2.21. Velocity Path Integral

The path integral [see (2.60 & 64)]

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = \int_{\mathbf{x}(t_a)=\mathbf{x}_a}^{\mathbf{x}(t_b)=\mathbf{x}_b} \mathcal{D}\,\mathbf{x}\, \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} M \dot{\mathbf{x}}^2 - V(\mathbf{x}) \right] \right\}$$

$$= \left(\frac{M}{2\,\pi\, i\, \hbar\, \epsilon} \right)^{d(N+1)/2} \prod_{n=1}^{N} \int d^d\, \mathbf{x}_n \exp\left\{ \frac{i}{\hbar} \sum_{n=1}^{N+1} \left[\frac{1}{2} M \frac{(\mathbf{x}_n - \mathbf{x}_{n-1})^2}{\epsilon} - \epsilon\, V(\mathbf{x}_n) \right] \right\}$$

can be written as

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = \int \mathcal{D}\,\mathbf{x}\, \delta\left[\mathbf{x}_b - \mathbf{x}_a - \int_{t_a}^{t_b} d\,t\, \dot{\mathbf{x}}(t) \right] \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} M \dot{\mathbf{x}}^2 - V(\mathbf{x}) \right] \right\} \qquad (2.712)$$

$$= \left(\frac{M}{2\,\pi\, i\, \hbar\, \epsilon} \right)^{d(N+1)/2} \prod_{n=1}^{N+1} \int d^d\, \mathbf{x}_n\, \delta\left[\mathbf{x}_b - \mathbf{x}_a - \int_{t_a}^{t_b} d\,t\, \dot{\mathbf{x}}(t) \right] \qquad (2.712a)$$

$$\times \exp\left\{ \frac{i}{\hbar} \sum_{n=1}^{N+1} \left[\frac{1}{2} M \frac{(\mathbf{x}_n - \mathbf{x}_{n-1})^2}{\epsilon} - \epsilon\, V(\mathbf{x}_n) \right] \right\}$$

where all time-sliced derivatives

$$\mathbf{v}_n = \frac{\mathbf{x}_n - \mathbf{x}_{n-1}}{\epsilon} \qquad n = 1, \ \ldots, \ N+1$$

are integrated over. This means these $\mathbf{v}_n$'s can replace $\mathbf{x}_n$ as independent fluctuation variables. Some possible ways to do so are

$$\mathbf{x}(t) = \mathbf{x}_b - \int_t^{t_b} d\,t\, \mathbf{v}(t) \qquad (2.713)$$

$$\mathbf{x}(t) = \mathbf{x}_a + \int_{t_a}^{t} d\,t\, \mathbf{v}(t) \qquad (2.714)$$

$$\mathbf{x}(t) = \mathbf{X} + \frac{1}{2} \int_{t_a}^{t_b} d\,t'\, \mathbf{v}(t')\, \epsilon(t'-t) \qquad (2.715)$$

where

$$\mathbf{X} = \frac{1}{2} (\mathbf{x}_b + \mathbf{x}_a) \qquad \epsilon(t) = \begin{cases} -1 & t > 0 \\ 1 & t < 0 \end{cases} \qquad (2.716)$$

The choice (2.713) turns (2.712) into a velocity path integral :

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = \int \mathcal{D}\,\mathbf{v}\, \delta\left[\mathbf{x}_b - \mathbf{x}_a - \int_{t_a}^{t_b} d\,t\, \mathbf{v}(t) \right] \qquad (2.717)$$

$$\times \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} M \mathbf{v}^2 - V\left(\mathbf{x}_b - \int_t^{t_b} d\,t\, \mathbf{v}(t) \right) \right] \right\}$$

The measure of the integration can be obtained by considering the case of a free particle. Using

$$\delta\left[\mathbf{x}_b - \mathbf{x}_a - \int_{t_a}^{t_b} d\,t\, \mathbf{v}(t) \right] = \int \frac{d\,\mathbf{p}}{(2\,\pi\, i\, \hbar)^d} \exp\left[\frac{i}{\hbar} \mathbf{p} \cdot \left(\mathbf{x}_b - \mathbf{x}_a - \int_{t_a}^{t_b} d\,t\, \mathbf{v}(t) \right) \right] \qquad (2.719)$$

(2.717) becomes

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a)_0 = \int \mathcal{D}\,\mathbf{v} \int \frac{d\,\mathbf{p}}{(2\,\pi\, i\, \hbar)^d} \exp\left[\frac{i}{\hbar} \mathbf{p} \cdot (\mathbf{x}_b - \mathbf{x}_a) \right]$$

$$\times \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} M \mathbf{v}^2 - \mathbf{p} \cdot \mathbf{v}(t) \right] \right\}$$

$$= \int \mathcal{D} \boldsymbol{v} \int \frac{d\,\boldsymbol{p}}{(2\,\pi\,i\,\hbar)^d}\ \exp\left[\frac{i}{\hbar}\left(\boldsymbol{p}\cdot(\boldsymbol{x}_b - \boldsymbol{x}_a) - \int_{t_a}^{t_b} d\,t\,\frac{\boldsymbol{p}^2}{2\,M}\right)\right]$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\ \frac{1}{2} M\left(\boldsymbol{v} - \frac{\boldsymbol{p}}{M}\right)^2\right\}$$

Comparing with [see (1.329)]

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a)_0 = \int \frac{d\,\boldsymbol{p}}{(2\,\pi\,i\,\hbar)^d}\ \exp\left[\frac{i}{\hbar}\left(\boldsymbol{p}\cdot(\boldsymbol{x}_b - \boldsymbol{x}_a) - \frac{\boldsymbol{p}^2}{2\,M}(t_b - t_a)\right)\right] \tag{2.720}$$

we have

$$\int \mathcal{D}\,\boldsymbol{v}\,\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\ \frac{1}{2} M\,\boldsymbol{v}^2\right) = 1 \tag{2.718}$$

The choice (2.715) leads to a more symmetric velocity path integral:

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \int \mathcal{D}\,\boldsymbol{v}\,\delta\left[\Delta\,\boldsymbol{x} - \int_{t_a}^{t_b} d\,t\,\boldsymbol{v}(t)\right]\ \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,\frac{1}{2} M\,\boldsymbol{v}^2\right) \tag{2.721}$$

$$\times \exp\left\{-\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,V\left(\boldsymbol{X} + \frac{1}{2}\int_{t_a}^{t_b} d\,t'\,\boldsymbol{v}(t')\,\epsilon(t' - t)\right)\right\}$$

where $\Delta\,\boldsymbol{x} = \boldsymbol{x}_b - \boldsymbol{x}_a$.

Using

$$\int d\,\boldsymbol{x}_a\,\delta\left[\Delta\,\boldsymbol{x} - \int_{t_a}^{t_b} d\,t\,\boldsymbol{v}(t)\right]\left(\boldsymbol{X} + \frac{1}{2}\int_{t_a}^{t_b} d\,t'\,\boldsymbol{v}(t')\,\epsilon(t' - t)\right)$$

$$= \int d\,\boldsymbol{x}_a\,\delta\left[\boldsymbol{x}_b - \boldsymbol{x}_a - \int_{t_a}^{t_b} d\,t\,\boldsymbol{v}(t)\right]\left(\frac{1}{2}(\boldsymbol{x}_b + \boldsymbol{x}_a) + \frac{1}{2}\int_{t_a}^{t_b} d\,t'\,\boldsymbol{v}(t')\,\epsilon(t' - t)\right)$$

$$= \boldsymbol{x}_b + \frac{1}{2}\int_{t_a}^{t_b} d\,t'\,\boldsymbol{v}(t')\left[-1 + \epsilon(t' - t)\right]$$

$$= \boldsymbol{x}_b - \int_{t}^{t_b} d\,t'\,\boldsymbol{v}(t') \qquad \text{[(2.716) used.]}$$

(2.271) becomes

$$\int d\,\boldsymbol{x}_a\,(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \int \mathcal{D}\,\boldsymbol{v}\ \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,\frac{1}{2} M\,\boldsymbol{v}^2\right) \tag{2.722}$$

$$\times \exp\left\{-\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,V\left(\boldsymbol{x}_b - \int_{t}^{t_b} d\,t'\,\boldsymbol{v}(t')\right)\right\}$$