2.19. Charged Particle in Magnetic Field plus Harmonic Potential

Consider the 2-D Hamiltonian that is the orthogonal part of (2.644) in which ω_B in the harmonic potential is replaced with ω :

$$H = \frac{\mathbf{p}^2}{2M} + \frac{1}{2} M \omega^2 \mathbf{x}^2 - \omega_B l_z \qquad \mathbf{x} = (x, y) \qquad (2.687)$$

where

$$l_z = (\mathbf{x} \times \mathbf{p})_z \qquad \omega_B = \frac{1}{2}\omega_L = \frac{eB}{2Mc} \qquad \mathbf{A} = \frac{1}{2} B(-y, x) \quad (2.687a)$$

The associated Euclidean action is

$$\mathcal{A}_e[\mathbf{x}, \mathbf{p}] \equiv -i\mathcal{A} = -i \int_{t_a}^{t_b} dt\, L \qquad \left(e^{i\mathcal{A}/\hbar} = e^{-\mathcal{A}_e/\hbar} \right)$$

$$= -i \int_{t_a}^{t_b} dt \left(\dot{\mathbf{x}} \cdot \mathbf{p} - H \right)$$

$$= \int_{\tau_a}^{\tau_b} d\tau \left(-i\mathbf{x}' \cdot \mathbf{p} + H \right) \qquad \tau = it \qquad \mathbf{x}' = \frac{d\mathbf{x}}{d\tau} \qquad (2.688)$$

$$\equiv \int_{\tau_a}^{\tau_b} d\tau\, L_e$$

$\rightarrow \qquad L_e = -L$

For the particle in magnetic field,

$$L = \frac{1}{2} M \dot{\mathbf{x}}^2 + \frac{e}{c} \dot{\mathbf{x}} \cdot \mathbf{A} \qquad \rightarrow \qquad \mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = M\dot{\mathbf{x}} + \frac{e}{c}\mathbf{A}$$

$$\& \qquad H = \dot{\mathbf{x}} \cdot \mathbf{p} - L = \frac{1}{2M}\left(\mathbf{p} - \frac{e}{c}\mathbf{A} \right)^2$$

The Euclidean version is

$$L_e = -L = \frac{1}{2} M \mathbf{x}'^2 - i\frac{e}{c}\mathbf{x}' \cdot \mathbf{A} \qquad \dot{\mathbf{x}} = i\mathbf{x}'$$

$$\rightarrow \qquad \mathbf{p}_e = \frac{\partial L_e}{\partial \mathbf{x}'} = M\mathbf{x}' - i\frac{e}{c}\mathbf{A} = -i\left(M\dot{\mathbf{x}} + \frac{e}{c}\mathbf{A} \right) = -i\mathbf{p}$$

$$\mathbf{x}' = \frac{1}{M}\left(\mathbf{p}_e + i\frac{e}{c}\mathbf{A} \right)$$

$$\therefore \qquad H_e = \mathbf{x}' \cdot \mathbf{p}_e - L_e = \frac{1}{2M}\left(\mathbf{p}_e + i\frac{e}{c}\mathbf{A} \right)^2$$

$$H = i\mathbf{x}' \cdot \mathbf{p} + L_e = -\mathbf{x}' \cdot \mathbf{p}_e + L_e = -H_e$$

Using

$$\mathbf{p} = iM\mathbf{x}' + \frac{e}{c}\mathbf{A} \qquad \mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{x} = \frac{1}{2} B(-y, x)$$

we have

$$\frac{e^2}{c^2}\mathbf{A}^2 = \frac{e^2 B^2}{4c^2}\mathbf{x}^2 = M^2 \omega_B^2 \mathbf{x}^2$$

$$\mathbf{p}^2 = -M^2 \mathbf{x}'^2 + 2i\frac{eM}{c}\mathbf{x}' \cdot \mathbf{A} + M^2 \omega_B^2 \mathbf{x}^2$$

$$\left(\mathbf{x} \times \frac{e}{c}\mathbf{A} \right)_z = \frac{eB}{2c}\mathbf{x}^2 = M\omega_B \mathbf{x}^2$$

$$l_z = (\mathbf{x} \times \mathbf{p})_z = iM(\mathbf{x} \times \mathbf{x}')_z + M\omega_B \mathbf{x}^2$$

(2.687) can therefore be written as

$$H = -\frac{1}{2} M \boldsymbol{x}'^2 + i \frac{e}{c} \boldsymbol{x}' \cdot \boldsymbol{A} + \frac{1}{2} M \left(\omega^2 - \omega_B^2\right) \boldsymbol{x}^2 - i M \omega_B (\boldsymbol{x} \times \boldsymbol{x}')_z$$

Using

$$-i \boldsymbol{x}' \cdot \boldsymbol{p} = M \boldsymbol{x}' - i \frac{e}{c} \boldsymbol{x}' \cdot \boldsymbol{A}$$

the action (2.688) takes the Lagrangian form

$$\begin{aligned} \mathcal{A}_e[\boldsymbol{x}] &= \int_0^{\beta\hbar} d\tau \left(-i \boldsymbol{x}' \cdot \boldsymbol{p} + H \right) \\ &= \int_0^{\beta\hbar} d\tau \left[\frac{1}{2} M \boldsymbol{x}'^2 + \frac{1}{2} M \left(\omega^2 - \omega_B^2\right) \boldsymbol{x}^2 - i M \omega_B (\boldsymbol{x} \times \boldsymbol{x}')_z \right] \qquad (2.689) \end{aligned}$$

Note that the harmonic potential becomes a repulsive barrier if $\omega^2 - \omega_B^2 < 0$, the system is stable only if $\omega \geq \omega_B$.

(2.689) can be written as

$$\begin{aligned} \mathcal{A}_e[\boldsymbol{x}] &= \frac{1}{2} M \int_0^{\beta\hbar} d\tau \left[(\boldsymbol{x} \cdot \boldsymbol{x}')' - \boldsymbol{x} \cdot \boldsymbol{x}'' + \left(\omega^2 - \omega_B^2\right) \boldsymbol{x}^2 - 2 i \omega_B (\boldsymbol{x} \times \boldsymbol{x}')_z \right] \qquad (2.690a) \\ &= \frac{1}{2} M \int_0^{\beta\hbar} d\tau \left[(\boldsymbol{x} \cdot \boldsymbol{x}')' + \boldsymbol{x}^T \boldsymbol{D}_{\omega,B}(\tau) \boldsymbol{x} \right] \qquad (2.690) \\ &= \frac{1}{2} M \boldsymbol{x} \cdot \boldsymbol{x}' \Big|_0^{\beta\hbar} + \frac{1}{2} M \int_0^{\beta\hbar} d\tau\, \boldsymbol{x}^T(\tau) \boldsymbol{D}_{\omega,B}(\tau) \boldsymbol{x}(\tau) \end{aligned}$$

where

$$(\boldsymbol{x} \times \boldsymbol{x}')_z = x y' - y x'$$

$$\boldsymbol{D}_{\omega,B}(\tau) = \begin{pmatrix} -\partial_\tau^2 + \omega^2 - \omega_B^2 & -2 i \omega_B \partial_\tau \\ 2 i \omega_B \partial_\tau & -\partial_\tau^2 + \omega^2 - \omega_B^2 \end{pmatrix} \qquad (2.691)$$

Checking the result, we have

$$\begin{aligned} \boldsymbol{x}^T \boldsymbol{D}_{\omega,B}(\tau) \boldsymbol{x} &= (x, y) \begin{pmatrix} -\partial_\tau^2 + \omega^2 - \omega_B^2 & -2 i \omega_B \partial_\tau \\ 2 i \omega_B \partial_\tau & -\partial_\tau^2 + \omega^2 - \omega_B^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (x, y) \begin{pmatrix} -x'' + \left(\omega^2 - \omega_B^2\right) x - 2 i \omega_B y' \\ 2 i \omega_B x' - y'' + \left(\omega^2 - \omega_B^2\right) y \end{pmatrix} \\ &= x \left[-x'' + \left(\omega^2 - \omega_B^2\right) x - 2 i \omega_B y' \right] + y \left[2 i \omega_B x' - y'' + \left(\omega^2 - \omega_B^2\right) y \right] \\ &= -(x x'' + y y'') + \left(\omega^2 - \omega_B^2\right)\left(x^2 + y^2\right) - 2 i \omega_B (x y' - y x') \end{aligned}$$

which indeed agrees with (2.690a).

Since the path integral is Gaussian, the partition function is simply [see §2.11]

$$Z = \frac{M}{2 \pi \hbar \sqrt{\det \boldsymbol{D}_{\omega,B}}} \qquad (2.692)$$

Owing to the periodic B.C. on $\boldsymbol{x}(\tau)$, we have

$$\boldsymbol{x}(\tau) = \sum_{m=-\infty}^{\infty} \boldsymbol{x}_m e^{-i \omega_m \tau} \qquad \omega_m = \frac{2 \pi m}{\beta \hbar} \qquad (2.692a)$$

so that

$$\int_0^{\beta\hbar} d\tau\, \boldsymbol{x}^T(\tau) \boldsymbol{D}_{\omega,B}(\tau) \boldsymbol{x}(\tau) = \sum_{m,n=-\infty}^{\infty} \int_0^{\beta\hbar} d\tau\, e^{-i \omega_n \tau} \boldsymbol{x}_n^T \boldsymbol{D}_{\omega,B}(\tau) e^{-i \omega_m \tau} \boldsymbol{x}_m$$

$$= \sum_{m,n=-\infty}^{\infty} \int_0^{\beta\hbar} d\tau\, e^{-i\omega_n \tau} (x_n, y_n) \begin{pmatrix} \omega_m^2 + \omega^2 - \omega_B^2 & -2\omega_B \omega_m \\ 2\omega_B \omega_m & \omega_m^2 + \omega^2 - \omega_B^2 \end{pmatrix} \begin{pmatrix} x_m \\ y_m \end{pmatrix} e^{-i\omega_m \tau}$$

$$= \beta\hbar \sum_{m=-\infty}^{\infty} (x_m^*, y_m^*) \begin{pmatrix} \omega_m^2 + \omega^2 - \omega_B^2 & -2\omega_B \omega_m \\ 2\omega_B \omega_m & \omega_m^2 + \omega^2 - \omega_B^2 \end{pmatrix} \begin{pmatrix} x_m \\ y_m \end{pmatrix} \quad (2.692b)$$

where we've used

$$\frac{1}{\beta\hbar} \int_0^{\beta\hbar} d\tau\, e^{-i(\omega_m - \omega_n)\tau} = \delta_{mn} \quad \& \quad \boldsymbol{x}_{-m} = \boldsymbol{x}_m^*$$

If we define the Fourier transform $\tilde{\boldsymbol{D}}_{\omega,B}(\omega_m)$ of the operator $\boldsymbol{D}_{\omega,B}(\tau)$ by

$$\frac{1}{\beta\hbar} \int_0^{\beta\hbar} d\tau\, \boldsymbol{x}^T(\tau)\, \boldsymbol{D}_{\omega,B}(\tau)\, \boldsymbol{x}(\tau) = \sum_{m=-\infty}^{\infty} \boldsymbol{x}_m^+ \tilde{\boldsymbol{D}}_{\omega,B}(\omega_m)\, \boldsymbol{x}_m \quad (2.693a)$$

we have

$$\tilde{\boldsymbol{D}}_{\omega,B}(\omega_m) = \begin{pmatrix} \omega_m^2 + \omega^2 - \omega_B^2 & -2\omega_B \omega_m \\ 2\omega_B \omega_m & \omega_m^2 + \omega^2 - \omega_B^2 \end{pmatrix} \quad (2.694)$$

(2.690) then becomes

$$\mathcal{A}_e[\boldsymbol{x}] = \frac{1}{2} M \boldsymbol{x} \cdot \boldsymbol{x}' \Big|_0^{\beta\hbar} + \frac{1}{2} M \beta\hbar \sum_{m=-\infty}^{\infty} \boldsymbol{x}_m^+ \tilde{\boldsymbol{D}}_{\omega,B}(\omega_m)\, \boldsymbol{x}_m \quad (2.694a)$$

which re-affirms the Gaussian nature of the path integral and hence (2.692).

(2.694) gives

$$\det \tilde{\boldsymbol{D}}_{\omega,B}(\omega_m) = \left(\omega_m^2 + \omega^2 - \omega_B^2\right)^2 + 4\omega_B^2 \omega_m^2 \quad (2.695)$$
$$= \omega_m^4 + 2\left(\omega^2 + \omega_B^2\right)\omega_m^2 + \left(\omega^2 - \omega_B^2\right)^2$$
$$= \left[\omega_m^2 + (\omega + \omega_B)^2\right]\left[\omega_m^2 + (\omega - \omega_B)^2\right]$$
$$= \left(\omega_m^2 + \omega_+^2\right)\left(\omega_m^2 + \omega_-^2\right) \quad (2.696)$$

where

$$\omega_\pm = \omega \pm \omega_B \quad (2.697)$$

Thus, the eigenvalues of $\tilde{\boldsymbol{D}}_{\omega,B}(\omega_m)$ are solutions of the secular equation

$$\left(\omega_m^2 + \omega^2 - \omega_B^2 - d\right)^2 + 4\omega_B^2 \omega_m^2 = 0$$

$$\rightarrow \quad d_\mp = \omega_m^2 + \omega^2 - \omega_B^2 \pm 2i\,\omega_B \omega_m$$
$$= (\omega_m \pm i\,\omega_B)^2 + \omega^2$$
$$= (\omega_m \pm i\,\omega_B + i\,\omega)(\omega_m \pm i\,\omega_B - i\,\omega)$$
$$= (\omega_m + i\,\omega_\pm)(\omega_m - i\,\omega_\mp) \quad (2.699)$$

[Caution: Kleinert called this $d_\pm$.]

The eigenvectors $\boldsymbol{e}_\pm$ corresponding to $d_\pm$ are given by

$$\begin{pmatrix} \pm 2i\,\omega_B \omega_m & -2\omega_B \omega_m \\ 2\omega_B \omega_m & \pm 2i\,\omega_B \omega_m \end{pmatrix} \begin{pmatrix} e_{x\pm} \\ e_{y\pm} \end{pmatrix} = 0$$

$$\rightarrow \quad e_{y\pm} = \pm i\, e_{x\pm}$$

The orthonormalized eigenvectors are therefore

$$\boldsymbol{e}_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \quad (2.698)$$

with

$$\boldsymbol{e}_\pm^+ \boldsymbol{e}_\pm = 1 \quad \& \quad \boldsymbol{e}_\pm^+ \boldsymbol{e}_\mp = 0$$

Thus, the right- and left- combinations $x_{m\pm} = \frac{1}{\sqrt{2}}(x_m \pm i\, y_m)$ diagonalize the quadrature for each m as [see (2.699)]

$$\boldsymbol{x}_m^+ \tilde{\boldsymbol{D}}_{\omega,B}(\omega_m)\, \boldsymbol{x}_m = x_{m+}^* (\omega_m + i\,\omega_+)(\omega_m - i\,\omega_-)\, x_{m+} + x_{m-}^* (\omega_m + i\,\omega_-)(\omega_m - i\,\omega_+)\, x_{m-}$$

Using (2.692a), we have,

$$\begin{aligned}\boldsymbol{x}^+(\tau)\, \tilde{\boldsymbol{D}}_{\omega,B}(\tau)\, \boldsymbol{x}(\tau) &= x_+^*(\tau)\,(i\,\partial_\tau + i\,\omega_+)(i\,\partial_\tau - i\,\omega_-)\, x_+(\tau) \\ &\quad + x_-^*(\tau)\,(i\,\partial_\tau + i\,\omega_-)(i\,\partial_\tau - i\,\omega_+)\, x_-(\tau) \\ &= -x_+^*(\tau)\,(\partial_\tau + \omega_+)(\partial_\tau - \omega_-)\, x_+(\tau) - x_-^*(\tau)\,(\partial_\tau + \omega_-)(\partial_\tau - \omega_+)\, x_-(\tau) \end{aligned} \qquad (2.700a)$$

where

$$x_\pm(\tau) = \frac{1}{\sqrt{2}}\left[x(\tau) \pm i\, y(\tau) \right] = x_\mp^*(\tau)$$

Hence,

$$x = \frac{1}{\sqrt{2}}(x_+ + x_-) \qquad\qquad y = \frac{1}{i\sqrt{2}}(x_+ - x_-)$$

$$\begin{aligned}\to \qquad \boldsymbol{x}\cdot\boldsymbol{x}' &= x\,x' + y\,y' = \frac{1}{2}\left[(x_+ + x_-)(x_+' + x_-') - (x_+ - x_-)(x_+' - x_-') \right] \\ &= x_+\, x_-' + x_-\, x_+' \\ &= x_-^*\, x_-' + x_+^*\, x_+' \end{aligned}$$

(2.690) thus becomes

$$\mathcal{A}_e[\boldsymbol{x}] = \frac{1}{2} M \int_0^{\beta\hbar} d\,\tau \left[(x_-^*\, x_-' + x_+^*\, x_+')' - x_+^*\,(\partial_\tau + \omega_+)(\partial_\tau - \omega_-)\, x_+ - x_-^*(\partial_\tau + \omega_-)(\partial_\tau - \omega_+)\, x_- \right] \qquad (2.700)$$

Continued back to real times, the components $x_\pm(t)$ oscillate independently with the frequencies $\omega_\pm$.

The factorization (2.696) makes (2.700) an action of two independent harmonic oscillators of frequencies $\omega_\pm$. The associated partition function has therefore the
product form [see §2.10]

$$Z = \frac{1}{2\sinh\left(\frac{\beta\hbar\omega_+}{2}\right)}\, \frac{1}{2\sinh\left(\frac{\beta\hbar\omega_-}{2}\right)} \qquad (2.701)$$

The system in §2.18 corresponds to $\omega = \omega_B$ so that

$$\omega_+ = 2\,\omega_B \qquad\qquad \omega_- = 0$$

and (2.701) diverges.

Including the free motion along the z-axis, we have free motion in 2-D. Extending the rule (2.359)

$$\frac{1}{\omega} \xrightarrow[\omega\to 0]{} \sqrt{\frac{\beta M}{2\pi}}\, L$$

to 2-D, we have

$$\frac{1}{\omega_-} \xrightarrow[\omega_-\to 0]{} \frac{\beta M}{2\pi} L^2 \qquad (2.702)$$

so that (2.701) becomes

$$Z \xrightarrow[\omega_-\to 0]{} \frac{1}{2\sinh(\beta\hbar\omega_B)}\, \frac{\beta M}{2\pi\hbar} L^2 \qquad (2.703)$$