1} \cdot \boldsymbol{x}_{N+1} + \sum_{n=1}^{N} (\boldsymbol{p}'_n - \boldsymbol{p}'_{n+1}) \cdot \boldsymbol{x}_n - \boldsymbol{p}'_1 \cdot \boldsymbol{x}_0 \right.$$

the evolution amplitude (2.652) contains a prefactor

$$\mathcal{P} = \prod_{n=1}^{N} \int d\, y_n \int d\, z_n \exp\left[\frac{i}{\hbar} \sum_{n=1}^{N+1} \boldsymbol{p}'_n \cdot (\boldsymbol{x}_n - \boldsymbol{x}_{n-1})\right]$$
$$= \exp\left[\frac{i}{\hbar} (\boldsymbol{p}'_{N+1} \cdot \boldsymbol{x}_{N+1} - \boldsymbol{p}'_1 \cdot \boldsymbol{x}_0)\right] \prod_{n=1}^{N} \mathcal{P}_n$$
$$= \exp\left[\frac{i}{\hbar} (\boldsymbol{p}'_{N+1} \cdot \boldsymbol{x}_{N+1} - \boldsymbol{p}'_1 \cdot \boldsymbol{x}_0)\right] \prod_{n=1}^{N} (2\pi\hbar)^2\, \delta(\boldsymbol{p}'_{n+1} - \boldsymbol{p}'_n)$$
$$= \exp\left[\frac{i}{\hbar} \boldsymbol{p}'_1 \cdot (\boldsymbol{x}_b - \boldsymbol{x}_a)\right] (2\pi\hbar)^{2N}\, \delta(\boldsymbol{p}'_{N+1} - \boldsymbol{p}'_N) \ldots \delta(\boldsymbol{p}'_2 - \boldsymbol{p}'_1) \quad (2.657)$$

where we've used the fact that the δ-functions force all $\boldsymbol{p}'_n$ to be equal.

The $\boldsymbol{p}'$ integrals in (2.652) then give

$$Q = \left(\prod_{k=1}^{N+1} \int \frac{d\,\boldsymbol{p}'_k}{(2\pi\hbar)^2}\right) \mathcal{P} \exp\left\{-\frac{i\epsilon}{2M\hbar} \sum_{n=1}^{N+1} \left[\left(p_{y\,n} - \frac{e}{c} B\, x_n\right)^2 + p_{z\,n}^2\right]\right\}$$
$$= \int \frac{d\,\boldsymbol{p}'_1}{(2\pi\hbar)^2} \exp\left\{\frac{i}{\hbar} \boldsymbol{p}'_1 \cdot (\boldsymbol{x}_b - \boldsymbol{x}_a) - \frac{i\epsilon}{2M\hbar} \sum_{n=1}^{N+1} \left[\left(p_{y\,1} - \frac{e}{c} B\, x_n\right)^2 + p_{z\,1}^2\right]\right\}$$
$$= \int \frac{d\,\boldsymbol{p}'}{(2\pi\hbar)^2} \exp \frac{i}{\hbar} \left\{ p_y (y_b - y_a) + p_z (z_b - z_a) - \frac{\epsilon}{2M} \sum_{n=1}^{N+1} \left[\left(p_y - \frac{e}{c} B\, x_n\right)^2 + p_z^2\right] \right\}$$

where we've set

$$\boldsymbol{p}' = (0, p_y, p_z) = \boldsymbol{p}'_1 = (0, p_{y\,1}, p_{z\,1})$$

Using

$$(N+1)\,\epsilon = t_b - t_a = \tau \qquad \Delta\, y = y_b - y_a \qquad \Delta\, z = z_b - z_a \qquad (2.657b)$$

we have

$$Q = \int \frac{d\,p_y\, d\,p_z}{(2\pi\hbar)^2} \exp \frac{i}{\hbar} \left\{ p_y\, \Delta\, y + p_z\, \Delta\, z - \frac{p_z^2}{2M} \tau - \frac{\epsilon}{2M} \sum_{n=1}^{N+1} \left(p_y - \frac{e}{c} B\, x_n\right)^2 \right\}$$

The evolution amplitude thus becomes

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \left(\prod_{n=1}^{N} \int d\, x_n\right) \left(\prod_{k=1}^{N+1} \int \frac{d\, p_{x\,k}}{2\pi\hbar}\right) Q \times \exp\left\{\frac{i}{\hbar} \sum_{n=1}^{N+1} \left[p_{x\,n} (x_n - x_{n-1}) - \frac{\epsilon}{2M} p_{x\,n}^2\right]\right\}$$
$$= \int \frac{d\,p_y\, d\,p_z}{(2\pi\hbar)^2} \left(\prod_{n=1}^{N} \int d\, x_n\right) \left(\prod_{k=1}^{N+1} \int \frac{d\, p_{x\,k}}{2\pi\hbar}\right) \times \exp\left[\frac{i}{\hbar}\left(p_y\, \Delta\, y + p_z\, \Delta\, z - \frac{p_z^2}{2M} \tau\right)\right] \exp\left(\frac{i}{\hbar} \mathcal{A}_x^N\right) \quad (2.658)$$

where

$$\mathcal{A}_x^N = \sum_{n=1}^{N+1} \left\{ p_{x\,n} (x_n - x_{n-1}) - \frac{\epsilon}{2M} \left[p_{x\,n}^2 + \left(p_y - \frac{e}{c} B\, x_n\right)^2 \right] \right\} \qquad (2.659)$$

(2.659) depicts a 1-D harmonic oscillator with frequency ω_B and centered at

$$x_0 = \frac{p_y\, c}{e\, B} = \frac{p_y}{M\, \omega_L} \qquad (2.660)$$

The path integral over $x(t)$ is given by (2.173) as

$$(x_b\, t_b \mid x_a\, t_a)_{x_0} = \sqrt{\frac{M\,\omega_L}{2\,\pi\, i\,\hbar \sin \omega_L\, \tau}}\ \exp\!\left(\frac{i}{\hbar}\,\mathcal{A}^{\mathrm{cl}}_{x_0}\right) \tag{2.661}$$

where

$$\mathcal{A}^{\mathrm{cl}}_{x_0} = \frac{M\,\omega_L}{2 \sin \omega_L\, \tau}\left\{\left[(x_b - x_0)^2 + (x_a - x_0)^2\right]\cos \omega_L\, \tau - 2\,(x_b - x_0)\,(x_a - x_0)\right\} \tag{2.661a}$$

Using

$$\int \frac{d\,p_z}{2\,\pi\,\hbar}\exp\!\left[\frac{i}{\hbar}\left(p_z\,\Delta\, z - \frac{p_z^2}{2\,M}\,\tau\right)\right] = \sqrt{\frac{M}{2\,\pi\, i\,\hbar\,\tau}}\ \exp\!\left[i\,\frac{M}{2\,\hbar}\,\frac{(\Delta\, z)^2}{\tau}\right]$$

(2.658) becomes

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \sqrt{\frac{M}{2\,\pi\, i\,\hbar\,\tau}}\ \exp\!\left[i\,\frac{M}{2\,\hbar}\,\frac{(\Delta\, z)^2}{\tau}\right]\left(\boldsymbol{x}_b^{\perp}\, t_b \mid \boldsymbol{x}_a^{\perp}\, t_a\right) \tag{2.662}$$

where [see “2.18_Code.nb”]

$$\begin{aligned}
\left(\boldsymbol{x}_b^{\perp}\, t_b \,\middle|\, \boldsymbol{x}_a^{\perp}\, t_a\right) &= \int \frac{d\,p_y}{2\,\pi\,\hbar}\exp\!\left(\frac{i}{\hbar}\,p_y\,\Delta\, y\right)(x_b\, t_b \mid x_a\, t_a)_{x_0} \\
&= \frac{M\,\omega_L}{2\,\pi\,\hbar}\int d\, x_0 \exp\!\left(\frac{i}{\hbar}\,M\,\omega_L\, x_0\,\Delta\, y\right)(x_b\, t_b \mid x_a\, t_a)_{x_0} \qquad (2.663) \\
&= \left(\frac{M\,\omega_L}{2\,\pi\,\hbar}\right)^{3/2}\frac{1}{\sqrt{i \sin \omega_L\, \tau}}\int d\, x_0 \exp\!\left(\frac{i}{\hbar}\left[M\,\omega_L\, x_0\,\Delta\, y\right.\right. \\
&\qquad + \frac{M\,\omega_L}{2 \sin \omega_L\, \tau}\left\{\left[(x_b - x_0)^2 + (x_a - x_0)^2\right]\cos(\omega_L\, \tau) \right. \\
&\qquad\qquad \left.\left.\left. - 2\,(x_b - x_0)\,(x_a - x_0)\right\}\right]\right) \\
&= \left(\frac{M\,\omega_L}{2\,\pi\,\hbar}\right)^{3/2}\frac{1}{\sqrt{i \sin \omega_L\, \tau}}\exp\left\{\frac{i}{\hbar}\,\frac{M\,\omega_L}{2 \sin(\omega_L\, \tau)}\left[\left(x_b^2 + x_a^2\right)\cos(\omega_L\, \tau) - 2\, x_b\, x_a\right]\right\} \\
&\qquad \times \int d\, x_0 \exp\left\{-\frac{i}{\hbar}\,M\,\omega_L \tan\!\left(\frac{\omega_L\, \tau}{2}\right)\left[x_0^2 - \left(2\,\bar{x} + \frac{\Delta\, y}{\tan\!\left(\frac{\omega_L\, \tau}{2}\right)}\right)x_0\right]\right\}
\end{aligned}$$

where

$$\bar{x} = \frac{1}{2}\,(x_b + x_a) \tag{2.663a}$$

The x_0 integral is a Gaussian that evaluates to

$$\sqrt{\frac{\pi\,\hbar}{i\,M\,\omega_L \tan\!\left(\frac{\omega_L\, \tau}{2}\right)}}\ \exp\!\left[\frac{i}{\hbar}\,M\,\omega_L \tan\!\left(\frac{\omega_L\, \tau}{2}\right)\left(\bar{x} + \frac{\Delta\, y}{2 \tan\!\left(\frac{\omega_L\, \tau}{2}\right)}\right)^2\right] \tag{2.665a}$$

so that [see “2.18_Code.nb”]

$$\left(\boldsymbol{x}_b^{\perp}\, t_b \mid \boldsymbol{x}_a^{\perp}\, t_a\right) = \left(\frac{M\,\omega_L}{4\,\pi\,\hbar\, i}\right) \frac{1}{\sin\left(\frac{\omega_L\,\tau}{2}\right)} \times \exp\left(i\,\frac{M\,\omega_L}{4\,\hbar} \left\{ \left[(\Delta x)^2 + (\Delta y)^2\right] \cot\left(\frac{\omega_L\,\tau}{2}\right) + 4\,\bar{x}\,\Delta y \right\} \right) \tag{2.665b}$$

Using

$$2\,\bar{x}\,\Delta y = (x_b + x_a)\,(y_b - y_a) \;= (x_b\, y_b - x_a\, y_a) + (x_a\, y_b - x_b\, y_a)$$

(2.662) becomes

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \left(\frac{M}{2\,\pi\, i\,\hbar\,\tau}\right)^{3/2} \frac{\omega_L\,\tau}{2\,\sin\left(\frac{\omega_L\,\tau}{2}\right)} \exp\left[\frac{i}{\hbar}\left(\mathcal{A}_{\rm cl} + \mathcal{A}_{\rm sf}\right)\right] \tag{2.666}$$

where

$$\mathcal{A}_{\rm cl} = \frac{M}{2}\,\frac{(\Delta z)^2}{\tau} + \frac{M\,\omega_L}{4}\left[(\Delta x)^2 + (\Delta y)^2\right] \cot\left(\frac{\omega_L\,\tau}{2}\right) + \frac{M\,\omega_L}{2}\,(x_a\, y_b - x_b\, y_a) \tag{2.667}$$

and the surface term

$$\mathcal{A}_{\rm sf} = \frac{M\,\omega_L}{2}\,(x_b\, y_b - x_a\, y_a) \;= \frac{e\,B}{2\,c}\, x\, y \Big|_a^b \qquad 2.668)$$

2.18.4. Classical Action

According to (2.655), $\Delta\,\mathcal{A}_{\rm cl} = \frac{e}{c}\left[\Lambda(\boldsymbol{x}_b) - \Lambda(\boldsymbol{x}_a)\right]$ if we switch to another gauge with $\boldsymbol{A}'(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}) + \nabla\Lambda$. By (2.640), the axially symmetric gauge (2.638) corresponds to

$$\Lambda = -\frac{1}{2}\, B\, x\, y$$

so that

$$\Delta\,\mathcal{A}_{\rm cl} = -\frac{e\,B}{2\,c}\,(x_b\, y_b - x_a\, y_a) = -\mathcal{A}_{\rm sf}$$

Therefore, the surface term is absent in the symmetric gauge. (2.666) then takes the form

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = F(\tau)\, \exp\left(\frac{i}{\hbar}\,\mathcal{A}_{\rm cl}\right)$$

as expected for a quadratic Hamiltonian (2.644), where $\mathcal{A}_{\rm cl}$ is the classical action

$$\mathcal{A}_{\rm cl} = \int_{t_a}^{t_b} d\,t\, L$$

Using (2.640a), the orthogonal (perpendicular to $\boldsymbol{B} = B\,\hat{\boldsymbol{z}}$) part of $\mathcal{A}_{\rm cl}$ is

$$\mathcal{A}_{\rm cl}^{\perp} = \int_{t_a}^{t_b} d\,t \left(\frac{1}{2}\, M\,\dot{\boldsymbol{x}}^2 + \frac{e}{c}\,\dot{\boldsymbol{x}}\cdot\boldsymbol{A}\right)_{x,\,y}$$

$$= \int_{t_a}^{t_b} d\,t \left[\frac{1}{2}\, M\left(\dot{x}^2 + \dot{y}^2\right) + \frac{e\,B}{2\,c}\,(-\dot{x}\, y + \dot{y}\, x)\right] \qquad \text{[(2.638) used.]}$$

Using

$$\dot{x}^2 = \frac{d}{d\,t}\,(x\,\dot{x}) - x\,\ddot{x}$$

we have

$$\mathcal{A}_{\rm cl}^{\perp} = \frac{1}{2} M\,(x\,\dot{x} + y\,\dot{y})\Big|_{t_a}^{t_b} \qquad (2.669)$$
$$+\frac{1}{2} M \int_{t_a}^{t_b} d\,t\,[-(x\,\ddot{x} + y\,\ddot{y}) + \omega_L\,(-\dot{x}\,y + \dot{y}\,x)]$$

For the Lagrangian

$$L^{\perp} = \frac{1}{2} M\left(\dot{x}^2 + \dot{y}^2\right) + \frac{e\,B}{2\,c}\,(-\dot{x}\,y + \dot{y}\,x) \qquad (2.669a)$$

we have

$$\frac{\partial L^{\perp}}{\partial \dot{x}} = M\,\dot{x} - \frac{e\,B}{2\,c}\,y \qquad \frac{\partial L^{\perp}}{\partial \dot{y}} = M\,\dot{y} + \frac{e\,B}{2\,c}\,x$$
$$\frac{\partial L^{\perp}}{\partial x} = \frac{e\,B}{2\,c}\,\dot{y} \qquad \frac{\partial L^{\perp}}{\partial y} = -\frac{e\,B}{2\,c}\,\dot{x}$$
$$\rightarrow \quad \ddot{x} = \omega_L\,\dot{y} \qquad \ddot{y} = -\omega_L\,\dot{x} \qquad (2.670)$$

Hence,

$$\mathcal{A}_{\rm cl}^{\perp} = \frac{1}{2} M\,(x\,\dot{x} + y\,\dot{y})\Big|_{t_a}^{t_b}$$
$$= \frac{1}{2} M\,(x_b\,\dot{x}_b - x_a\,\dot{x}_a + y_b\,\dot{y}_b - y_a\,\dot{y}_a) \qquad (2.671)$$

(2.670) can be written as

$$\dddot{x} = \omega_L\,\ddot{y} = -\omega_L^2\,\dot{x} \qquad \dddot{y} = -\omega_L\,\ddot{x} = -\omega_L^2\,\dot{y} \qquad (2.672)$$

Since (2.672) are 3rd order equations, there are 3 pairs of integration constants. Two of these pairs can be fixed by the B.C.

$$x(t_b) = x_b \qquad x(t_a) = x_a$$
$$y(t_b) = y_b \qquad y(t_a) = y_a$$

It is easily proved by direct calculation that the position B.C.'s are satisfied by the solutions

$$x - x_0 = \frac{1}{\sin\omega_L\,\tau}[\,(x_b - x_0)\sin\omega_L(t - t_a) - (x_a - x_0)\sin\omega_L(t - t_b)\,] \qquad (2.673)$$
$$y - y_0 = \frac{1}{\sin\omega_L\,\tau}[\,(y_b - y_0)\sin\omega_L(t - t_a) - (y_a - y_0)\sin\omega_L(t - t_b)\,] \qquad (2.674)$$

where the 3rd pair of constants are denoted by x_0 & y_0.

Using

$$\ddot{x} = -\omega_L^2\,(x - x_0) \qquad \ddot{y} = -\omega_L^2\,(y - y_0)$$

while substituting (2.673-4) into (2.670), we get, at $t = t_b$,

$$-(x_b - x_0)\sin\omega_L\,\tau = (y_b - y_0)\cos\omega_L\,\tau - (y_a - y_0) \qquad (2.674a)$$
$$(y_b - y_0)\sin\omega_L\,\tau = (x_b - x_0)\cos\omega_L\,\tau - (x_a - x_0) \qquad (2.674b)$$

The solutions are [see "2.18_Code.nb"]

$$x_0 = \overline{x} + \frac{1}{2}\Delta\,y\cot\left(\frac{\omega_L\,\tau}{2}\right) \qquad (2.675)$$
$$y_0 = \overline{y} - \frac{1}{2}\Delta\,x\cot\left(\frac{\omega_L\,\tau}{2}\right) \qquad (2.676)$$

where

$$\overline{x} = \frac{1}{2}(x_b + x_a) \quad \overline{y} = \frac{1}{2}(y_b + y_a) \quad \tau = t_b - t_a$$

$$\Delta x = x_b - x_a \qquad \Delta y = y_b - y_a$$

Using (2.673-4), we calculate $\mathcal{A}_{\text{cl}}^{\perp}$ in (2.671) as follows

$$x_b\,\dot{x}_b = \frac{\omega_L}{\sin(\omega_L\,\tau)}\,x_b\left[x_0 - x_a + (x_b - x_0)\cos(\omega_L\,\tau)\right] \tag{2.677}$$

$$x_a\,\dot{x}_a = -\frac{\omega_L}{\sin(\omega_L\,\tau)}\,x_a\left[x_0 - x_b + (x_a - x_0)\cos(\omega_L\,\tau)\right] \tag{2.678}$$

$$\to \quad x_b\,\dot{x}_b - x_a\,\dot{x}_a = \frac{\omega_L}{\sin(\omega_L\,\tau)}\left[\left(x_b^2 + x_a^2\right)\cos(\omega_L\,\tau) - 2\,x_b\,x_a + 4\,x_0\,\bar{x}\sin^2\left(\frac{\omega_L\,\tau}{2}\right)\right]$$

$$= \frac{1}{2}\,\omega_L\left[(\Delta\,x)^2\cot\left(\frac{\omega_L\,\tau}{2}\right) + 2\,\bar{x}\,\Delta\,y\right] \tag{2.679}$$

Similarly,

$$y_b\,\dot{y}_b - y_a\,\dot{y}_a = \frac{\omega_L}{\sin(\omega_L\,\tau)}\left[\left(y_b^2 + y_a^2\right)\cos(\omega_L\,\tau) - 2\,y_b\,y_a + 4\,y_0\,\bar{y}\sin^2\left(\frac{\omega_L\,\tau}{2}\right)\right]$$

$$= \frac{1}{2}\,\omega_L\left[(\Delta\,y)^2\cot\left(\frac{\omega_L\,\tau}{2}\right) - 2\,\bar{y}\,\Delta\,x\right] \tag{2.680}$$

(2.671) thus becomes

$$\mathcal{A}_{\text{cl}}^{\perp} = \frac{1}{4}\,M\,\omega_L\left\{\left[(\Delta\,x)^2 + (\Delta\,y)^2\right]\cot\left(\frac{\omega_L\,\tau}{2}\right) + 2\,(\bar{x}\,\Delta\,y - \bar{y}\,\Delta\,x)\right\}$$

$$= \frac{1}{4}\,M\,\omega_L\left\{\left[(\Delta\,x)^2 + (\Delta\,y)^2\right]\cot\left(\frac{\omega_L\,\tau}{2}\right) + 2\,(x_a\,y_b - x_b\,y_a)\right\} \tag{2.681}$$

which is indee the orthogonal part of $\mathcal{A}_{\text{cl}}$ in (2.667).

2.18.5. Translational Invariance

We now consider the effect on the evolution amplitude by a translation

$$\boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{d} \tag{2.683}$$

First of all,

$$\Delta\,\boldsymbol{x} = \boldsymbol{x}_b - \boldsymbol{x}_a \;\to\; \Delta\,\boldsymbol{x}' = \boldsymbol{x}_b + \boldsymbol{d} - (\boldsymbol{x}_a + \boldsymbol{d}) = \Delta\,\boldsymbol{x}$$

so that the change of $\mathcal{A}_{\text{cl}}$ in (2.667) is

$$\begin{aligned}
\delta\,\mathcal{A}_{\text{cl}} &= \mathcal{A}_{\text{cl}}' - \mathcal{A}_{\text{cl}} \\
&= \frac{M\,\omega_L}{2}\left[(x_a + d_x)\,(y_b + d_y) - (x_b + d_x)\,(y_a + d_y) - (x_a\,y_b - x_b\,y_a)\right] \\
&= \frac{M\,\omega_L}{2}\,(d_x\,\Delta\,y - d_y\,\Delta\,x) \\
&= \frac{M\,\omega_L}{2}\,(\boldsymbol{d}\times\Delta\,\boldsymbol{x})_z \\
&= \frac{M\,\omega_L}{2}\,(\boldsymbol{d}\times\boldsymbol{x})_z\Big|_{t_a}^{t_b}
\end{aligned} \tag{2.684}$$

The evolution amplitude (2.666) thus transforms as

$$(\boldsymbol{x}_b\,t_b \mid \boldsymbol{x}_a\,t_a) \;\to\; (\boldsymbol{x}_b\,t_b \mid \boldsymbol{x}_a\,t_a)\,e^{i\,\delta\mathcal{A}_{\text{cl}}/\hbar}$$

which can be written as a gauge transformation with [see (2.655-6)]

$$(\boldsymbol{x}_b\,t_b \mid \boldsymbol{x}_a\,t_a) \;\to\; \exp\left[i\,\frac{e}{\hbar\,c}\,\Lambda(\boldsymbol{x}_b)\right](\boldsymbol{x}_b\,t_b \mid \boldsymbol{x}_a\,t_a)\exp\left[-i\,\frac{e}{\hbar\,c}\,\Lambda(\boldsymbol{x}_a)\right]$$

where

$$\Lambda(\boldsymbol{x}) = \frac{c\, M\, \omega_L}{2\, e} (\, \boldsymbol{d} \times \boldsymbol{x}\,)_z \tag{2.686}$$

Since observables are gauge-invariant, this means all observables of the system are translationally invariant.