2.23. Heisenberg Operator Approach to Time Evolution Amplitude

2.23.1. Free Particle

Consider the matrix elements of the evolution operator

$$(\boldsymbol{x}\,t \mid \boldsymbol{x}'\,0) = \left\langle \boldsymbol{x} \left| e^{-i\hat{H}t/\hbar} \right| \boldsymbol{x}' \right\rangle \tag{2.755}$$

where

$$\hat{H} = H(\hat{\boldsymbol{p}}) = \frac{\hat{\boldsymbol{p}}^2}{2M} \tag{2.756}$$

Taking the time derivatives on both sides, (2.755) becomes

$$\begin{aligned} i\hbar\partial_t (\boldsymbol{x}\,t \mid \boldsymbol{x}'\,0) &= \left\langle \boldsymbol{x} \left| \hat{H} e^{-i\hat{H}t/\hbar} \right| \boldsymbol{x}' \right\rangle \\ &= \left\langle \boldsymbol{x} \left| e^{-i\hat{H}t/\hbar} e^{i\hat{H}t/\hbar} \hat{H} e^{-i\hat{H}t/\hbar} \right| \boldsymbol{x}' \right\rangle \\ &= \left\langle \boldsymbol{x}\,t \left| e^{i\hat{H}t/\hbar} \hat{H} e^{-i\hat{H}t/\hbar} \right| \boldsymbol{x}'\,0 \right\rangle \qquad \mid t\,\rangle = e^{i\hat{H}t/\hbar} \left| 0 \right\rangle \\ &= \left\langle \boldsymbol{x}\,t \left| \hat{H}(t) \right| \boldsymbol{x}'\,0 \right\rangle \qquad \hat{O}(t) = e^{i\hat{H}t/\hbar}\,\hat{O}\,e^{-i\hat{H}t/\hbar} \\ &= \left\langle \boldsymbol{x}\,t \mid H\left[\hat{\boldsymbol{p}}(t)\right] \mid \boldsymbol{x}'\,0 \right\rangle \end{aligned} \tag{2.757}$$

Now,

$$\hat{\boldsymbol{x}}(t) \mid \boldsymbol{x}\,t\,\rangle = e^{i\hat{H}t/\hbar}\hat{\boldsymbol{x}}\,e^{-i\hat{H}t/\hbar} \left| \boldsymbol{x}\,t \right\rangle = e^{i\hat{H}t/\hbar}\hat{\boldsymbol{x}} \left| \boldsymbol{x} \right\rangle = e^{i\hat{H}t/\hbar} \left| \boldsymbol{x} \right\rangle \boldsymbol{x} = \mid \boldsymbol{x}\,t\,\rangle \boldsymbol{x}$$

$$\rightarrow \qquad \left\langle \boldsymbol{x}\,t \mid \hat{\boldsymbol{x}}(t) = \boldsymbol{x}\,\langle\, \boldsymbol{x}\,t \mid \right. \tag{2.759}$$

$$\hat{\boldsymbol{x}}(0) \mid \boldsymbol{x}'\,0\,\rangle = \hat{\boldsymbol{x}} \mid \boldsymbol{x}'\,\rangle = \boldsymbol{x}' \mid \boldsymbol{x}'\,\rangle = \boldsymbol{x}' \mid \boldsymbol{x}'\,0\,\rangle \tag{2.759a}$$

Therefore, if we can express $H\left[\hat{\boldsymbol{p}}(t)\right]$ in a time-ordered form

$$\hat{H} = H\left[\hat{\boldsymbol{x}}(t),\ \hat{\boldsymbol{x}}(0)\,;\ t\right] \tag{2.758}$$

in which all $\hat{\boldsymbol{x}}(t)$'s stand to the left of all $\hat{\boldsymbol{x}}(0)$'s, then the R.H.S. of (2.757) can be evaluated using (2.759-a) as

$$\left\langle \boldsymbol{x}\,t \mid H\left[\hat{\boldsymbol{x}}(t),\ \hat{\boldsymbol{x}}(0)\,;\ t\right] \mid \boldsymbol{x}'\,0 \right\rangle = H(\boldsymbol{x},\ \boldsymbol{x}'\,;\ t)\,\langle\, \boldsymbol{x}\,t \mid \boldsymbol{x}'\,0\,\rangle \tag{2.760}$$

This turns (2.757) into a differential eq.

$$i\hbar\partial_t (\boldsymbol{x}\,t \mid \boldsymbol{x}'\,0) = H(\boldsymbol{x},\ \boldsymbol{x}'\,;\ t)\,\langle\, \boldsymbol{x}\,t \mid \boldsymbol{x}'\,0\,\rangle \tag{2.761}$$

with solution

$$\begin{aligned} (\boldsymbol{x}\,t \mid \boldsymbol{x}'\,0) &= C(\boldsymbol{x},\ \boldsymbol{x}')\,E(\boldsymbol{x},\ \boldsymbol{x}'\,;t) \\ &= C(\boldsymbol{x},\ \boldsymbol{x}')\exp\left[-\frac{i}{\hbar}\int^t dt'\,H(\boldsymbol{x},\ \boldsymbol{x}'\,;\ t')\right] \end{aligned} \tag{2.762}$$

where $C(\boldsymbol{x},\ \boldsymbol{x}')$ is an integration constant.

The time-ordered form (2.758) can be obtained by solving the Heisenberg eqs. of motion

$$\frac{d\,\hat{\boldsymbol{x}}(t)}{dt} = \frac{1}{i\hbar}\left[\hat{\boldsymbol{x}}(t)\,,\ \hat{H}\right] = \frac{1}{i\hbar}e^{i\hat{H}t/\hbar}\left[\hat{\boldsymbol{x}},\ \frac{\hat{\boldsymbol{p}}^2}{2M}\right]e^{-i\hat{H}t/\hbar} = \frac{\hat{\boldsymbol{p}}(t)}{M} \tag{2.763}$$

$$\frac{d\,\hat{\boldsymbol{p}}(t)}{dt} = \frac{1}{i\hbar}\left[\hat{\boldsymbol{p}}(t)\,,\ \hat{H}\right] = \frac{1}{i\hbar}e^{i\hat{H}t/\hbar}\left[\hat{\boldsymbol{p}},\ \frac{\hat{\boldsymbol{p}}^2}{2M}\right]e^{-i\hat{H}t/\hbar} = 0 \tag{2.764}$$

(2.764) gives

$$\hat{\boldsymbol{p}}(t) = \hat{\boldsymbol{p}}(0) = \hat{\boldsymbol{p}} \tag{2.765}$$

so that the solution to (2.763) is simply

$$\hat{\boldsymbol{x}}(t) = \hat{\boldsymbol{x}}(0) + \frac{\hat{\boldsymbol{p}}}{M} t \tag{2.766}$$

(2.756) thus becomes

$$\hat{H} = \frac{1}{2} M \frac{\left[\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}(0)\right]^2}{t^2} \tag{2.767}$$

$$= \frac{M}{2t^2}\left[\hat{\boldsymbol{x}}(t)^2 - \hat{\boldsymbol{x}}(t)\cdot\hat{\boldsymbol{x}}(0) - \hat{\boldsymbol{x}}(t)\cdot\hat{\boldsymbol{x}}(0) + \hat{\boldsymbol{x}}(0)^2\right]$$

$$= \frac{M}{2t^2}\left\{\hat{\boldsymbol{x}}(t)^2 - 2\,\hat{\boldsymbol{x}}(t)\cdot\hat{\boldsymbol{x}}(0) + \hat{\boldsymbol{x}}(0)^2 + \sum_{j=1}^{D}\left[\hat{x}_j(t),\ \hat{x}_j(0)\right]\right\} \tag{2.768}$$

Using (2.766), we have

$$\left[\hat{x}_i(t),\ \hat{x}_j(0)\right] = \left[\hat{x}_i + \frac{\hat{p}_j}{M} t,\ \hat{x}_j\right] = -i\frac{\hbar t}{M}\delta_{ij} \tag{2.769a}$$

$$\rightarrow \qquad \sum_{j=1}^{D}\left[\hat{x}_j(t),\ \hat{x}_j(0)\right] = -i\frac{\hbar t}{M} D \tag{2.769}$$

(2.768) thus takes the time-ordered form

$$H\left[\hat{\boldsymbol{x}}(t),\ \hat{\boldsymbol{x}}(0);\ t\right] = \frac{M}{2t^2}\left\{\hat{\boldsymbol{x}}(t)^2 - 2\,\hat{\boldsymbol{x}}(t)\cdot\hat{\boldsymbol{x}}(0) + \hat{\boldsymbol{x}}(0)^2\right\} - i\hbar\frac{D}{2t} \tag{2.770}$$

According to (2.760), we have

$$H(\boldsymbol{x},\ \boldsymbol{x}';\ t) = H\left[\hat{\boldsymbol{x}}(t),\ \hat{\boldsymbol{x}}(0);\ t\right]\Big|_{\hat{\boldsymbol{x}}(t)\to\boldsymbol{x},\ \hat{\boldsymbol{x}}(0)\to\boldsymbol{x}'}$$

$$= \frac{M}{2t^2}\left(\boldsymbol{x}^2 - 2\,\boldsymbol{x}\cdot\boldsymbol{x}' + \boldsymbol{x}'^2\right) - i\hbar\frac{D}{2t}$$

$$= \frac{M}{2t^2}\left(\boldsymbol{x} - \boldsymbol{x}'\right)^2 - i\hbar\frac{D}{2t} \tag{2.771}$$

The exponential factor in (2.762) thus becomes

$$E(\boldsymbol{x},\ \boldsymbol{x}';\ t) = \exp\left[-\frac{i}{\hbar}\int^t dt'\, H(\boldsymbol{x},\ \boldsymbol{x}';\ t')\right]$$

$$= \exp\left\{-\frac{i}{\hbar}\int^t dt'\left[\frac{M}{2t'^2}\left(\boldsymbol{x} - \boldsymbol{x}'\right)^2 - i\hbar\frac{D}{2t'}\right]\right\}$$

$$= \exp\left\{-\frac{i}{\hbar}\left[-\frac{M}{2t}\left(\boldsymbol{x} - \boldsymbol{x}'\right)^2 - i\hbar\frac{D}{2}\ln t\right]\right\}$$

$$= \exp\left\{\frac{i}{\hbar}\frac{M}{2t}\left(\boldsymbol{x} - \boldsymbol{x}'\right)^2 - \frac{D}{2}\ln t\right\} \tag{2.772}$$

$$= \frac{1}{t^{D/2}}\exp\left[\frac{i}{\hbar}\frac{M}{2t}\left(\boldsymbol{x} - \boldsymbol{x}'\right)^2\right]$$

which turns (2.762) into

$$(\boldsymbol{x}\,t \mid \boldsymbol{x}'\,0) = C(\boldsymbol{x},\ \boldsymbol{x}')\,\frac{1}{t^{D/2}}\exp\left[\frac{i}{\hbar}\frac{M}{2t}\left(\boldsymbol{x} - \boldsymbol{x}'\right)^2\right] \tag{2.773}$$

Note that rigorously, there should be an integration constant from the t'-integral in (2.772). We shall simply merge it into $C(\boldsymbol{x},\ \boldsymbol{x}')$.

$C(\boldsymbol{x},\ \boldsymbol{x}')$ can be determined by considering the following differential eqs. obtained by spatial differentiation of (2.755):

$$\frac{\hbar}{i}\nabla\,(\boldsymbol{x}\,t \mid \boldsymbol{x}'\,0) = \frac{\hbar}{i}\nabla\left\langle \boldsymbol{x}\ \middle|\ e^{-i\hat{H}t/\hbar}\ \middle|\ \boldsymbol{x}'\right\rangle$$

$$= \left\langle \boldsymbol{x} \,\middle|\, \hat{\boldsymbol{p}}\, e^{-i\hat{H}t/\hbar} \,\middle|\, \boldsymbol{x}' \right\rangle \qquad \frac{\hbar}{i}\nabla\langle \boldsymbol{x} \mid \phi\rangle = \left\langle \boldsymbol{x} \mid \hat{\boldsymbol{p}} \mid \phi \right\rangle$$

$$= \left\langle \boldsymbol{x} \,\middle|\, e^{-i\hat{H}t/\hbar} e^{i\hat{H}t/\hbar} \hat{\boldsymbol{p}}\, e^{-i\hat{H}t/\hbar} \,\middle|\, \boldsymbol{x}' \right\rangle$$

$$= \left\langle \boldsymbol{x}\, t \mid \hat{\boldsymbol{p}}(t) \mid \boldsymbol{x}'\, 0 \right\rangle$$

$$\frac{\hbar}{i}\nabla'(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = \frac{\hbar}{i}\nabla'\left\langle \boldsymbol{x} \,\middle|\, e^{-i\hat{H}t/\hbar} \,\middle|\, \boldsymbol{x}' \right\rangle = \left(-\frac{\hbar}{i}\nabla'\left\langle \boldsymbol{x}' \,\middle|\, e^{i\hat{H}t/\hbar} \,\middle|\, \boldsymbol{x} \right\rangle \right)^*$$

$$= \left(-\left\langle \boldsymbol{x}' \,\middle|\, \hat{\boldsymbol{p}}\, e^{i\hat{H}t/\hbar} \,\middle|\, \boldsymbol{x} \right\rangle \right)^*$$

$$= -\left\langle \boldsymbol{x} \,\middle|\, e^{-i\hat{H}t/\hbar} \hat{\boldsymbol{p}} \,\middle|\, \boldsymbol{x}' \right\rangle$$

$$= -\left\langle \boldsymbol{x}\, t \mid \hat{\boldsymbol{p}} \mid \boldsymbol{x}'\, 0 \right\rangle \tag{2.774}$$

Replacing $\hat{\boldsymbol{p}}$ with (2.766), we have

$$\frac{\hbar}{i}\nabla(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = \left\langle \boldsymbol{x}\, t \,\middle|\, M\frac{\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}}{t} \,\middle|\, \boldsymbol{x}'\, 0 \right\rangle$$

$$= \frac{M}{t}\left\langle \boldsymbol{x}\, t \,\middle|\, \left(e^{i\hat{H}t/\hbar} \hat{\boldsymbol{x}}\, e^{-i\hat{H}t/\hbar} - \boldsymbol{x}' \right) \,\middle|\, \boldsymbol{x}'\, 0 \right\rangle$$

$$= \frac{M}{t}(\boldsymbol{x} - \boldsymbol{x}')(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0)$$

$$\frac{\hbar}{i}\nabla'(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = -\left\langle \boldsymbol{x}\, t \,\middle|\, M\frac{\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}}{t} \,\middle|\, \boldsymbol{x}'\, 0 \right\rangle$$

$$= -\frac{M}{t}(\boldsymbol{x} - \boldsymbol{x}')(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) \tag{2.775}$$

Differentiating (2.773), we have

$$\frac{\hbar}{i}\nabla(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = \left\{ \frac{\hbar}{i}\frac{\nabla C}{C} + \frac{M}{t}\left(\boldsymbol{x} - \boldsymbol{x}'\right) \right\}(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0)$$

$$\frac{\hbar}{i}\nabla'(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = \left\{ \frac{\hbar}{i}\frac{\nabla' C}{C} - \frac{M}{t}\left(\boldsymbol{x} - \boldsymbol{x}'\right) \right\}(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0)$$

Comparing with (2.775) then gives

$$\nabla C = 0 \qquad \text{and} \qquad \nabla' C = 0 \tag{2.776}$$

so that C is a true constant independent of $\boldsymbol{x}$ and $\boldsymbol{x}'$. Its value is fixed by the I.C.

$$\lim_{t\to 0}(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = \langle \boldsymbol{x} \mid \boldsymbol{x}' \rangle = \delta(\boldsymbol{x} - \boldsymbol{x}') \tag{2.777}$$

Using

$$\delta(\boldsymbol{x} - \boldsymbol{x}') = \lim_{a\to 0} \frac{1}{(\pi a)^{D/2}} e^{-\left(\boldsymbol{x}-\boldsymbol{x}'\right)^2/a}$$

we can write (2.773) as

$$\lim_{t\to 0}(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = \lim_{t\to 0} C\frac{1}{t^{D/2}}\exp\left[\frac{i}{\hbar}\frac{M}{2t}\left(\boldsymbol{x} - \boldsymbol{x}'\right)^2 \right]$$

$$= C\left(\frac{i2\pi\hbar}{M}\right)^{D/2} \delta(\boldsymbol{x} - \boldsymbol{x}')$$

Comparing with (2.777) then gives

$$C = \left(\frac{M}{i2\pi\hbar}\right)^{D/2} \tag{2.778}$$

(2.773) thus becomes

$$(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = \left(\frac{M}{i\, 2\, \pi\, \hbar\, t} \right)^{D/2} \exp\left[\frac{i}{\hbar} \frac{M}{2\, t} \left(\boldsymbol{x} - \boldsymbol{x}' \right)^2 \right] \tag{2.779}$$

in agreement with the previous result (2.130).

Note that in this approach, the fluctuation factor $t^{-D/2}$ arises from the commutator relation (2.769).

2.23.2. Harmonic Oscillator

The Hamiltonian

$$\hat{H} = H(\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}) = \frac{\hat{\boldsymbol{p}}^2}{2\, M} + \frac{1}{2} M\, \omega^2\, \hat{\boldsymbol{x}}^2 \tag{2.780}$$

gives rise to eqs. of motion

$$\frac{d\, \hat{\boldsymbol{x}}(t)}{d\, t} = \frac{1}{i\, \hbar} \left[\hat{\boldsymbol{x}}(t), \frac{\hat{\boldsymbol{p}}(t)^2}{2\, M} \right] = \frac{\hat{\boldsymbol{p}}(t)}{M} \tag{2.781}$$

$$\frac{d\, \hat{\boldsymbol{p}}(t)}{d\, t} = \frac{1}{i\, \hbar} \left[\hat{\boldsymbol{p}}(t), \frac{1}{2} M\, \omega^2\, \hat{\boldsymbol{x}}(t)^2 \right] = -M\, \omega^2\, \hat{\boldsymbol{x}}(t) \tag{2.782}$$

The solution is [c.f. (2.156)]

$$\hat{\boldsymbol{p}}(t) = M \frac{d\, \hat{\boldsymbol{x}}(t)}{d\, t} = \frac{M\, \omega}{\sin \omega\, t} \left[\hat{\boldsymbol{x}}(t) \cos \omega\, t - \hat{\boldsymbol{x}}(0) \right] \tag{2.783}$$

To verify this claim, we differentiate (2.783) to get

$$\begin{aligned} \frac{d\, \hat{\boldsymbol{p}}(t)}{d\, t} &= -\frac{M\, \omega^2 \cos \omega\, t}{\sin^2 \omega\, t} \left[\hat{\boldsymbol{x}}(t) \cos \omega\, t - \hat{\boldsymbol{x}}(0) \right] + \frac{M\, \omega}{\sin \omega\, t} \left[\frac{d\, \hat{\boldsymbol{x}}(t)}{d\, t} \cos \omega\, t - \omega\, \hat{\boldsymbol{x}}(t) \sin \omega\, t \right] \\ &= -\frac{M\, \omega^2 \cos \omega\, t}{\sin^2 \omega\, t} \left[\hat{\boldsymbol{x}}(t) \cos \omega\, t - \hat{\boldsymbol{x}}(0) \right] \\ &\quad + \frac{M\, \omega}{\sin \omega\, t} \left\{ \frac{\omega}{\sin \omega\, t} \left[\hat{\boldsymbol{x}}(t) \cos \omega\, t - \hat{\boldsymbol{x}}(0) \right] \cos \omega\, t - \omega\, \hat{\boldsymbol{x}}(t) \sin \omega\, t \right\} \\ &= -M\, \omega^2\, \hat{\boldsymbol{x}}(t) \end{aligned}$$

so that (2.783) is indeed a solution of (2.782).

(2.780) then becomes

$$\hat{H} = \frac{M\, \omega^2}{2 \sin^2 \omega\, t} \left\{ \left[\hat{\boldsymbol{x}}(t) \cos \omega\, t - \hat{\boldsymbol{x}}(0) \right]^2 + \sin^2 \omega\, t\, \hat{\boldsymbol{x}}(t)^2 \right\} \tag{2.784}$$

$$= \frac{M\, \omega^2}{2 \sin^2 \omega\, t} \left\{ \hat{\boldsymbol{x}}(t)^2 - 2\, \hat{\boldsymbol{x}}(t) \cdot \hat{\boldsymbol{x}}(0) \cos \omega\, t + \hat{\boldsymbol{x}}(0)^2 + \cos \omega\, t \sum_{j=1}^{D} \left[\hat{x}_j(t), \hat{x}_j(0) \right] \right\} \tag{2.785}$$

From (2.783), we have

$$\begin{aligned} \left[\hat{x}_i(t), \hat{p}_j(t) \right] &= i\, \hbar\, \delta_{ij} \\ &= \left[\hat{x}_i(t), \frac{M\, \omega}{\sin \omega\, t} \left\{ \hat{x}_j(t) \cos \omega\, t - \hat{x}_j(0) \right\} \right] \\ &= -\frac{M\, \omega}{\sin \omega\, t} \left[\hat{x}_i(t), \hat{x}_j(0) \right] \end{aligned} \tag{2.786a}$$

so that

$$\sum_{j=1}^{D} \left[\hat{x}_j(t), \hat{x}_j(0) \right] = -\frac{i\, \hbar \sin \omega\, t}{M\, \omega} D \tag{2.786}$$

(2.785) thus implies [c.f.(2.771)]

$$H(\mathbf{x},\mathbf{x}';t)=\frac{M\omega^2}{2\sin^2\omega t}\left(\mathbf{x}^2-2\,\mathbf{x}\cdot\mathbf{x}'\cos\omega t+\mathbf{x}'^2-\frac{i\hbar\sin\omega t\cos\omega t}{M\omega}D\right)$$
$$=\frac{M\omega^2}{2\sin^2\omega t}\left(\mathbf{x}^2+\mathbf{x}'^2-2\,\mathbf{x}\cdot\mathbf{x}'\cos\omega t\right)-\frac{i\hbar\omega\cot\omega t}{2}D \qquad (2.787)$$

Using

$$\int dt\,\frac{1}{\sin^2\omega t}=-\frac{\cot\omega t}{\omega} \qquad \int dt\,\frac{\cos\omega t}{\sin^2\omega t}=-\frac{1}{\omega\sin\omega t}$$
$$\int dt\cot\omega t=\frac{1}{\omega}\ln(\sin\omega t)$$

we get

$$\int dt\,H(\mathbf{x},\mathbf{x}';t)=-\frac{M\omega}{2\sin\omega t}\left[\left(\mathbf{x}^2+\mathbf{x}'^2\right)\cos\omega t-2\,\mathbf{x}\cdot\mathbf{x}'\right]-\frac{i\hbar}{2}D\ln(\sin\omega t) \qquad (2.788)$$

Inserting this into (2.762) gives a $(\mathbf{x}\,t\,|\,\mathbf{x}'\,0)$ that agrees with (2.175) aside from the prefactor $C(\mathbf{x},\mathbf{x}')$, which can be determined from differential eqs similiar to (2.774).

2.23.3. Charged Particle in Magnetic Field

The Hamiltonian is

$$\hat{H}=\frac{1}{2M}\hat{\mathbf{P}}^2 \qquad (2.790)$$

where

$$\hat{\mathbf{P}}=\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A}(\hat{\mathbf{x}}) \qquad (2.789)$$

is the covariant momentum with non-commuting components:

$$[P_i,\,P_j]=\left[p_i,\,-\frac{e}{c}A_j\right]+\left[-\frac{e}{c}A_i,\,p_j\right]$$
$$=i\frac{e\hbar}{c}(\partial_i A_j-\partial_j A_i)$$
$$=i\frac{e\hbar}{c}B_{ij} \qquad (2.791)$$

where

$$B_{ij}=\epsilon_{ijk}B_k=\begin{pmatrix}0 & B_3 & -B_2\\ -B_3 & 0 & B_1\\ B_2 & -B_1 & 0\end{pmatrix}=\mathbb{B} \qquad (2.791a)$$

is the anti-symmetric tensor representation of the magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$.

The eqs. of motion are

$$\frac{d\,\hat{\mathbf{x}}(t)}{dt}=\frac{1}{i\hbar}\left[\hat{\mathbf{x}}(t),\,\frac{\hat{\mathbf{P}}(t)^2}{2M}\right]=\frac{\hat{\mathbf{P}}(t)}{M} \qquad (2.792)$$
$$\frac{d\,\hat{P}_i(t)}{dt}=\frac{1}{i\hbar}\left[\hat{P}_i(t),\,\frac{\hat{P}_j(t)\,\hat{P}_j(t)}{2M}\right]$$
$$=\frac{1}{2i\hbar M}\left(\left[\hat{P}_i(t),\,\hat{P}_j(t)\right]\hat{P}_j(t)+\hat{P}_j(t)\left[\hat{P}_i(t),\,\hat{P}_j(t)\right]\right)$$
$$=\frac{e}{2Mc}\left(B_{ij}\,\hat{P}_j(t)+\hat{P}_j(t)\,B_{ij}\right) \qquad (2.793a)$$

Using

$$\hat{P}_j(t)\,B_{ij}\,\psi=\frac{\hbar}{i}\partial_j(B_{ij}\,\psi)-\frac{e}{c}A_j\,B_{ij}\,\psi=B_{ij}\left(\frac{\hbar}{i}\partial_j-\frac{e}{c}A_j\right)\psi+\frac{\hbar}{i}(\partial_j B_{ij})\,\psi$$

$$= B_{ij}\,\hat{P}_j(t)\ \psi + \frac{\hbar}{i}\,(\partial_j B_{ij})\ \psi$$

(2.793a) becomes

$$\frac{d\,\hat{P}_i(t)}{d\,t} = \frac{e}{M\,c}\,B_{ij}\,\hat{P}_j(t)\ + \frac{e\,\hbar}{2\,M\,c\,i}\,(\,\partial_j B_{ij}\,)$$
$$= \frac{e}{M\,c}\,B_{ij}\,\hat{P}_j(t)\ + i\,\frac{e\,\hbar}{2\,M\,c}\,(\,\partial_j B_{j\,i}\,) \qquad (\,B_{ij} = -B_{j\,i}\,) \qquad (2.793)$$

or in vector form

$$\frac{d\,\hat{\mathbf{P}}(t)}{d\,t} = \frac{e}{M\,c}\,\mathbb{B}\cdot\hat{\mathbf{P}}(t) + i\,\frac{e\,\hbar}{2\,M\,c}\,(\,\nabla\cdot\mathbb{B}\,) \qquad (2.793b)$$

In the following, we shall restrict ourselves to a constant field so that $\nabla\cdot\mathbb{B} = 0$ and (2.793) reduces to

$$\frac{d\,\hat{P}_i(t)}{d\,t} = \frac{e}{M\,c}\,B_{ij}\,\hat{P}_j(t)$$

with solution

$$\hat{P}_i(t) = \exp\!\left(\frac{e}{M\,c}\,B_{ij}\,t\right)\ \hat{P}_j(0)\ = e^{W_{L\,ij}\,t}\,\hat{P}_j(0) \qquad (2.794a)$$

where

$$W_{L\,ij} = \frac{e}{M\,c}\,B_{ij} \qquad (2.795)$$

are components of the Landau frequency matrix

$$\mathbb{W}_L = \frac{e}{M\,c}\,\mathbb{B} \qquad (2.795a)$$

Note: we've used $W_{L\,ij}$ instead of Kleinert's $\Omega_{L\,ij}$ because there is no double-stroke font for Ω.

(2.794a) can be written in vector form as

$$\hat{\mathbf{P}}(t) = e^{\mathbb{W}_L\,t}\cdot\hat{\mathbf{P}}(0) \qquad (2.794)$$

Using the Landau frequency vector

$$\boldsymbol{\omega}_L = \frac{e}{M\,c}\,\boldsymbol{B} \qquad (2.796)$$

and the 3-D generators $\mathbb{L}_k$ of the rotation group with components

$$(\,L_k)_{ij} = -i\,\epsilon_{k\,ij} = \mathbb{L}_k \qquad (2.797)$$

(2.795) becomes

$$W_{L\,ij} = \frac{e}{M\,c}\,\epsilon_{ij\,k}\,B_k\ = \frac{e}{M\,c}\,i\,(\,L_k)_{ij}\,B_k\ = i\,(\,L_k)_{ij}\,\omega_{L\,k} \qquad (2.797a)$$

or, in matrix form,

$$\mathbb{W}_L = i\,\mathbb{L}_k\,\omega_{L\,k} = i\,\mathbb{L}\cdot\boldsymbol{\omega}_L \qquad (2.798)$$

where $\mathbb{L}$ is a vector with components $\mathbb{L}_k$ that are themselves matrices with components $(\,L_k)_{ij}$.

Inserting (2.794) into (2.792) gives

$$\frac{d\,\hat{\mathbf{x}}(t)}{d\,t} = \frac{1}{M}\,e^{\mathbb{W}_L\,t}\cdot\hat{\mathbf{P}}(0) \qquad (2.798a)$$

Using

$$e^{\mathbb{W}_L\,t} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\,\mathbb{W}_L^n \qquad (2.798b)$$

we have

$$\frac{d\,e^{\mathbb{W}_L t}}{d\,t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!}\,\mathbb{W}_L^n = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,\mathbb{W}_L^{n+1} = \mathbb{W}_L \cdot e^{\mathbb{W}_L t} \tag{2.798c}$$

The solution of (2.798a) is then easily proved to be

$$\begin{aligned}\hat{\boldsymbol{x}}(t) &= \hat{\boldsymbol{x}}(0) + \frac{1}{M}\,\mathbb{W}_L^{-1} \cdot \left(e^{\mathbb{W}_L t} - \mathbb{I}\right) \cdot \hat{\boldsymbol{P}}(0) \\ &= \hat{\boldsymbol{x}}(0) + \frac{1}{M}\left(e^{\mathbb{W}_L t} - \mathbb{I}\right) \cdot \mathbb{W}_L^{-1} \cdot \hat{\boldsymbol{P}}(0) \\ &= \hat{\boldsymbol{x}}(0) + \frac{1}{M}\,\frac{e^{\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L} \cdot \hat{\boldsymbol{P}}(0)\end{aligned} \tag{2.799}$$

where $\mathbb{I}$ is the 3×3 unit matrix.

Using (2.798b), we have

$$\begin{aligned}\mathbb{W}_L^{-1} \cdot \left(e^{\mathbb{W}_L t} - 1\right) = \mathbb{W}_L^{-1} \cdot \left(\sum_{n=1}^{\infty} \frac{t^n}{n!}\,\mathbb{W}_L^n\right) &= \sum_{n=1}^{\infty} \frac{t^n}{n!}\,\mathbb{W}_L^{n-1} \\ &= t\,\mathbb{I} + \frac{t^2}{2!}\,\mathbb{W}_L + \frac{t^3}{3!}\,\mathbb{W}_L^2 + \ldots\end{aligned} \tag{2.800}$$

Since the terms involving $\mathbb{W}_L$ all commute among themselves, (2.799) can be inverted to give

$$\begin{aligned}\frac{\hat{\boldsymbol{P}}(0)}{M} &= \frac{\mathbb{W}_L}{e^{\mathbb{W}_L t} - \mathbb{I}} \cdot \left[\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}(0)\right] \\ &= \frac{\mathbb{W}_L}{e^{\mathbb{W}_L t/2}\left(e^{\mathbb{W}_L t/2} - e^{-\mathbb{W}_L t/2}\right)} \cdot \left[\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}(0)\right] \\ &= \frac{\mathbb{W}_L \cdot e^{-\mathbb{W}_L t/2}}{2 \sinh \frac{\mathbb{W}_L t}{2}} \cdot \left[\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}(0)\right]\end{aligned} \tag{2.801}$$

(2.794) then becomes

$$\begin{aligned}\hat{\boldsymbol{P}}(t) &= M\,\frac{\mathbb{W}_L \cdot e^{\mathbb{W}_L t/2}}{2 \sinh \frac{\mathbb{W}_L t}{2}} \cdot \left[\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}(0)\right] \\ &= M\,\mathbb{N}(\mathbb{W}_L t) \cdot \left[\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}(0)\right]\end{aligned} \tag{2.802}$$

where

$$\mathbb{N}(\mathbb{W}_L t) = \frac{\mathbb{W}_L \cdot e^{\mathbb{W}_L t/2}}{2 \sinh \frac{\mathbb{W}_L t}{2}} \tag{2.803}$$

(2.802) implies

$$\begin{aligned}\hat{H} = \frac{\hat{\boldsymbol{P}}(t)^2}{2M} &= \frac{\hat{\boldsymbol{P}}(t)^T \cdot \hat{\boldsymbol{P}}(t)}{2M} \\ &= \frac{1}{2} M \left[\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}(0)\right]^T \cdot \mathbb{N}^T(\mathbb{W}_L t) \cdot \mathbb{N}(\mathbb{W}_L t) \cdot \left[\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}(0)\right] \\ &= \frac{1}{2} M \left[\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}(0)\right]^T \cdot \mathbb{K}(\mathbb{W}_L t) \cdot \left[\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}(0)\right]\end{aligned} \tag{2.804}$$

where

$$\mathbb{K}(\mathbb{W}_L t) = \mathbb{N}^T(\mathbb{W}_L t) \cdot \mathbb{N}(\mathbb{W}_L t) \tag{2.805}$$

From (2.795a), we see that

$$\mathbb{B}^T = -\mathbb{B} \qquad \rightarrow \qquad \mathbb{W}_L^T = -\mathbb{W}_L \tag{2.805a}$$

Therefore, (2.803) gives

$$\mathbb{N}^T(\mathbb{W}_L t) = \mathbb{N}(-\mathbb{W}_L t) = \frac{\mathbb{W}_L \cdot e^{-\mathbb{W}_L t/2}}{2\sinh\frac{\mathbb{W}_L t}{2}}$$

$$\to \qquad \mathbb{K}(\mathbb{W}_L t) = \frac{\mathbb{W}_L^2}{4\sinh^2\frac{\mathbb{W}_L t}{2}} \tag{2.806}$$

Using

$$\left[\hat{x}_i, \hat{P}_j\right] = \left[\hat{x}_i, \hat{p}_j - \frac{e}{c}A_j(\hat{\boldsymbol{x}})\right] = \left[\hat{x}_i, \hat{p}_j\right] = i\hbar\,\delta_{ij} \tag{2.806a}$$

(2.799) implies

$$\left[\hat{x}_i(t), \hat{x}_j(0)\right] = \left[\frac{1}{M}\left(\frac{e^{\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L}\right)_{ik}\hat{P}_k, \hat{x}_j\right]$$

$$= -i\hbar\frac{1}{M}\left(\frac{e^{\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L}\right)_{ij} \tag{2.807}$$

so that

$$\left[\hat{x}_i(t), \hat{x}_j(0)\right] + \left[\hat{x}_j(t), \hat{x}_i(0)\right] = -i\hbar\frac{1}{M}\left\{\left(\frac{e^{\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L}\right)_{ij} + \left(\frac{e^{\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L}\right)_{ji}\right\}$$

$$= -i\hbar\frac{1}{M}\left\{\frac{e^{\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L} + \left(\frac{e^{\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L}\right)^T\right\}_{ij}$$

$$= -i\hbar\frac{1}{M}\left(\frac{e^{\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L} - \frac{e^{-\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L}\right)_{ij}$$

$$= -i\hbar\frac{1}{M}\left(\frac{e^{\mathbb{W}_L t} - e^{-\mathbb{W}_L t}}{\mathbb{W}_L}\right)_{ij}$$

$$= -i\hbar\frac{2}{M}\left(\frac{\sinh\mathbb{W}_L t}{\mathbb{W}_L}\right)_{ij} \tag{2.808}$$

Expanding (2.804), we have

$$\hat{H} = \frac{1}{2}M\left\{\hat{\boldsymbol{x}}^T(t)\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}^T(t)\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\hat{\boldsymbol{x}}(0) - \hat{\boldsymbol{x}}^T(0)\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\hat{\boldsymbol{x}}(t) + \hat{\boldsymbol{x}}^T(0)\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\hat{\boldsymbol{x}}(0)\right\} \tag{2.808a}$$

Using

$$\hat{\boldsymbol{x}}^T(t)\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\hat{\boldsymbol{x}}(0) - \hat{\boldsymbol{x}}^T(0)\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\hat{\boldsymbol{x}}(t)$$

$$= \left(\hat{\boldsymbol{x}}^T - \frac{1}{M}\hat{\boldsymbol{P}}^T\cdot\frac{e^{-\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L}\right)\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\hat{\boldsymbol{x}} - \hat{\boldsymbol{x}}^T\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\left(\hat{\boldsymbol{x}} + \frac{1}{M}\frac{e^{\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L}\cdot\hat{\boldsymbol{P}}\right)$$

$$= -\frac{1}{M}\hat{\boldsymbol{P}}^T\cdot\frac{e^{-\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L}\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\hat{\boldsymbol{x}} - \frac{1}{M}\hat{\boldsymbol{x}}^T\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\frac{e^{\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L}\cdot\hat{\boldsymbol{P}}$$

$$= -\frac{1}{M}\hat{\boldsymbol{P}}^T\cdot\frac{e^{-\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L}\cdot\frac{\mathbb{W}_L^2}{4\sinh^2\frac{\mathbb{W}_L t}{2}}\cdot\hat{\boldsymbol{x}} - \frac{1}{M}\hat{\boldsymbol{x}}^T\cdot\frac{\mathbb{W}_L^2}{4\sinh^2\frac{\mathbb{W}_L t}{2}}\cdot\frac{e^{\mathbb{W}_L t} - \mathbb{I}}{\mathbb{W}_L}\cdot\hat{\boldsymbol{P}}$$

$$= \frac{1}{M}\hat{\boldsymbol{P}}^T\cdot\frac{e^{-\mathbb{W}_L t/2}\cdot\mathbb{W}_L}{2\sinh\frac{\mathbb{W}_L t}{2}}\cdot\hat{\boldsymbol{x}} - \frac{1}{M}\hat{\boldsymbol{x}}^T\cdot\frac{\mathbb{W}_L\cdot e^{\mathbb{W}_L t/2}}{2\sinh\frac{\mathbb{W}_L t}{2}}\cdot\hat{\boldsymbol{P}}$$

$$= \frac{1}{M}\left(\frac{e^{-\mathbb{W}_L t/2}\cdot\mathbb{W}_L}{2\sinh\frac{\mathbb{W}_L t}{2}}\right)_{ij}\hat{P}_i\hat{x}_j - \frac{1}{M}\left(\frac{\mathbb{W}_L\cdot e^{\mathbb{W}_L t/2}}{2\sinh\frac{\mathbb{W}_L t}{2}}\right)_{ji}\hat{x}_j\hat{P}_i$$

$$= \frac{1}{M}\left(\frac{e^{-\mathbb{W}_L t/2}\cdot \mathbb{W}_L}{2\sinh\frac{\mathbb{W}_L t}{2}}\right)_{ij} \hat{P}_i\,\hat{x}_j - \frac{1}{M}\left(\frac{e^{-\mathbb{W}_L t/2}\cdot \mathbb{W}_L}{2\sinh\frac{\mathbb{W}_L t}{2}}\right)_{ij} \hat{x}_i\,\hat{P}_j \qquad [\ \mathbb{W}_L^T = -\mathbb{W}_L \text{ used. }]$$

$$= \frac{1}{M}\left(\frac{e^{-\mathbb{W}_L t/2}\cdot \mathbb{W}_L}{2\sinh\frac{\mathbb{W}_L t}{2}}\right)_{ij} (-i\,\hbar\,\delta_{ij}) \qquad [\ (2.806a) \text{ used.}]$$

$$= -\frac{i\,\hbar}{M}\left(\frac{e^{-\mathbb{W}_L t/2}\cdot \mathbb{W}_L}{2\sinh\frac{\mathbb{W}_L t}{2}}\right)_{ii}$$

$$= -\frac{i\,\hbar}{M}\left(\frac{\left(e^{-\mathbb{W}_L t/2}+e^{\mathbb{W}_L t/2}\right)\cdot \mathbb{W}_L}{4\sinh\frac{\mathbb{W}_L t}{2}}\right)_{ii} \qquad [\ \mathbb{A}_{ii} = \mathbb{A}_{ii}^T \text{ used. }]$$

$$= -\frac{i\,\hbar}{M}\left(\frac{1}{2}\,\mathbb{W}_L\cdot\coth\frac{\mathbb{W}_L t}{2}\right)_{ii}$$

$$= -\frac{i\,\hbar}{M}\,\mathrm{tr}\left(\frac{1}{2}\,\mathbb{W}_L\cdot\coth\frac{\mathbb{W}_L t}{2}\right) \qquad (2.808b)$$

(2.808a) becomes

$$\hat{H} = \frac{1}{2}\,M\Big\{\hat{\boldsymbol{x}}^T(t)\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\hat{\boldsymbol{x}}(t) - 2\,\hat{\boldsymbol{x}}^T(t)\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\hat{\boldsymbol{x}}(0) \qquad (2.809)$$
$$+\hat{\boldsymbol{x}}^T(0)\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\hat{\boldsymbol{x}}(0) - \frac{i\,\hbar}{M}\,\mathrm{tr}\left(\frac{1}{2}\,\mathbb{W}_L\cdot\coth\frac{\mathbb{W}_L t}{2}\right)\Big\}$$

Using the recipe prescribed in (2.771), we get

$$H(\boldsymbol{x},\,\boldsymbol{x}';\,t) = \frac{1}{2}\,M\Big\{\boldsymbol{x}^T\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\boldsymbol{x} - 2\,\boldsymbol{x}^T\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\boldsymbol{x}' + \boldsymbol{x}'^T\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\boldsymbol{x}'\Big\}$$
$$-\frac{i\,\hbar}{2}\,\mathrm{tr}\left(\frac{1}{2}\,\mathbb{W}_L\cdot\coth\frac{\mathbb{W}_L t}{2}\right)$$
$$= \frac{1}{2}\,M\,(\boldsymbol{x}-\boldsymbol{x}')^T\cdot\mathbb{K}(\mathbb{W}_L t)\cdot(\boldsymbol{x}-\boldsymbol{x}') - \frac{i\,\hbar}{2}\,\mathrm{tr}\left(\frac{1}{2}\,\mathbb{W}_L\cdot\coth\frac{\mathbb{W}_L t}{2}\right) \qquad (2.809a)$$

where we've used

$$\boldsymbol{x}^T\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\boldsymbol{x}' = \boldsymbol{x}'^T\cdot\mathbb{K}(\mathbb{W}_L t)\cdot\boldsymbol{x}$$

since (2.806) gives

$$\mathbb{K}^T(\mathbb{W}_L t) = \mathbb{K}(\mathbb{W}_L t)$$

The next step is to calculate

$$E(\boldsymbol{x},\,\boldsymbol{x}';\,t) = \exp\left[-\frac{i}{\hbar}\int^t d\,t'\,H(\boldsymbol{x},\,\boldsymbol{x}';\,t')\right]$$

From (2.806), we have

$$\int d\,t\,\mathbb{K}(\mathbb{W}_L t) = \int d\,t\,\frac{\mathbb{W}_L^2}{4\sinh^2\frac{\mathbb{W}_L t}{2}} \qquad \left[\ \int d\,t\,\frac{1}{\sinh^2(a\,t)} = -\frac{1}{a}\coth(a\,t)\ \right]$$
$$= -\frac{1}{2}\,\mathbb{W}_L\cdot\coth\left(\frac{\mathbb{W}_L t}{2}\right) \qquad (2.810)$$

Let w_n be the eigenvalues of $\mathbb{W}_L$. Using

$$\int d\,t\,\coth(a\,t) = \frac{1}{a}\ln[\sinh(a\,t)]$$

we have

$$\int dt\,\mathrm{tr}\left(\frac{1}{2}\mathbb{W}_L\cdot\coth\frac{\mathbb{W}_L t}{2}\right)=\int dt\sum_n\frac{1}{2}w_n\coth\frac{w_n t}{2}$$
$$=\sum_n\ln\left[\sinh\left(\frac{w_n t}{2}\right)\right]+C \qquad (2.810a)$$

where C is a constant.

Now

$$\mathbb{W}_L^T=-\mathbb{W}_L \qquad \to \qquad \mathrm{tr}\,\mathbb{W}_L^T=\mathrm{tr}\,\mathbb{W}_L=-\mathrm{tr}\,\mathbb{W}_L=0=\sum_n w_n \qquad (2.810b)$$

Hence, for some n, $w_n<0$, thus making $\ln\left[\sinh\left(\frac{w_n t}{2}\right)\right]$ imaginary. However, the diagonal elements $E(\boldsymbol{x},\ \boldsymbol{x};t)$ should be real. We therefore guarantee the argument of each logarithm to be positive by setting

$$C=-\sum_n\ln\left(\frac{w_n}{2}\right)$$

so that

$$\int dt\,\mathrm{tr}\left(\frac{1}{2}\mathbb{W}_L\cdot\coth\frac{\mathbb{W}_L t}{2}\right)=\sum_n\ln\left[\frac{\sinh(w_n t/2)}{w_n/2}\right]$$
$$=\sum_n\ln\left[\frac{\sinh(w_n t/2)}{w_n t/2}\right]+\sum_{n=1}^{D}\ln t$$
$$=\mathrm{tr}\ln\left[\frac{\sinh(\mathbb{W}_L t/2)}{\mathbb{W}_L t/2}\right]+D\ln t \qquad (2.811)$$

where D is the dimension of the system.

Let a_n be the eigenvalues of $\mathbb{A}$. Then

$$\mathrm{tr}\ln\mathbb{A}=\sum_n\ln a_n=\ln\left(\prod_n a_n\right)=\ln\det\mathbb{A}$$

Hence,

$$\int dt\,\mathrm{tr}\left(\frac{1}{2}\mathbb{W}_L\cdot\coth\frac{\mathbb{W}_L t}{2}\right)=\ln\left[\det\left(\frac{\sinh(\mathbb{W}_L t/2)}{\mathbb{W}_L t/2}\right)\right]+D\ln t$$
$$=\ln\left[t^D\det\left(\frac{\sinh(\mathbb{W}_L t/2)}{\mathbb{W}_L t/2}\right)\right] \qquad (2.811a)$$

(2.809a) then gives

$$\int dt\,H(\boldsymbol{x},\boldsymbol{x}';t)=-\frac{1}{2}M(\boldsymbol{x}-\boldsymbol{x}')^T\cdot\frac{1}{2}\mathbb{W}_L\cdot\coth\left(\frac{\mathbb{W}_L t}{2}\right)\cdot(\boldsymbol{x}-\boldsymbol{x}') \qquad (2.811b)$$
$$-\frac{i\hbar}{2}\left\{\mathrm{tr}\ln\left[\frac{\sinh(\mathbb{W}_L t/2)}{\mathbb{W}_L t/2}\right]+D\ln t\right\}$$
$$=-\frac{1}{2}M(\boldsymbol{x}-\boldsymbol{x}')^T\cdot\frac{1}{2}\mathbb{W}_L\cdot\coth\left(\frac{\mathbb{W}_L t}{2}\right)\cdot(\boldsymbol{x}-\boldsymbol{x}') \qquad (2.811c)$$
$$-\frac{i\hbar}{2}\ln\left[t^D\det\left(\frac{\sinh(\mathbb{W}_L t/2)}{\mathbb{W}_L t/2}\right)\right]$$

and

$$E(\boldsymbol{x}, \boldsymbol{x}'; t) = t^{-D/2} \exp\left\{ -\frac{1}{2} \operatorname{tr} \ln\left[\frac{\sinh(\mathbb{W}_L t/2)}{\mathbb{W}_L t/2} \right] \right\} \tag{2.812}$$
$$\times \exp\left[\frac{i}{\hbar} \frac{1}{2} M (\boldsymbol{x}-\boldsymbol{x}')^T \cdot \frac{1}{2} \mathbb{W}_L \cdot \coth\left(\frac{\mathbb{W}_L t}{2} \right) \cdot (\boldsymbol{x}-\boldsymbol{x}') \right]$$
$$= t^{-D/2} \left[\det\left(\frac{\sinh(\mathbb{W}_L t/2)}{\mathbb{W}_L t/2} \right) \right]^{-1/2} \tag{2.813}$$
$$\times \exp\left[\frac{i}{\hbar} \frac{1}{2} M (\boldsymbol{x}-\boldsymbol{x}')^T \cdot \frac{1}{2} \mathbb{W}_L \cdot \coth\left(\frac{\mathbb{W}_L t}{2} \right) \cdot (\boldsymbol{x}-\boldsymbol{x}') \right]$$

Next, we need to fix $C(\boldsymbol{x}, \boldsymbol{x}')$ of (2.762) by solving the differential eqs. obtained by spatial differentiation of the evolution amplitude

$$(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = \left\langle \boldsymbol{x} \left| e^{-i \hat{H} t/\hbar} \right| \boldsymbol{x}' \right\rangle$$

where

$$\hat{H} = \frac{1}{2M} \hat{\boldsymbol{P}}^2 \qquad\qquad \hat{\boldsymbol{P}} = \hat{\boldsymbol{p}} - \frac{e}{c} \boldsymbol{A}(\hat{\boldsymbol{x}})$$

Owing to the form of $\hat{H}$, it is easier to work with the covariant derivatives so that

$$\left(\frac{\hbar}{i} \nabla - \frac{e}{c} \boldsymbol{A}(\boldsymbol{x}) \right) (\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = \left\langle \boldsymbol{x} \left| \hat{\boldsymbol{P}}\, e^{-i \hat{H} t/\hbar} \right| \boldsymbol{x}' \right\rangle$$
$$= \left\langle \boldsymbol{x} \left| e^{-i \hat{H} t/\hbar} e^{i \hat{H} t/\hbar} \hat{\boldsymbol{P}}\, e^{-i \hat{H} t/\hbar} \right| \boldsymbol{x}' \right\rangle$$
$$= \left\langle \boldsymbol{x}\, t \left| \hat{\boldsymbol{P}}(t) \right| \boldsymbol{x}' \right\rangle$$
$$= M \mathbb{N}(\mathbb{W}_L t) \cdot \left\langle \boldsymbol{x}\, t \left| \left[\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}(0) \right] \right| \boldsymbol{x}' \right\rangle \qquad \text{[(2.802) used.]}$$
$$= M \mathbb{N}(\mathbb{W}_L t) \cdot (\boldsymbol{x} - \boldsymbol{x}') (\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) \tag{2.814}$$

Similarly,

$$\left(\frac{\hbar}{i} \nabla' + \frac{e}{c} \boldsymbol{A}(\boldsymbol{x}') \right) (\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = -\left\langle \boldsymbol{x} \left| e^{-i \hat{H} t/\hbar} \hat{\boldsymbol{P}} \right| \boldsymbol{x}' \right\rangle$$
$$= -\left\langle \boldsymbol{x}\, t \left| \hat{\boldsymbol{P}}(0) \right| \boldsymbol{x}' \right\rangle$$
$$= -M \mathbb{N}^T(\mathbb{W}_L t) \cdot \left\langle \boldsymbol{x}\, t \left| \left[\hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}(0) \right] \right| \boldsymbol{x}' \right\rangle \qquad \text{[(2.801) used.]}$$
$$= -M \mathbb{N}^T(\mathbb{W}_L t) \cdot (\boldsymbol{x} - \boldsymbol{x}') (\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) \tag{2.815}$$

Also,

$$(\boldsymbol{x}-\boldsymbol{x}')^T \cdot \mathbb{W}_L \cdot \coth\left(\frac{\mathbb{W}_L t}{2} \right) = (x_i - x_i')\, W_{L\,ij} \coth\left(\frac{\mathbb{W}_L t}{2} \right)_{jk}$$
$$\coth\left(\frac{\mathbb{W}_L t}{2} \right) \cdot \mathbb{W}_L \cdot (\boldsymbol{x}-\boldsymbol{x}') = \coth\left(\frac{\mathbb{W}_L t}{2} \right)_{kj} W_{L\,ji} (x_i - x_i')$$
$$= \left[-\coth\left(\frac{\mathbb{W}_L t}{2} \right)_{jk} \right] (-W_{L\,ij}) (x_i - x_i')$$
$$\to \qquad (\boldsymbol{x}-\boldsymbol{x}')^T \cdot \mathbb{W}_L \cdot \coth\left(\frac{\mathbb{W}_L t}{2} \right) = \coth\left(\frac{\mathbb{W}_L t}{2} \right) \cdot \mathbb{W}_L \cdot (\boldsymbol{x}-\boldsymbol{x}')$$

(2.813) then gives

$$\frac{\hbar}{i} \nabla E = M \frac{\mathbb{W}_L}{2} \cdot \coth\left(\frac{\mathbb{W}_L t}{2} \right) \cdot (\boldsymbol{x}-\boldsymbol{x}')\, E \tag{2.815a}$$
$$\frac{\hbar}{i} \nabla' E = -M \frac{\mathbb{W}_L}{2} \cdot \coth\left(\frac{\mathbb{W}_L t}{2} \right) \cdot (\boldsymbol{x}-\boldsymbol{x}')\, E \tag{2.815b}$$

From (2.762)

$$(\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = C(\boldsymbol{x},\ \boldsymbol{x}')\, E(\boldsymbol{x},\ \boldsymbol{x}'; t)$$

we get

$$\frac{\hbar}{i} \nabla (\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = \left(\frac{\hbar}{i} \nabla C\right) E + C \frac{\hbar}{i} \nabla E$$

$$= \left(\frac{\hbar}{i} \nabla C\right) E + C\, M \frac{\mathbb{W}_L}{2} \cdot \coth\left(\frac{\mathbb{W}_L\, t}{2}\right) \cdot (\boldsymbol{x} - \boldsymbol{x}')\, E \qquad (2.815c)$$

$$\frac{\hbar}{i} \nabla' (\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = \left(\frac{\hbar}{i} \nabla' C\right) E + C \frac{\hbar}{i} \nabla' E$$

$$= \left(\frac{\hbar}{i} \nabla' C\right) E - C\, M \frac{\mathbb{W}_L}{2} \cdot \coth\left(\frac{\mathbb{W}_L\, t}{2}\right) \cdot (\boldsymbol{x} - \boldsymbol{x}')\, E \qquad (2.815d)$$

(2.814-5) thus become

$$\left\{ \frac{\hbar}{i} \nabla - \frac{e}{c} \boldsymbol{A}(\boldsymbol{x}) + M \frac{\mathbb{W}_L}{2} \cdot \left[\coth\left(\frac{\mathbb{W}_L\, t}{2}\right) - \mathbb{N}(\mathbb{W}_L\, t) \right] \cdot (\boldsymbol{x} - \boldsymbol{x}') \right\} C = 0 \qquad (2.816a)$$

$$\left\{ \frac{\hbar}{i} \nabla' + \frac{e}{c} \boldsymbol{A}(\boldsymbol{x}') + M \frac{\mathbb{W}_L}{2} \cdot \left[\coth\left(\frac{\mathbb{W}_L\, t}{2}\right) - \mathbb{N}^T (\mathbb{W}_L\, t) \right] \cdot (\boldsymbol{x}' - \boldsymbol{x}) \right\} C = 0 \qquad (2.816b)$$

Using (2.803),

$$\mathbb{N}(\mathbb{W}_L\, t) = \frac{\mathbb{W}_L \cdot e^{\mathbb{W}_L t/2}}{2 \sinh \frac{\mathbb{W}_L t}{2}} \qquad \mathbb{N}^T (\mathbb{W}_L\, t) = \frac{\mathbb{W}_L \cdot e^{-\mathbb{W}_L t/2}}{2 \sinh \frac{\mathbb{W}_L t}{2}}$$

we have

$$\frac{\mathbb{W}_L}{2} \cdot \coth\left(\frac{\mathbb{W}_L\, t}{2}\right) - \mathbb{N}(\mathbb{W}_L\, t) = \frac{\mathbb{W}_L \cdot \left(e^{\mathbb{W}_L t/2} + e^{-\mathbb{W}_L t/2}\right)}{4 \sinh \frac{\mathbb{W}_L t}{2}} - \frac{\mathbb{W}_L \cdot e^{\mathbb{W}_L t/2}}{2 \sinh \frac{\mathbb{W}_L t}{2}}$$

$$= \frac{\mathbb{W}_L \cdot \left(-e^{\mathbb{W}_L t/2} + e^{-\mathbb{W}_L t/2}\right)}{4 \sinh \frac{\mathbb{W}_L t}{2}} = -\frac{\mathbb{W}_L}{2}$$

$$\frac{\mathbb{W}_L}{2} \cdot \coth\left(\frac{\mathbb{W}_L\, t}{2}\right) - \mathbb{N}^T(\mathbb{W}_L\, t) = \frac{\mathbb{W}_L \cdot \left(e^{\mathbb{W}_L t/2} + e^{-\mathbb{W}_L t/2}\right)}{4 \sinh \frac{\mathbb{W}_L t}{2}} - \frac{\mathbb{W}_L \cdot e^{-\mathbb{W}_L t/2}}{2 \sinh \frac{\mathbb{W}_L t}{2}}$$

$$= \frac{\mathbb{W}_L \cdot \left(e^{\mathbb{W}_L t/2} - e^{-\mathbb{W}_L t/2}\right)}{4 \sinh \frac{\mathbb{W}_L t}{2}} = \frac{\mathbb{W}_L}{2}$$

Hence, (2.816a-b) become

$$\left[\frac{\hbar}{i} \nabla - \frac{e}{c} \boldsymbol{A}(\boldsymbol{x}) - M \frac{\mathbb{W}_L}{2} \cdot (\boldsymbol{x} - \boldsymbol{x}') \right] C = 0 \qquad (2.817)$$

$$\left[\frac{\hbar}{i} \nabla' + \frac{e}{c} \boldsymbol{A}(\boldsymbol{x}') + M \frac{\mathbb{W}_L}{2} \cdot (\boldsymbol{x}' - \boldsymbol{x}) \right] C = 0 \qquad (2.818)$$

Remembering that we are dealing with a constant field $\boldsymbol{B}$ so that $\mathbb{W}_L$ is also constant, the solution is

$$C(\boldsymbol{x}, \boldsymbol{x}') = C \exp\left\{ \frac{i}{\hbar} \int_{\boldsymbol{x}'}^{\boldsymbol{x}} d\, \boldsymbol{\xi} \cdot \left[\frac{e}{c} \boldsymbol{A}(\boldsymbol{\xi}) + \frac{1}{2} M\, \mathbb{W}_L \cdot (\boldsymbol{\xi} - \boldsymbol{x}') \right] \right\} \qquad (2.819)$$

where C is some constant since

$$\frac{\hbar}{i} \nabla C = \left\{ \nabla \int_{\boldsymbol{x}'}^{\boldsymbol{x}} d\, \boldsymbol{\xi} \cdot \left[\frac{e}{c} \boldsymbol{A}(\boldsymbol{\xi}) + \frac{1}{2} M\, \mathbb{W}_L \cdot (\boldsymbol{\xi} - \boldsymbol{x}') \right] \right\} C$$

$$= \left[\frac{e}{c} \boldsymbol{A}(\boldsymbol{\xi}) + \frac{1}{2} M\, \mathbb{W}_L \cdot (\boldsymbol{\xi} - \boldsymbol{x}') \right]_{\boldsymbol{\xi} = \boldsymbol{x}} C$$

$$= \left[\frac{e}{c} \boldsymbol{A}(\boldsymbol{x}) + \frac{1}{2} M \mathbb{W}_L \cdot (\boldsymbol{x} - \boldsymbol{x}') \right] C$$

$$\frac{\hbar}{i} \nabla' C = \left\{ \nabla' \int_{\boldsymbol{x}'}^{\boldsymbol{x}} d\boldsymbol{\xi} \cdot \left[\frac{e}{c} \boldsymbol{A}(\boldsymbol{\xi}) + \frac{1}{2} M \mathbb{W}_L \cdot (\boldsymbol{\xi} - \boldsymbol{x}') \right] \right\} C$$

$$= \left\{ - \left[\frac{e}{c} \boldsymbol{A}(\boldsymbol{\xi}) + \frac{1}{2} M \mathbb{W}_L \cdot (\boldsymbol{\xi} - \boldsymbol{x}') \right]_{\boldsymbol{\xi} = \boldsymbol{x}'} - \int_{\boldsymbol{x}'}^{\boldsymbol{x}} d\xi_j \frac{1}{2} M W_{L\,jk} (\nabla' x_k') \right\} C$$

$$= \left\{ - \frac{e}{c} \boldsymbol{A}(\boldsymbol{x}') - \frac{1}{2} \left[M (\boldsymbol{x} - \boldsymbol{x}')^T \cdot \mathbb{W}_L \right]^T \right\} C$$

$$= \left\{ - \frac{e}{c} \boldsymbol{A}(\boldsymbol{x}') - \frac{1}{2} M \mathbb{W}_L^T \cdot (\boldsymbol{x} - \boldsymbol{x}') \right\} C$$

$$= \left\{ - \frac{e}{c} \boldsymbol{A}(\boldsymbol{x}') - \frac{1}{2} M \mathbb{W}_L \cdot (\boldsymbol{x}' - \boldsymbol{x}) \right\} C$$

Let

$$\boldsymbol{A}'(\boldsymbol{\xi}) = \boldsymbol{A}(\boldsymbol{\xi}) + \frac{1}{2} \frac{c}{e} M \mathbb{W}_L \cdot (\boldsymbol{\xi} - \boldsymbol{x}') \qquad (2.820a)$$

Using (2.797a),

$$W_{L\,ij} = \frac{e}{M c} \epsilon_{ijk} B_k$$

we have

$$A_i' = A_i + \frac{1}{2} \epsilon_{ijk} B_k (\xi_j - x_j') \qquad (2.820b)$$

or

$$\boldsymbol{A}'(\boldsymbol{\xi}) = \boldsymbol{A}(\boldsymbol{\xi}) - \frac{1}{2} \boldsymbol{B} \times (\boldsymbol{\xi} - \boldsymbol{x}') \qquad (2.820)$$

Since $\boldsymbol{B}$ is constant, we have

$$\nabla_{\xi} \times [\boldsymbol{B} \times (\boldsymbol{\xi} - \boldsymbol{x}')] = \boldsymbol{B} [\nabla_{\xi} \cdot (\boldsymbol{\xi} - \boldsymbol{x}')] - (\boldsymbol{B} \cdot \nabla_{\xi}) (\boldsymbol{\xi} - \boldsymbol{x}') = 3 \boldsymbol{B} - \boldsymbol{B} = 2 \boldsymbol{B}$$

so that

$$\nabla_{\xi} \times \boldsymbol{A}'(\boldsymbol{\xi}) = \nabla_{\xi} \times \boldsymbol{A}(\boldsymbol{\xi}) - \boldsymbol{B} = 0$$

Therefore, the line integral in (2.819) is path-independent. Choosing the striaght line connecting $\boldsymbol{x}'$ to $\boldsymbol{x}$ as the integration path, we have

$$d\boldsymbol{\xi} \;//\; (\boldsymbol{\xi} - \boldsymbol{x}')$$

$$\rightarrow \qquad d\boldsymbol{\xi} \cdot \mathbb{W}_L \cdot (\boldsymbol{\xi} - \boldsymbol{x}') \propto d\boldsymbol{\xi} \cdot \left[\boldsymbol{B} \times (\boldsymbol{\xi} - \boldsymbol{x}') \right] = 0$$

Therefore, (2.819) reduces to

$$C(\boldsymbol{x}, \boldsymbol{x}') = \mathcal{C} \exp\left\{ \frac{i}{\hbar} \frac{e}{c} \int_{\boldsymbol{x}'}^{\boldsymbol{x}} d\boldsymbol{\xi} \cdot \boldsymbol{A}(\boldsymbol{\xi}) \right\} \qquad (2.821)$$

$\mathcal{C}$ is fixed by the I.C. [see (2.777)]

$$\lim_{t \to 0} (\boldsymbol{x}\, t \mid \boldsymbol{x}'\, 0) = \langle \boldsymbol{x} \mid \boldsymbol{x}' \rangle = \delta(\boldsymbol{x} - \boldsymbol{x}')$$

and take the value [see (2.778)]

$$\mathcal{C} = \left(\frac{M}{i 2 \pi \hbar} \right)^{D/2}$$

Putting (2.813) & (2.821) together, we have

$$(\boldsymbol{x}\,t \mid \boldsymbol{x}'\,0) = \left(\frac{M}{i 2\pi\hbar t}\right)^{D/2} \left[\det\left(\frac{\sinh(\mathbb{W}_L t/2)}{\mathbb{W}_L t/2}\right)\right]^{-1/2} \exp\left\{\frac{i}{\hbar}\frac{e}{c}\int_{x'}^{x} d\boldsymbol{\xi}\cdot\boldsymbol{A}(\boldsymbol{\xi})\right\} \times \exp\left[\frac{i}{\hbar}\frac{1}{2} M (\boldsymbol{x}-\boldsymbol{x}')^T \cdot \frac{1}{2}\mathbb{W}_L \cdot \coth\left(\frac{\mathbb{W}_L t}{2}\right)\cdot(\boldsymbol{x}-\boldsymbol{x}')\right] \tag{2.822}$$

These expressions can be simplified by assuming $\boldsymbol{B} = B\,\hat{\boldsymbol{z}}$. (2.791a) & (2.795a) thus become

$$\mathbb{B} = B\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbb{W}_L = \omega_L \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.823}$$

$$\rightarrow \qquad \mathbb{W}_L^2 = -\omega_L^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\cos\frac{\mathbb{W}_L t}{2} = \mathbb{I} + \sum_{n=1}^{\infty} \frac{(-)^n}{(2n)!}\left(\frac{\mathbb{W}_L t}{2}\right)^{2n} = \mathbb{I} + \sum_{n=1}^{\infty} \frac{\left(-\mathbb{W}_L^2\right)^n}{(2n)!}\left(\frac{t}{2}\right)^{2n}$$

$$= \mathbb{I} + \sum_{n=1}^{\infty} \frac{1}{(2n)!}\left(\frac{\omega_L t}{2}\right)^{2n} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \mathbb{I} + \left[\cosh\left(\frac{\omega_L t}{2}\right) - 1\right]\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\left(\frac{\omega_L t}{2}\right) & 0 & 0 \\ 0 & \cosh\left(\frac{\omega_L t}{2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.824}$$

$$\sinh\frac{\mathbb{W}_L t}{2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{\mathbb{W}_L t}{2}\right)^{2n+1}$$

$$= \frac{\mathbb{W}_L t}{2} + \frac{\mathbb{W}_L t}{2}\cdot \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}\left(\frac{\mathbb{W}_L t}{2}\right)^{2n}$$

$$= \frac{\mathbb{W}_L t}{2} + \frac{\mathbb{W}_L t}{2}\cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sum_{n=1}^{\infty} \frac{(-)^n}{(2n+1)!}\left(\frac{\omega_L t}{2}\right)^{2n}$$

$$= \frac{\mathbb{W}_L t}{2} + \frac{\mathbb{W}_L t}{2}\cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left[\frac{\sin(\omega_L t / 2)}{\omega_L t / 2} - 1\right]$$

Altough $\mathbb{W}_L^{-1}$ does not actually exist, it is legitimate to write

$$\frac{\sinh(\mathbb{W}_L t/2)}{\mathbb{W}_L t/2} = \mathbb{I} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left[\frac{\sin(\omega_L t / 2)}{\omega_L t / 2} - 1\right]$$

$$= \begin{pmatrix} \frac{\sin(\omega_L t/2)}{\omega_L t/2} & 0 & 0 \\ 0 & \frac{\sin(\omega_L t/2)}{\omega_L t/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.825}$$

$$\rightarrow \qquad \det\frac{\sinh(\mathbb{W}_L t/2)}{\mathbb{W}_L t/2} = \left(\frac{\sin(\omega_L t/2)}{\omega_L t / 2}\right)^2 \tag{2.826}$$

Alternatively, the above results can be obtained easily using the fact $\mathbb{W}_L^2$ is diagonal so that any function $F\left(\mathbb{W}_L^2\right)$ must be diagonal. The eigenvalues of $\mathbb{W}_L$ are $\pm i\,\omega_L$ and 0, with respective eigenvectors

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\\ 0\end{pmatrix} \quad \& \quad \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Any function $f(\mathbb{W}_L)$ that can be put into the form $F(\mathbb{W}_L^2)$ can therefore be written as

$$f(\mathbb{W}_L) = F(\mathbb{W}_L^2) = \begin{pmatrix} f(\pm i\,\omega_L) & 0 & 0 \\ 0 & f(\mp i\,\omega_L) & 0 \\ 0 & 0 & f(0) \end{pmatrix} \tag{2.826a}$$

where the result is independent of the signs we choose since $(\pm i\,\omega_L)^2 = -\omega_L^2$.

Consider now the line integral

$$\mathcal{I} = \int_{\boldsymbol{x}'}^{\boldsymbol{x}} d\,\boldsymbol{\xi}\cdot\boldsymbol{A}\,(\boldsymbol{\xi})$$

for the gauge with

$$\boldsymbol{A}(\boldsymbol{x}) = B\,x\,\hat{\boldsymbol{y}}$$

along the straight-line path

$$\boldsymbol{\xi} = \boldsymbol{x}' + s\,(\boldsymbol{x} - \boldsymbol{x}') \qquad s \in [0, 1] \tag{2.827}$$

Hence,

$$\begin{aligned}
\mathcal{I} &= \int_0^1 d\,s\;(\boldsymbol{x}-\boldsymbol{x}')\cdot\left(B\,\xi_x\,\hat{\boldsymbol{y}}\right) \\
&= B\,(y-y')\int_0^1 d\,s\;\left[x' + s\,(x-x')\right] \\
&= B\,(y-y')\left[x' + \frac{1}{2}(x-x')\right] \\
&= \frac{1}{2}B\,(y-y')\,(x+x') \\
&= \frac{1}{2}B\,(x\,y - x'\,y') + \frac{1}{2}B\,(x'\,y - x\,y')
\end{aligned} \tag{2.828}$$

Thus, $\frac{e}{c}\mathcal{I}$ provides $\mathcal{A}_{\mathrm{sf}}$ in (2.668) as well as the last term of $\mathcal{A}_{\mathrm{cl}}$ in (2.667).

Using (2.826a), we have

$$\begin{aligned}
\mathbb{W}_L\cdot\coth\left(\frac{\mathbb{W}_L\,t}{2}\right) &= \begin{pmatrix} (\pm i\,\omega_L)\coth\left(\frac{\pm i\,\omega_L\,t}{2}\right) & 0 & 0 \\ 0 & (\mp i\,\omega_L)\coth\left(\frac{\mp i\,\omega_L\,t}{2}\right) & 0 \\ 0 & 0 & \lim_{w\to 0} w\coth\frac{w\,t}{2} \end{pmatrix} \\
&= \begin{pmatrix} \omega_L\cot\left(\frac{\omega_L\,t}{2}\right) & 0 & 0 \\ 0 & \omega_L\cot\left(\frac{\omega_L\,t}{2}\right) & 0 \\ 0 & 0 & \frac{2}{t} \end{pmatrix}
\end{aligned}$$

Hence, the other term in (2.822) that reads

$$\begin{aligned}
&\frac{1}{2}M\,(\boldsymbol{x}-\boldsymbol{x}')^T\cdot\frac{1}{2}\mathbb{W}_L\cdot\coth\left(\frac{\mathbb{W}_L\,t}{2}\right)\cdot(\boldsymbol{x}-\boldsymbol{x}') \\
&= \frac{1}{2}M\left\{\frac{1}{2}\omega_L\cot\left(\frac{\omega_L\,t}{2}\right)\left[(x-x')^2 + (y-y')^2\right] + \frac{(z-z')^2}{t}\right\}
\end{aligned}$$

gives the rest of $\mathcal{A}_{\mathrm{cl}}$ in (2.667).

Therefore, (2.822) agrees with (2.666), as it should.