2.20. Gauge Invariance and Alternative Path Integral Representation

Consider a particle in a magnetic field as well as an ordinary potential $V(\boldsymbol{x}, t)$. By (2.633), the action is

$$\mathcal{A}[\boldsymbol{x}] = \int_{t_a}^{t_b} d\,t \left(\frac{1}{2} M \dot{\boldsymbol{x}}^2 + \frac{e}{c} \dot{\boldsymbol{x}} \cdot \boldsymbol{A} - V \right) \qquad (2.704a)$$

Under a gauge transformation

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \nabla \Lambda(\boldsymbol{x}, t)$$

the action (2.704a) becomes

$$\begin{aligned} \mathcal{A}[\boldsymbol{x}] \to \mathcal{A}'[\boldsymbol{x}] &= \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} M \dot{\boldsymbol{x}}^2 + \frac{e}{c} \dot{\boldsymbol{x}} \cdot (\boldsymbol{A} + \nabla \Lambda) - V \right] \\ &= \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} M \dot{\boldsymbol{x}}^2 + \frac{e}{c} \dot{\boldsymbol{x}} \cdot (\boldsymbol{A} + \nabla \Lambda) - V + \frac{e}{c} \frac{d\Lambda}{d\,t} \right] \\ &\quad - \frac{e}{c} \left[\Lambda(\boldsymbol{x}_b\, t_b) - \Lambda(\boldsymbol{x}_a, t_a) \right] \end{aligned} \qquad (2.704)$$

Recall now the Hamilton-Jacobi eq. (1.65) which uses the action-derived function [see (1.61a)]

$$\mathfrak{A}(\boldsymbol{x}, t) \equiv \mathfrak{A}(\boldsymbol{x}, t\,;\, \boldsymbol{x}_a, t_a) = \int_{t_a}^{t} d\,t' \left[\boldsymbol{p} \cdot \dot{\boldsymbol{x}} - H(\boldsymbol{x}, \boldsymbol{p}, t') \right] \qquad (2.704b)$$

as the generator:

$$\begin{aligned} \partial_t \mathfrak{A} &= -H \\ &= -\frac{1}{2\,M} \left(\boldsymbol{p} - \frac{e}{c} \boldsymbol{A} \right)^2 - V \\ &= -\frac{1}{2\,M} \left(\nabla \mathfrak{A} - \frac{e}{c} \boldsymbol{A} \right)^2 - V \end{aligned} \qquad (2.705)$$

where [see (1.62)]

$$\boldsymbol{p} = \nabla \mathfrak{A} \qquad (2.705a)$$

was used.

Note that, owing to the integration over t', the $\boldsymbol{x}$ in $\mathfrak{A}(\boldsymbol{x}, t)$ is to be treated as a variable, not a function of time. In other words,

$$\frac{d\,\mathfrak{A}}{d\,t} = \partial_t \mathfrak{A} \qquad (2.705b)$$

Setting $\Lambda = -\frac{c}{e} \mathfrak{A}$, we have

$$-V + \frac{e}{c} \frac{d\Lambda}{d\,t} = -V - \partial_t \mathfrak{A} = \frac{1}{2\,M} \left(\nabla \mathfrak{A} - \frac{e}{c} \boldsymbol{A} \right)^2$$

so that (2.704) becomes

$$\begin{aligned} \mathcal{A}'[\boldsymbol{x}] &= \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} M \dot{\boldsymbol{x}}^2 + \dot{\boldsymbol{x}} \cdot \left(\frac{e}{c} \boldsymbol{A} - \nabla \mathfrak{A} \right) + \frac{1}{2\,M} \left(\nabla \mathfrak{A} - \frac{e}{c} \boldsymbol{A} \right)^2 \right] \\ &\quad + \mathfrak{A}(\boldsymbol{x}_b\, t_b) - \mathfrak{A}(\boldsymbol{x}_a, t_a) \\ &= \int_{t_a}^{t_b} d\,t \frac{1}{2\,M} \left(M \dot{\boldsymbol{x}} - \nabla \mathfrak{A} + \frac{e}{c} \boldsymbol{A} \right)^2 + \mathfrak{A}(\boldsymbol{x}_b\, t_b) \end{aligned} \qquad (2.706)$$

where we've used

$$\mathfrak{A}(\boldsymbol{x}_a, t_a) = \mathfrak{A}(\boldsymbol{x}_a, t_a\,;\, \boldsymbol{x}_a, t_a) = 0 \qquad (2.706a)$$

The classical velocity field is defined as

$$\boldsymbol{v} \equiv \frac{\partial H}{\partial \boldsymbol{p}} = \frac{1}{M}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right) = \frac{1}{M}\left(\nabla \mathfrak{A} - \frac{e}{c}\boldsymbol{A}\right) \tag{2.708}$$

For two infinitesimal solutions of (2.705), their difference $\delta\mathfrak{A}$ satisfies

$$\begin{aligned}\partial_t\, \delta\mathfrak{A} &= -\delta H \\ &= -\frac{1}{M}\left(\nabla \mathfrak{A} - \frac{e}{c}\boldsymbol{A}\right)\cdot\nabla\delta\mathfrak{A} \\ &= -\boldsymbol{v}\cdot\nabla\delta\mathfrak{A} \qquad (2.707) \\ &= -\boldsymbol{v}\cdot\delta\boldsymbol{p} \qquad \text{[(2.705a) used.]}\end{aligned}$$

Identifying H with the particle energy E, (2.707) shows

$$\delta E = \boldsymbol{v}\cdot\nabla\delta\mathfrak{A} \;\; = \boldsymbol{v}\cdot\delta\boldsymbol{p} \tag{2.708a}$$

Under the same gauge change, (2.706) gives

$$\begin{aligned}\delta\mathcal{A}'[\boldsymbol{x}] &= \int_{t_a}^{t_b} d\,t\, \frac{1}{M}\left(M\dot{\boldsymbol{x}} - \nabla\mathfrak{A} + \frac{e}{c}\boldsymbol{A}\right)\cdot(-\nabla\delta\mathfrak{A}) \;\; + \delta\mathfrak{A}(\boldsymbol{x}_b\, t_b) \\ &= -\int_{t_a}^{t_b} d\,t\,\left(\dot{\boldsymbol{x}}\cdot\nabla\delta\mathfrak{A} + \partial_t\delta\mathfrak{A}\right) + \delta\mathfrak{A}(\boldsymbol{x}_b\, t_b) \\ &= -\delta\mathfrak{A}(\boldsymbol{x}_b\, t_b)\end{aligned}$$

as expected. However,

$$\int_{t_a}^{t_b} d\,t\, \frac{1}{2M}\left(M\dot{\boldsymbol{x}} - \nabla\mathfrak{A} + \frac{e}{c}\boldsymbol{A}\right)^2$$

changes by $-2\,\delta\mathfrak{A}(\boldsymbol{x}_b\, t_b)$ and is not gauge invariant as claimed by Kleinert.

The evolution amplitude for the action (2.706) is

$$\begin{aligned}(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = e^{i\,\mathfrak{A}(\boldsymbol{x}_b\, t_b\,;\, x_a\, t_a)/\hbar} \int_{\boldsymbol{x}(t_a)=\boldsymbol{x}_a}^{\boldsymbol{x}(t_b)=\boldsymbol{x}_b} \mathcal{D}\,\boldsymbol{x} \\ \times\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\, \frac{1}{2M}\left(M\dot{\boldsymbol{x}} - \nabla\mathfrak{A} + \frac{e}{c}\boldsymbol{A}\right)^2\right\}\end{aligned} \tag{2.709}$$

Using (2.708), we have

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = e^{i\,\mathfrak{A}(\boldsymbol{x}_b\, t_b\,;\, x_a\, t_a)/\hbar} \int_{\boldsymbol{x}(t_a)=\boldsymbol{x}_a}^{\boldsymbol{x}(t_b)=\boldsymbol{x}_b} \mathcal{D}\,\boldsymbol{x}\, \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\, \frac{1}{2} M\left(\dot{\boldsymbol{x}} - \boldsymbol{v}\right)^2\right\} \tag{2.710}$$

The fluctuations are now controlled by the velocity deviation $\dot{\boldsymbol{x}} - \boldsymbol{v}$. Since the path integral attempts to keep the deviations as small as possible, $\boldsymbol{v}(\boldsymbol{x}, t)$ may be called the desired velocity of the particle at $\boldsymbol{x}$ and t.

Introducing the momentum variables so that their Gaussian integrals reproduce (2.710), we have

$$\begin{aligned}(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = e^{i\,\mathfrak{A}(\boldsymbol{x}_b\, t_b\,;\, x_a\, t_a)/\hbar} \int_{\boldsymbol{x}(t_a)=\boldsymbol{x}_a}^{\boldsymbol{x}(t_b)=\boldsymbol{x}_b} \mathcal{D}\,\boldsymbol{x} \int \frac{\mathcal{D}\,\boldsymbol{p}}{(2\pi\hbar)^d} \\ \times\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\left[\boldsymbol{p}\cdot\left(\dot{\boldsymbol{x}} - \boldsymbol{v}\right) - \frac{\boldsymbol{p}^2}{2M}\right]\right\}\end{aligned} \tag{2.711}$$