2.22. Path Integral Representation of the Scattering Matrix

2.22.1. General Development

The scattering matrix was defined in (1.471) as

$$\left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle = \lim_{t_b - t_a \to \infty} e^{i(E_b t_b - E_a t_a)/\hbar} \left\langle \boldsymbol{p}_b \,\middle|\, e^{-i\hat{H}(t_b - t_a)/\hbar} \,\middle|\, \boldsymbol{p}_a \right\rangle$$

which is related to the evolution amplitude by a double Fourier transform:

$$\left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle = \lim_{t_b - t_a \to \infty} e^{i(E_b t_b - E_a t_a)/\hbar} \int d\boldsymbol{x}_b \int d\boldsymbol{x}_a \, e^{-i(\boldsymbol{p}_b \cdot \boldsymbol{x}_b - \boldsymbol{p}_a \cdot \boldsymbol{x}_a)/\hbar} (\boldsymbol{x}_b \, t_b \mid \boldsymbol{x}_a \, t_a) \tag{2.723}$$

Using

$$\boldsymbol{q} = \boldsymbol{p}_b - \boldsymbol{p}_a$$

$$\to \quad \boldsymbol{p}_b \cdot \boldsymbol{x}_b - \boldsymbol{p}_a \cdot \boldsymbol{x}_a = \boldsymbol{q} \cdot \boldsymbol{x}_b + \boldsymbol{p}_a \cdot (\boldsymbol{x}_b - \boldsymbol{x}_a)$$

$$\boldsymbol{p}_a \cdot (\boldsymbol{x}_b - \boldsymbol{x}_a) = \int_{t_a}^{t_b} dt \, \boldsymbol{p}_a \cdot \dot{\boldsymbol{x}}$$

and

$$(\boldsymbol{x}_b \, t_b \mid \boldsymbol{x}_a \, t_a) = \int \mathcal{D}\boldsymbol{x} \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} M \dot{\boldsymbol{x}}^2 - V(\boldsymbol{x}) \right] \right\}$$

the integrand in (2.723) becomes

$$\begin{aligned} & e^{-i(\boldsymbol{p}_b \cdot \boldsymbol{x}_b - \boldsymbol{p}_a \cdot \boldsymbol{x}_a)/\hbar} (\boldsymbol{x}_b \, t_b \mid \boldsymbol{x}_a \, t_a) \\ &= e^{-i\boldsymbol{q} \cdot \boldsymbol{x}_b/\hbar} \int \mathcal{D}\boldsymbol{x} \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} M \dot{\boldsymbol{x}}^2 - \boldsymbol{p}_a \cdot \dot{\boldsymbol{x}} - V(\boldsymbol{x}) \right] \right\} \end{aligned} \tag{2.724}$$

Setting

$$\boldsymbol{y}(t) = \boldsymbol{x}(t) - \frac{\boldsymbol{p}_a}{M} t \tag{2.275}$$

we get

$$\dot{\boldsymbol{y}} = \dot{\boldsymbol{x}} - \frac{\boldsymbol{p}_a}{M} \quad \to \quad \frac{1}{2} M \dot{\boldsymbol{x}}^2 - \boldsymbol{p}_a \cdot \dot{\boldsymbol{x}} = \frac{1}{2} M \dot{\boldsymbol{y}}^2 - \frac{\boldsymbol{p}_a^2}{2M}$$

and (2.724) becomes

$$\begin{aligned} & e^{-i(\boldsymbol{p}_b \cdot \boldsymbol{x}_b - \boldsymbol{p}_a \cdot \boldsymbol{x}_a)/\hbar} (\boldsymbol{x}_b \, t_b \mid \boldsymbol{x}_a \, t_a) \\ &= e^{-i\boldsymbol{q} \cdot \boldsymbol{x}_b/\hbar} e^{-i\boldsymbol{p}_a^2 (t_b - t_a)/2M\hbar} \int \mathcal{D}\boldsymbol{y} \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} M \dot{\boldsymbol{y}}^2 - V\left(\boldsymbol{y} + \frac{\boldsymbol{p}_a}{M} t \right) \right] \right\} \end{aligned} \tag{2.726}$$

Together with

$$\begin{aligned} & -\boldsymbol{q} \cdot \boldsymbol{x}_b + E_b t_b - E_a t_a - \frac{\boldsymbol{p}_a^2}{2M}(t_b - t_a) \\ &= -\boldsymbol{q} \cdot \left(\boldsymbol{y}_b + \frac{\boldsymbol{p}_a}{M} t_b \right) + \frac{\boldsymbol{p}_b^2}{2M} t_b - \frac{\boldsymbol{p}_a^2}{2M} t_a - \frac{\boldsymbol{p}_a^2}{2M}(t_b - t_a) \\ &= -\boldsymbol{q} \cdot \boldsymbol{y}_b + \frac{\boldsymbol{p}_b^2 - 2\boldsymbol{q} \cdot \boldsymbol{p}_a - \boldsymbol{p}_a^2}{2M} t_b \\ &= -\boldsymbol{q} \cdot \boldsymbol{y}_b + \frac{\boldsymbol{q}^2}{2M} t_b \end{aligned}$$

(2.723) becomes

$$\left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle = \lim_{t_b - t_a \to \infty} e^{i\boldsymbol{q}^2 t_b/2M\hbar} \int d\boldsymbol{y}_b\, e^{-i\boldsymbol{q}\cdot\boldsymbol{y}_b/\hbar} \int d\,\boldsymbol{y}_a \tag{2.727}$$
$$\times \int \mathcal{D}\boldsymbol{y} \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} M \dot{\boldsymbol{y}}^2 - V\!\left(\boldsymbol{y} + \frac{\boldsymbol{p}_a}{M} t\right) \right] \right\}$$

For $V = 0$, (2.71) gives

$$\int \mathcal{D}\boldsymbol{y} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} dt \frac{1}{2} M \dot{\boldsymbol{y}}^2 \right) = \left(\frac{M}{2\pi i \hbar (t_b - t_a)} \right)^{d/2} \exp\left[\frac{i}{\hbar} \frac{M}{2} \frac{(\boldsymbol{y}_b - \boldsymbol{y}_a)^2}{t_b - t_a} \right]$$

$$\rightarrow \qquad \int d\,\boldsymbol{y}_a \int \mathcal{D}\boldsymbol{y} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} dt \frac{1}{2} M \dot{\boldsymbol{y}}^2 \right) = 1 \tag{2.728}$$

$$\therefore \qquad \left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle_{V=0} = \lim_{t_b - t_a \to \infty} e^{i\boldsymbol{q}^2 t_b/2M\hbar} \int d\boldsymbol{y}_b\, e^{-i\boldsymbol{q}\cdot\boldsymbol{y}_b/\hbar}$$
$$= \lim_{t_b - t_a \to \infty} e^{i\boldsymbol{q}^2 t_b/2M\hbar} (2\pi\hbar)^d\, \delta(\boldsymbol{q})$$
$$= \lim_{t_b - t_a \to \infty} (2\pi\hbar)^d\, \delta(\boldsymbol{q})$$
$$= \lim_{t_b - t_a \to \infty} (2\pi\hbar)^d\, \delta(\boldsymbol{p}_b - \boldsymbol{p}_a) \tag{2.729}$$

which represents the contribution of the unscattered beam.

Contributions of V can be studied by expanding $\exp\left\{ -\frac{i}{\hbar} \int_{t_a}^{t_b} dt\, V\!\left(\boldsymbol{y} + \frac{\boldsymbol{p}_a}{M} t\right) \right\}$ as a power series in V.

The 1st order term is

$$\left\langle \boldsymbol{p}_b \mid \hat{S}_1 \mid \boldsymbol{p}_a \right\rangle = \lim_{t_b - t_a \to \infty} e^{i\boldsymbol{q}^2 t_b/2M\hbar} \int d\boldsymbol{y}_b\, e^{-i\boldsymbol{q}\cdot\boldsymbol{y}_b/\hbar} \int d\,\boldsymbol{y}_a \tag{2.730a}$$
$$\times \int \mathcal{D}\boldsymbol{y} \left[-\frac{i}{\hbar} \int_{t_a}^{t_b} dt'\, V\!\left(\boldsymbol{y} + \frac{\boldsymbol{p}_a}{M} t'\right) \right] \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} dt \frac{1}{2} M \dot{\boldsymbol{y}}^2 \right)$$

Introducing the Fourier transform

$$V\!\left(\boldsymbol{y} + \frac{\boldsymbol{p}_a}{M} t'\right) = \int \frac{d\boldsymbol{Q}}{(2\pi\hbar)^d}\, e^{i\boldsymbol{Q}\cdot(\boldsymbol{y} + \frac{\boldsymbol{p}_a}{M} t')/\hbar}\, V(\boldsymbol{Q})$$
$$= \int \frac{d\boldsymbol{Q}}{(2\pi\hbar)^d}\, e^{i\boldsymbol{Q}\cdot\frac{\boldsymbol{p}_a}{M} t'/\hbar}\, V(\boldsymbol{Q}) \exp\left[\frac{i}{\hbar} \int_{t_a}^{t_b} dt\, \boldsymbol{Q}\cdot\boldsymbol{y}\, \delta(t - t') \right]$$

we have

$$\left\langle \boldsymbol{p}_b \mid \hat{S}_1 \mid \boldsymbol{p}_a \right\rangle = -\frac{i}{\hbar} \lim_{t_b - t_a \to \infty} e^{i\boldsymbol{q}^2 t_b/2M\hbar} \int d\boldsymbol{y}_b\, e^{-i\boldsymbol{q}\cdot\boldsymbol{y}_b/\hbar} \int d\,\boldsymbol{y}_a \int \frac{d\boldsymbol{Q}}{(2\pi\hbar)^d} V(\boldsymbol{Q}) \tag{2.730}$$
$$\times \int_{t_a}^{t_b} dt'\, e^{i\boldsymbol{Q}\cdot\frac{\boldsymbol{p}_a}{M} t'/\hbar} \int \mathcal{D}\boldsymbol{y} \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} M \dot{\boldsymbol{y}}^2 + \boldsymbol{Q}\cdot\boldsymbol{y}\, \delta(t - t') \right] \right\}$$

The path integral of a harmonic oscillator in the presence of an external source $\boldsymbol{j}(t)$ is given by [see §3.6]

$$(\boldsymbol{x}_b t_b \mid \boldsymbol{x}_a t_a)^j_\omega = \int \mathcal{D}\boldsymbol{x} \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} M \left(\dot{\boldsymbol{x}}^2 - \omega^2 \boldsymbol{x}^2 \right) + \boldsymbol{j}\cdot\boldsymbol{x} \right] \right\}$$
$$= e^{i\mathcal{A}_{j,\mathrm{cl}}/\hbar} F_{\omega,j}(t_b, t_a) \tag{3.168}$$

where

$$\mathcal{A}_{j,\mathrm{cl}} = \frac{M\omega}{2\sin\omega(t_b - t_a)} \left[(\boldsymbol{x}_b^2 + \boldsymbol{x}_a^2) \cos\omega(t_b - t_a) - 2\,\boldsymbol{x}_b\cdot\boldsymbol{x}_a \right] \tag{3.169}$$
$$+ \frac{1}{\sin\omega(t_b - t_a)} \int_{t_a}^{t_b} dt \left[\boldsymbol{x}_a \sin\omega(t_b - t) + \boldsymbol{x}_b \sin\omega(t - t_a) \right] \cdot \boldsymbol{j}(t)$$

$$F_{\omega,j}(t_b, t_a) = \left(\frac{M}{2\pi i \hbar} \frac{\omega}{\sin \omega(t_b - t_a)} \right)^{d/2} \exp\left\{ - \frac{i}{\hbar M \omega \sin \omega(t_b - t_a)} \times \int_{t_a}^{t_b} dt \int_{t_a}^{t} dt' \sin \omega(t_b - t) \sin \omega(t' - t_a)\, \boldsymbol{j}(t) \cdot \boldsymbol{j}(t') \right\} \tag{3.170}$$

Setting

$$\omega = 0 \qquad \boldsymbol{j}(t) = \boldsymbol{Q}\, \delta(t - t')$$

we have

$$\mathcal{J} = \int \mathcal{D}\boldsymbol{y} \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} M \dot{\boldsymbol{y}}^2 + \boldsymbol{Q} \cdot \boldsymbol{y}\, \delta(t - t') \right] \right\}$$
$$= e^{i \mathcal{A}_{j,\text{cl}} / \hbar} F_{\omega,j}(t_b, t_a) \tag{2.731a}$$

where

$$\mathcal{A}_{j,\text{cl}} = \frac{M}{2(t_b - t_a)} (\boldsymbol{y}_b - \boldsymbol{y}_a)^2 + \frac{1}{t_b - t_a} \left[\boldsymbol{y}_a (t_b - t') + \boldsymbol{y}_b (t' - t_a) \right] \cdot \boldsymbol{Q} \tag{2.731b}$$

$$F_{\omega,j}(t_b, t_a) = \left(\frac{M}{2\pi i \hbar (t_b - t_a)} \right)^{d/2} \exp\left\{ - \frac{i}{\hbar M (t_b - t_a)} \times \int_{t_a}^{t_b} dt \int_{t_a}^{t} dt'' (t_b - t)(t'' - t_a) \boldsymbol{Q}^2 \delta(t - t') \delta(t'' - t') \right\}$$

Let

$$\mathcal{I} = (t_b - t)(t'' - t_a) \boldsymbol{Q}^2 \delta(t - t') \delta(t'' - t')$$

Then

$$\int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'' \mathcal{I} = (t_b - t')(t' - t_a) \boldsymbol{Q}^2$$
$$= \int_{t_a}^{t_b} dt \left(\int_{t}^{t_b} + \int_{t_a}^{t} \right) dt'' \mathcal{I}$$

By symmetry, we expect

$$\int_{t_a}^{t_b} dt \int_{t}^{t_b} dt'' \mathcal{I} = \int_{t_a}^{t_b} dt \int_{t_a}^{t} dt'' \mathcal{I} \propto (t_b - t')(t' - t_a) \boldsymbol{Q}^2$$

Hence

$$\int_{t_a}^{t_b} dt \int_{t_a}^{t} dt'' \mathcal{I} = \frac{1}{2} (t_b - t')(t' - t_a) \boldsymbol{Q}^2$$

and

$$F_{\omega,j}(t_b, t_a) = \left(\frac{M}{2\pi i \hbar (t_b - t_a)} \right)^{d/2} \exp\left\{ - \frac{i (t_b - t')(t' - t_a)}{2 \hbar M (t_b - t_a)} \boldsymbol{Q}^2 \right\} \tag{2.731c}$$

Combining (2.731a-c), we have

$$\mathcal{J} = \left(\frac{M}{2\pi i \hbar (t_b - t_a)} \right)^{d/2} \exp\left\{ \frac{i}{\hbar} \frac{M}{2(t_b - t_a)} (\boldsymbol{y}_b - \boldsymbol{y}_a)^2 \right\} \times \exp\left\{ \frac{i}{\hbar} \frac{1}{t_b - t_a} \left(\left[\boldsymbol{y}_a (t_b - t') + \boldsymbol{y}_b (t' - t_a) \right] \cdot \boldsymbol{Q} - \frac{(t_b - t')(t' - t_a)}{2M} \boldsymbol{Q}^2 \right) \right\} \tag{2.731}$$

For the $\boldsymbol{y}_a$ integral, we use

$$\int d\, \boldsymbol{y}_a \exp\left\{ \frac{i}{\hbar} \frac{1}{t_b - t_a} \left[\frac{M}{2} (\boldsymbol{y}_b - \boldsymbol{y}_a)^2 + \boldsymbol{y}_a \cdot \boldsymbol{Q} (t_b - t') \right] \right\}$$
$$= \left(\frac{2\pi \hbar i (t_b - t_a)}{M} \right)^{d/2} \exp\left[- \frac{i}{\hbar} \frac{\boldsymbol{Q}^2}{2M(t_b - t_a)} (t_b - t')^2 + \frac{i}{\hbar} \boldsymbol{y}_b \cdot \boldsymbol{Q} \left(\frac{t_b - t'}{t_b - t_a} \right) \right] \tag{2.732a}$$

to get

$$\int d\,\boldsymbol{y}_a\, \mathcal{J} = \exp\left\{\frac{i}{\hbar}\frac{1}{t_b - t_a}\left[\boldsymbol{y}_b\,(t_b - t_a)\cdot\boldsymbol{Q} - \frac{(t_b - t')\,(t_b - t_a)}{2M}\boldsymbol{Q}^2\right]\right\}$$
$$= \exp\left\{\frac{i}{\hbar}\left[\boldsymbol{y}_b\cdot\boldsymbol{Q} - \frac{t_b - t'}{2M}\boldsymbol{Q}^2\right]\right\} \qquad (2.732)$$

For the $\boldsymbol{y}_b$ integral, we use

$$\int d\,\boldsymbol{y}_b\, e^{-i\boldsymbol{q}\cdot\boldsymbol{y}_b/\hbar}\ \exp\left\{\frac{i}{\hbar}\boldsymbol{y}_b\cdot\boldsymbol{Q}\right\} = (2\pi\hbar)^d\,\delta(\boldsymbol{q}-\boldsymbol{Q})$$

to get

$$\int d\,\boldsymbol{y}_b\, e^{-i\boldsymbol{q}\cdot\boldsymbol{y}_b/\hbar}\int d\,\boldsymbol{y}_a\,\mathcal{J} = (2\pi\hbar)^d\,\delta(\boldsymbol{q}-\boldsymbol{Q})\exp\left\{-\frac{i}{\hbar}\frac{(t_b - t')}{2M}\boldsymbol{Q}^2\right\} \qquad (2.732b)$$

(2.730) thus becomes

$$\left\langle \boldsymbol{p}_b \mid \hat{S}_1 \mid \boldsymbol{p}_a\right\rangle = -\frac{i}{\hbar}\lim_{t_b - t_a\to\infty} e^{i\boldsymbol{q}^2 t_b/2M\hbar}\int\frac{d\,\boldsymbol{Q}}{(2\pi\hbar)^d}V(\boldsymbol{Q})$$
$$\times\int_{t_a}^{t_b} d\,t'\,e^{i\boldsymbol{Q}\cdot\frac{\boldsymbol{p}_a}{M}t'/\hbar}(2\pi\hbar)^d\,\delta(\boldsymbol{q}-\boldsymbol{Q})\exp\left\{-\frac{i}{\hbar}\frac{(t_b - t')}{2M}\boldsymbol{Q}^2\right\}$$
$$= -\frac{i}{\hbar}\lim_{t_b - t_a\to\infty} V(\boldsymbol{q})\int_{t_a}^{t_b} d\,t'\,e^{i\boldsymbol{q}\cdot\frac{\boldsymbol{p}_a}{M}t'/\hbar}\exp\left\{\frac{i}{\hbar}\frac{t'}{2M}\boldsymbol{q}^2\right\}$$
$$= -2\pi i\ V(\boldsymbol{q})\,\delta\left(\boldsymbol{q}\cdot\frac{\boldsymbol{p}_a}{M} + \frac{1}{2M}\boldsymbol{q}^2\right) \qquad (2.732c)$$

Using

$$\boldsymbol{q}\cdot\frac{\boldsymbol{p}_a}{M} + \frac{1}{2M}\boldsymbol{q}^2 = \frac{1}{2M}\boldsymbol{q}\cdot(\boldsymbol{p}_b + \boldsymbol{p}_a) = \frac{1}{2M}\left(\boldsymbol{p}_b^2 - \boldsymbol{p}_a^2\right) = E_b - E_a \qquad (2.732d)$$

(2.732c) becomes the well-known Born-approximation

$$\left\langle \boldsymbol{p}_b \mid \hat{S}_1 \mid \boldsymbol{p}_a\right\rangle = -2\pi i\ V(\boldsymbol{q})\,\delta(E_b - E_a) \qquad (2.733)$$

Using the definition of the T-matrix

$$\left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a\right\rangle = (2\pi\hbar)^d\,\delta(\boldsymbol{p}_b - \boldsymbol{p}_a) - 2\pi\hbar i\,\delta(E_b - E_a)\left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a\right\rangle \qquad (1.474)$$

we get from (2.727) and (2.729) a definition in terms of the path integral as

$$2\pi\hbar i\,\delta(E_b - E_a)\left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a\right\rangle$$
$$\equiv -\lim_{t_b - t_a\to\infty} e^{i\boldsymbol{q}^2 t_b/2M\hbar}\int d\,\boldsymbol{y}_b\,e^{-i\boldsymbol{q}\cdot\boldsymbol{y}_b/\hbar}\int d\,\boldsymbol{y}_a\int\mathcal{D}\,\boldsymbol{y}\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,\frac{1}{2}M\dot{\boldsymbol{y}}^2\right) \qquad (2.734)$$
$$\times\left\{\exp\left[-\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,V\!\left(\boldsymbol{y} + \frac{\boldsymbol{p}_a}{M}t\right)\right] - 1\right\}$$

A definition free of $\delta(E_b - E_a)$ can be obtained as follows.

Consider a transformation

$$t \to t + t_0 \qquad (2.734a)$$

Since $t_b - t_a \to \infty$, everything on the R.H.S. of (2.734) remains unchanged except for

$$e^{i\boldsymbol{q}^2 t_b/2M\hbar} \to e^{i\boldsymbol{q}^2 t_b/2M\hbar}\,e^{i\boldsymbol{q}^2 t_0/2M\hbar}$$
$$V\!\left(\boldsymbol{y} + \frac{\boldsymbol{p}_a}{M}t\right) \to V\!\left[\boldsymbol{y} + \frac{\boldsymbol{p}_a}{M}(t + t_0)\right]$$

This shift in V can be nullified by the transform

$$\boldsymbol{y} \to \boldsymbol{y} - \frac{\boldsymbol{p}_a}{M}t_0 \qquad (2.734b)$$

which, however, induces the change

$$e^{-i\boldsymbol{q}\cdot\boldsymbol{y}_b/\hbar} \to e^{-i\boldsymbol{q}\cdot\boldsymbol{y}_b/\hbar} \exp\left(\frac{i}{\hbar}\frac{\boldsymbol{q}\cdot\boldsymbol{p}_a}{M}t_0\right)$$

Therefore, the combined transform (2.734a-b) merely adds a prefactor

$$\exp\left[\frac{i}{\hbar}\left(\frac{\boldsymbol{q}\cdot\boldsymbol{p}_a}{M}+\frac{\boldsymbol{q}^2}{2M}\right)t_0\right]$$

to the R.H.S. of (2.734).

Using (2.732d), we have

$$\int_{-\infty}^{\infty} d t_0 \exp\left[\frac{i}{\hbar}\left(\frac{\boldsymbol{q}\cdot\boldsymbol{p}_a}{M}+\frac{\boldsymbol{q}^2}{2M}\right)t_0\right] = 2\pi\hbar\,\delta(E_b-E_a) \tag{2.734c}$$

Using

$$\int_{-\infty}^{\infty} d t_0\, \delta\left[\hat{\boldsymbol{p}}_a\cdot\left(\boldsymbol{y}_b+\frac{\boldsymbol{p}_a}{M}t_0\right)\right] = \frac{M}{\hat{\boldsymbol{p}}_a\cdot\boldsymbol{p}_a}\int_{-\infty}^{\infty} d t_0\, \delta\left(t_0+\frac{M\hat{\boldsymbol{p}}_a\cdot\boldsymbol{y}_b}{\hat{\boldsymbol{p}}_a\cdot\boldsymbol{p}_a}\right) = \frac{M}{|\boldsymbol{p}_a|}$$

we can insert the identity

$$1 = \frac{|\boldsymbol{p}_a|}{M}\int_{-\infty}^{\infty} d t_0\, \delta\left[\hat{\boldsymbol{p}}_a\cdot\left(\boldsymbol{y}_b+\frac{\boldsymbol{p}_a}{M}t_0\right)\right]$$

into the R.H.S. of (2.734). Together with (2.734c), we have

$$\begin{aligned}
& i\int_{-\infty}^{\infty} d t_0 \exp\left[\frac{i}{\hbar}\left(\frac{\boldsymbol{q}\cdot\boldsymbol{p}_a}{M}+\frac{\boldsymbol{q}^2}{2M}\right)t_0\right]\left\langle\boldsymbol{p}_b\mid\hat{T}\mid\boldsymbol{p}_a\right\rangle \\
&= -\lim_{t_b-t_a\to\infty}\frac{|\boldsymbol{p}_a|}{M}e^{i\boldsymbol{q}^2 t_b/2M\hbar}\int_{-\infty}^{\infty} d t_0 \int d\boldsymbol{y}_b\,\delta\left[\hat{\boldsymbol{p}}_a\cdot\left(\boldsymbol{y}_b+\frac{\boldsymbol{p}_a}{M}t_0\right)\right]e^{-i\boldsymbol{q}\cdot\boldsymbol{y}_b/\hbar} \\
&\qquad\times\int d\boldsymbol{y}_a\int\mathcal{D}\boldsymbol{y}\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} d t\,\frac{1}{2}M\dot{\boldsymbol{y}}^2\right)\left\{\exp\left[-\frac{i}{\hbar}\int_{t_a}^{t_b} d t\, V\left(\boldsymbol{y}+\frac{\boldsymbol{p}_a}{M}t\right)\right]-1\right\} \\
&= -\lim_{t_b-t_a\to\infty}\frac{|\boldsymbol{p}_a|}{M}\int_{-\infty}^{\infty} d t_0 \exp\left[\frac{i}{\hbar}\left(\frac{\boldsymbol{q}\cdot\boldsymbol{p}_a}{M}+\frac{\boldsymbol{q}^2}{2M}\right)t_0\right] \\
&\qquad e^{i\boldsymbol{q}^2 t_b/2M\hbar}\int d\boldsymbol{y}_b\,\delta(\hat{\boldsymbol{p}}_a\cdot\boldsymbol{y}_b)\,e^{-i\boldsymbol{q}\cdot\boldsymbol{y}_b/\hbar}\int d\boldsymbol{y}_a\int\mathcal{D}\boldsymbol{y}\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} d t\,\frac{1}{2}M\dot{\boldsymbol{y}}^2\right) \\
&\qquad\times\left\{\exp\left[-\frac{i}{\hbar}\int_{t_a}^{t_b} d t\, V\left(\boldsymbol{y}+\frac{\boldsymbol{p}_a}{M}t\right)\right]-1\right\}
\end{aligned}$$

where the last expression was obtained by Implementing the transforms (2.734a-b).

Equating the integrand of the t_0-integral, we have

$$\begin{aligned}
\left\langle\boldsymbol{p}_b\mid\hat{T}\mid\boldsymbol{p}_a\right\rangle &= i\frac{|\boldsymbol{p}_a|}{M}\lim_{t_b-t_a\to\infty}e^{i\boldsymbol{q}^2 t_b/2M\hbar}\int d\boldsymbol{y}_b\,\delta(\hat{\boldsymbol{p}}_a\cdot\boldsymbol{y}_b)\,e^{-i\boldsymbol{q}\cdot\boldsymbol{y}_b/\hbar}\int d\boldsymbol{y}_a \\
&\qquad\times\int\mathcal{D}\boldsymbol{y}\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} d t\,\frac{1}{2}M\dot{\boldsymbol{y}}^2\right)\left\{\exp\left[-\frac{i}{\hbar}\int_{t_a}^{t_b} d t\, V\left(\boldsymbol{y}+\frac{\boldsymbol{p}_a}{M}t\right)\right]-1\right\}
\end{aligned} \tag{2.737a}$$

Setting

$$\boldsymbol{p} = \boldsymbol{p}_a \qquad\qquad p = |\boldsymbol{p}_a|$$

we have

$$\begin{aligned}
\left\langle\boldsymbol{p}_b\mid\hat{T}\mid\boldsymbol{p}_a\right\rangle &= i\frac{p}{M}\lim_{t_b-t_a\to\infty}e^{i\boldsymbol{q}^2 t_b/2M\hbar}\int d\boldsymbol{y}_b\,\delta(\hat{\boldsymbol{p}}\cdot\boldsymbol{y}_b)\,e^{-i\boldsymbol{q}\cdot\boldsymbol{y}_b/\hbar}\int d\boldsymbol{y}_a \\
&\qquad\times\int\mathcal{D}\boldsymbol{y}\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} d t\,\frac{1}{2}M\dot{\boldsymbol{y}}^2\right)\left\{\exp\left[-\frac{i}{\hbar}\int_{t_a}^{t_b} d t\, V\left(\boldsymbol{y}+\frac{\boldsymbol{p}}{M}t\right)\right]-1\right\}
\end{aligned} \tag{2.737}$$

Going over to the velocity path integral by using (2.722), we have

$$\left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a \right\rangle = i \frac{p}{M} \lim_{t_b - t_a \to \infty} e^{i \boldsymbol{q}^2 t_b / 2 M \hbar} \int d\boldsymbol{y}_b \, \delta\left(\hat{\boldsymbol{p}} \cdot \boldsymbol{y}_b\right) e^{-i \boldsymbol{q} \cdot \boldsymbol{y}_b / \hbar} \times \int \mathcal{D} \boldsymbol{v} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} dt \frac{1}{2} M \boldsymbol{v}^2 \right) \times \left\{ \exp\left[-\frac{i}{\hbar} \int_{t_a}^{t_b} dt \, V\left(\boldsymbol{y}_b + \frac{\boldsymbol{p}}{M} t - \int_t^{t_b} dt' \, \boldsymbol{v}(t') \right) \right] - 1 \right\} \tag{2.738a}$$

Using

$$\boldsymbol{y}_b = \left(\hat{\boldsymbol{p}} \cdot \boldsymbol{y}_b \right) \hat{\boldsymbol{p}} + \boldsymbol{b} \qquad \text{with} \qquad \boldsymbol{b} \cdot \boldsymbol{p} = 0 \tag{2.738}$$

we have

$$d\boldsymbol{y}_b = d y_{//} \, d\boldsymbol{b} \qquad\qquad y_{//} = \hat{\boldsymbol{p}} \cdot \boldsymbol{y}_b$$

Owing to $\delta\left(\hat{\boldsymbol{p}} \cdot \boldsymbol{y}_b \right) = \delta\left(y_{//} \right)$, the $y_{//}$-integral amounts to replacing $\boldsymbol{y}_b$ with $\boldsymbol{b}$. Therefore

$$\left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a \right\rangle = i \frac{p}{M} \lim_{t_b - t_a \to \infty} e^{i \boldsymbol{q}^2 t_b / 2 M \hbar} \int d\boldsymbol{b} \, e^{-i \boldsymbol{q} \cdot \boldsymbol{b} / \hbar} \times \int \mathcal{D} \boldsymbol{v} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} dt \frac{1}{2} M \boldsymbol{v}^2 \right) \left\{ \exp\left(i \chi_{\boldsymbol{b}, \boldsymbol{p}}[\boldsymbol{v}] \right) - 1 \right\} \tag{2.739}$$

where

$$\chi_{\boldsymbol{b}, \boldsymbol{p}}[\boldsymbol{v}] = -\frac{1}{\hbar} \int_{t_a}^{t_b} dt \, V\left(\boldsymbol{b} + \frac{\boldsymbol{p}}{M} t - \int_t^{t_b} dt' \, \boldsymbol{v}(t') \right) \tag{2.740}$$

We can go back to a more conventional path integral by replacing the velocity paths $\boldsymbol{v}(t)$ by

$$\boldsymbol{y}(t) = -\int_t^{t_b} dt' \, \boldsymbol{v}(t')$$

which vanishes at $t = t_b$. For periodic paths, we can use

$$\boldsymbol{z}(t) = z_b + \boldsymbol{y}(t) \qquad \text{with} \qquad \int_{t_a}^{t_b} dt' \, \boldsymbol{v}(t') = 0$$

According to (1.494), the scattering amplitude is

$$f_{\boldsymbol{p}_b \boldsymbol{p}_a} = -\frac{M}{2 \pi \hbar} \left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a \right\rangle \tag{2.740a}$$

while the scattering cross section is, by (1.493),

$$\frac{d\sigma}{d\Omega} = \left| f_{\boldsymbol{p}_b \boldsymbol{p}_a} \right|^2$$

In the 1st order approximation, (2.739) becomes

$$\left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a \right\rangle = i \frac{p}{M} \lim_{t_b - t_a \to \infty} e^{i \boldsymbol{q}^2 t_b / 2 M \hbar} \int d\boldsymbol{b} \, e^{-i \boldsymbol{q} \cdot \boldsymbol{b} / \hbar} \times \int \mathcal{D} \boldsymbol{v} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} dt \frac{1}{2} M \boldsymbol{v}^2 \right) \left\{ -\frac{1}{\hbar} \int_{t_a}^{t_b} dt'' \, V\left(\boldsymbol{b} + \frac{\boldsymbol{p}}{M} t'' - \int_{t''}^{t_b} dt' \, \boldsymbol{v}(t') \right) \right\} \tag{2.740b}$$

Introducing the Fourier transform

$$V(\boldsymbol{y}) = \int \frac{d\boldsymbol{Q}}{(2 \pi \hbar)^d} e^{i \boldsymbol{Q} \cdot \boldsymbol{y} / \hbar} V(\boldsymbol{Q})$$

we have

$$V\left(\boldsymbol{b} + \frac{\boldsymbol{p}}{M} t'' - \int_{t''}^{t_b} dt' \, \boldsymbol{v}(t') \right) = \int \frac{d\boldsymbol{Q}}{(2 \pi \hbar)^d} e^{i \boldsymbol{Q} \cdot \left(\boldsymbol{b} + \frac{\boldsymbol{p}}{M} t'' \right) / \hbar} V(\boldsymbol{Q}) \times \exp\left[-\frac{i}{\hbar} \int_{t_a}^{t_b} dt \, \theta(t - t'') \, \boldsymbol{Q} \cdot \boldsymbol{v}(t) \right]$$

The velocity path integral portion of (2.740b) thus becomes

$$\mathcal{P} = \int \mathcal{D} \boldsymbol{v} \exp\left[\frac{i}{\hbar} \int_{t_a}^{t_b} dt \left(\frac{1}{2} M \boldsymbol{v}^2 - \theta(t - t'') \, \boldsymbol{Q} \cdot \boldsymbol{v} \right) \right]$$

which is easily evaluated using

$$\int \mathcal{D}\,\boldsymbol{v}\,\exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,\frac{1}{2}\,M\,\boldsymbol{v}^2\right] = 1 \tag{2.718}$$

to give

$$\mathcal{P} = \exp\left\{-\frac{i}{\hbar}\frac{1}{2\,M}\int_{t_a}^{t_b} d\,t\,[\theta(t-t'')]^2\,\boldsymbol{Q}^2\right\}$$
$$= \exp\left[-\frac{i}{\hbar}\frac{\boldsymbol{Q}^2}{2\,M}\,(\,t_b - t''\,)\right] \tag{2.741}$$

The $\boldsymbol{b}$-integral in (2.740b) thus evaluates to $(2\,\pi\,\hbar)^d\,\delta(\boldsymbol{q}-\boldsymbol{Q})$ so that the $\boldsymbol{Q}$-integral becomes trivial, giving us

$$\left\langle \boldsymbol{p}_b\mid\hat{T}\mid\boldsymbol{p}_a\right\rangle = -\frac{i}{\hbar}\frac{p}{M}\lim_{t_b-t_a\to\infty} V(\boldsymbol{q})\int_{t_a}^{t_b} d\,t''\,\exp\left\{\frac{i}{\hbar}\left(\boldsymbol{q}\cdot\frac{\boldsymbol{p}}{M}+\frac{q^2}{2\,M}\right)t''\right\}$$
$$= -2\,\pi\,i\,\frac{p}{M}\,V(\boldsymbol{q})\,\delta\,(E_b - E_a) \qquad \text{[(2.734c) used.]} \tag{2.741a}$$

2 .22.2. Improved Formulation

Using (2.741), we can write

$$\exp\left(\frac{i}{\hbar}\frac{\boldsymbol{q}^2}{2\,M}\,t_b\right) = \int \mathcal{D}\,\boldsymbol{w}\,\exp\left\{-\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\left(\frac{1}{2}\,M\,\boldsymbol{w}^2 - \theta(t)\,\boldsymbol{q}\cdot\boldsymbol{w}\right)\right\} \tag{2.742}$$

Combining (2.740b) and (2.739) then gives

$$f_{\boldsymbol{p}_b\,\boldsymbol{p}_a} = -i\,\frac{p}{2\,\pi\,\hbar}\lim_{t_b-t_a\to\infty}\int \mathcal{D}\,\boldsymbol{w}\int d\,\boldsymbol{b}\,\exp\left[-\frac{i}{\hbar}\,\boldsymbol{q}\cdot\left(\boldsymbol{b}+\int_{t_a}^{t_b} d\,t\,\theta(t)\,\boldsymbol{w}\right)\right] \tag{2.743a}$$
$$\times\int \mathcal{D}\,\boldsymbol{v}\,\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,\frac{1}{2}\,M\left(\boldsymbol{v}^2 - \boldsymbol{w}^2\right)\right)\left\{\exp(\,i\,\chi_{\boldsymbol{b},\boldsymbol{p}}[\boldsymbol{v}]\,) - 1\right\}$$

Setting

$$b_{\boldsymbol{w}} = \boldsymbol{b} - \int_{t_a}^{t_b} d\,t\,\theta(t)\,\boldsymbol{w} \tag{2.743b}$$

gives

$$f_{\boldsymbol{p}_b\,\boldsymbol{p}_a} = -i\,\frac{p}{2\,\pi\,\hbar}\lim_{t_b-t_a\to\infty}\int d\,\boldsymbol{b}_{\boldsymbol{w}}\,\exp\left(-\frac{i}{\hbar}\,\boldsymbol{q}\cdot\boldsymbol{b}_{\boldsymbol{w}}\right)\int \mathcal{D}\,\boldsymbol{w} \tag{2.743}$$
$$\times\int \mathcal{D}\,\boldsymbol{v}\,\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,\frac{1}{2}\,M\left(\boldsymbol{v}^2 - \boldsymbol{w}^2\right)\right)\left\{\exp\left(\,i\,\tilde{\chi}_{\boldsymbol{b}_w,\boldsymbol{p}}[\boldsymbol{v}]\right) - 1\right\}$$

where

$$\tilde{\chi}_{\boldsymbol{b}_{\boldsymbol{w}},\boldsymbol{p}}[\boldsymbol{v}] = \chi_{\boldsymbol{b},\boldsymbol{p}}[\boldsymbol{v}]\Big|_{\boldsymbol{b}=\boldsymbol{b}_{\boldsymbol{w}}+\int d\,t'\,\theta(t')\,\boldsymbol{w}}$$
$$= -\frac{1}{\hbar}\int_{t_a}^{t_b} d\,t\,V\left(\boldsymbol{b}_{\boldsymbol{w}} + \frac{\boldsymbol{p}}{M}\,t - \int_{t_a}^{t_b} d\,t'\left[\,\theta(t'-t)\,\boldsymbol{v}(t') - \theta(t')\,\boldsymbol{w}(t')\,\right]\right) \tag{2.744}$$

One can go back to the conventional path integrals by setting

$$\boldsymbol{y}(t) = -\int_t^{t_b} d\,t'\,\boldsymbol{v}(t') \qquad\qquad \boldsymbol{z}(t) = -\int_t^{t_b} d\,t'\,\boldsymbol{w}(t')$$

and turns (2.743) into

$$f_{p_b\,p_a} = -i\,\frac{p}{2\,\pi\,\hbar}\lim_{t_b-t_a\to\infty}\int d\,\boldsymbol{b}\;\exp\left(-\frac{i}{\hbar}\,\boldsymbol{q}\cdot\boldsymbol{b}\right)\;\int d\,\boldsymbol{y}_a\int d\,\boldsymbol{z}_a \tag{2.745}$$
$$\times\int\mathcal{D}\boldsymbol{z}\int\mathcal{D}\boldsymbol{y}\;\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,\frac{1}{2}\,M\left(\dot{\boldsymbol{y}}^2-\dot{\boldsymbol{z}}^2\right)\right)\left\{\exp\left(i\,\tilde{\chi}_{\boldsymbol{b}_z,\boldsymbol{p}}[\boldsymbol{y}]\right)-1\right\}$$

where we have dropped the subscript $\boldsymbol{w}$ from the dummy variable $\boldsymbol{b}_{\boldsymbol{w}}$. Furthermore, using

$$\int_{t_a}^{t_b} d\,t'\left[\,\theta\,(t'-t)\,\boldsymbol{v}(t')-\theta(t')\,\boldsymbol{w}\,(t')\,\right] = \int_{t}^{t_b} d\,t'\,\dot{\boldsymbol{y}}(t') - \int_{0}^{t_b} d\,t'\,\dot{\boldsymbol{z}}(t')$$
$$= \boldsymbol{y}_b-\boldsymbol{y}(t)-z_b+\boldsymbol{z}(0)$$

we have, for paths with $\boldsymbol{y}_b = \boldsymbol{z}_b = 0$,

$$\tilde{\chi}_{\boldsymbol{b}_z,\boldsymbol{p}}[\boldsymbol{y}] = -\frac{1}{\hbar}\int_{t_a}^{t_b} d\,t\;V\!\left(\boldsymbol{b}+\frac{\boldsymbol{p}}{M}\,t+\boldsymbol{y}(t)-\boldsymbol{z}(0)\right) \tag{2.746}$$

2.22.3. Eikonal Approximation to the Scattering Amplitude

Setting $\boldsymbol{Q} = 0$ in (2.731a-b), we get

$$\int d\,\boldsymbol{y}_a\int\mathcal{D}\,\boldsymbol{y}\,\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,\frac{1}{2}\,M\,\dot{\boldsymbol{y}}^2\right)$$
$$= \left(\frac{M}{2\,\pi\,\hbar\,i\,(\,t_b-t_a\,)}\right)^{d/2}\int d\,\boldsymbol{y}_a\,\exp\left[\frac{i}{\hbar}\,\frac{M\,(\,\boldsymbol{y}_b-\boldsymbol{y}_a\,)^2}{2\,(\,t_b-t_a\,)}\right]$$
$$= 1 \tag{2.747}$$

Thus, if we ignore the effects of the fluctuations $\boldsymbol{y}$ & $\boldsymbol{z}$ on V and set

$$\tilde{\chi}_{\boldsymbol{b}_z,\boldsymbol{p}}[\boldsymbol{y}] \approx \tilde{\chi}^{\text{ei}}_{\boldsymbol{b},\boldsymbol{p}} = -\frac{1}{\hbar}\int_{t_a}^{t_b} d\,t\;V\!\left(\boldsymbol{b}+\frac{\boldsymbol{p}}{M}\,t\right) \tag{2.749}$$

(2.745) reduces to

$$f_{p_b\,p_a} = -i\,\frac{p}{2\,\pi\,\hbar}\lim_{t_b-t_a\to\infty}\int d\,\boldsymbol{b}\;\exp\left(-\frac{i}{\hbar}\,\boldsymbol{q}\cdot\boldsymbol{b}\right)\left\{\exp\left(i\,\tilde{\chi}^{\text{ei}}_{\boldsymbol{b},\boldsymbol{p}}\right)-1\right\} \tag{2.748}$$

Introducing

$$z = \frac{p}{M}\,t \to \qquad d\,z = \frac{p}{M}\,d\,t$$

(2.749) becomes

$$\tilde{\chi}^{\text{ei}}_{\boldsymbol{b},\boldsymbol{p}} = -\frac{M}{p\,\hbar}\int_{-\infty}^{\infty} d\,z\;V(\boldsymbol{b}+z\,\hat{\boldsymbol{p}}) \tag{2.750}$$

For a potential with rotational symmetry, $V(\boldsymbol{x}) = V(r)$, where $r = |\,\boldsymbol{x}\,|$. Since $\boldsymbol{b}\cdot\boldsymbol{p} = 0$ [see (2.738)], (2.750) reduces to

$$\tilde{\chi}^{\text{ei}}_{\boldsymbol{b},\boldsymbol{p}} = -\frac{M}{p\,\hbar}\int_{-\infty}^{\infty} d\,z\;V\!\left(\sqrt{b^2+z^2}\right) \tag{2.751}$$

Using

$$\frac{1}{2\,\pi}\int_0^{2\,\pi} d\,\theta\;\exp(\,-i\,a\cos\theta\,) = J_0(a) \tag{2.752}$$

where J_0 is a Bessel function, (2.748) becomes

$$f_{p_b\,p_a} = -i\,\frac{p}{2\,\pi\,\hbar}\int_0^{\infty} b\,d\,b\int_0^{2\,\pi} d\,\theta\;\exp\left(-\frac{i}{\hbar}\,q\,b\cos\theta\right)\left\{\exp\left(i\,\tilde{\chi}^{\text{ei}}_{\boldsymbol{b},\boldsymbol{p}}\right)-1\right\}$$
$$= -i\,\frac{p}{\hbar}\int_0^{\infty} b\,d\,b\,J_0\!\left(\frac{q\,b}{\hbar}\right)\left\{\exp\left(i\,\tilde{\chi}^{\text{ei}}_{\boldsymbol{b},\boldsymbol{p}}\right)-1\right\} \tag{2.753}$$

Comparing with the eikonal approximation (1.497), we see that b is just the impact parameter and

$$\tilde{\chi}^{\text{ei}}_{\boldsymbol{b},\boldsymbol{p}} = 2\,\delta_l(p) \qquad l = \frac{p\,b}{\hbar} \tag{2.754}$$