Appendix 2A. Baker-Campbell-Hausdorff Formula and Magnus Expansion

The Adjoint Representation

This subsection aims to introduce just enough terminology to understand the adjoint representations of a Lie group & its Lie algebra.

More details can be found in the book

1. R.Aldrovandi, J.G.Pereira, "An Introduction to Geometrical Physics", World Scientific (1995). See in particular Chap 8.
2. T.Frankel, “The Geometry of Physics”, 2nd ed., Cambridge Univ. Press (2003). See in particular §18.2c.

A 1-1 onto mapping is called an isomorphism.
Isomorphism of a set into itself is called an automorphism.

Automorphisms on a vector space V are sometimes taken as operators on it.
The set of all automorphisms on V is denoted by $\text{Aut}(V)$.

A mapping that preserves group multiplication is called a homomorphism.

A representation of a group G is a homomorphism to another group H.

Many authors use a more restrictive definition and set $H = \text{Aut}(V)$ so that the representation of G is a homomorphism of G to the operators on V.

The set $\text{Aut}(G)$ of all automorphisms on a group G is itself a group.

A Lie group is an analytic (or C^∞) manifold, or a union of such, that is also a group.
For our purposes, one can treat an analytic manifold as a continuum in which any two points can be joined by an infinitely differentiable line.

Let G be a Lie group & $\mathfrak{g}$ its Lie algebra. The vector space of the algebra is $T_e\, G$, the tangent space at the identity e of G.

The close relationship betwen G and $\mathfrak{g}$ arises from the following observation.

Let $g \in G$, then there is a $X \in \mathfrak{g}$ such that

$$g = e^X$$

For proof, consider a curve on G given by

$$g(t) = e^{X t}$$

then

$$g(0) = e \qquad\qquad g(1) = g$$

where e is the identity of G. The curve passes through g at $t = 1$ and its tangent at e is

$$\left.\frac{d\,g(t)}{d\,t}\right|_{t=0} \equiv g'(0) = X$$

i.e., $X \in \mathfrak{g}$ as claimed.

It is straightforward to verify that $g(t)$ is a 1-parameter Abelian subgroup of G.

Given a map $f : M \to N$ between two manifolds, it induces a map $f_* : T_x M \to T_{f(x)} N$ between the corresponding tangent spaces. f_* is called the differential map and sometimes denoted as $d\,f$.

Thus,

$$d : \quad \text{Aut}(G) \;\to\; \text{Aut}(\mathfrak{g})$$

$$j \mapsto d\,j = j_*$$

Since

$$d(j \circ k) = d\,j \circ d\,k \qquad \forall\; j, k \in \text{Aut}(G)$$

d is a homomorphism of the group Aut(G). Consequently, it is a representation of Aut(G) on the vector space $\mathfrak{g}$.

From the following diagrams

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{j_* = d\,j} & \mathfrak{g} \\ \text{exp} \downarrow & & \text{exp} \downarrow \\ G & \xrightarrow[j]{} & G \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{j_* = d\,j} & j_* X \\ \text{exp} \downarrow & & \text{exp} \downarrow \\ e^X & \xrightarrow[j]{} & j\,e^X = e^{j_* X} \end{array}$$

we see that

$$j\left(e^X\right) = e^{j_* X} \qquad \text{(a)}$$

An important subgroup of Aut(G) is the set of inner automorphisms

$$\begin{aligned} j_g :\quad & G \to G \\ & h \mapsto j_g(h) = g\,h\,g^{-1} \qquad g \in G \end{aligned}$$

Each j_g is also a diffeomorphism (differentiable isomorphism). Since

$$j_g(h\,k) = g\,h\,k\,g^{-1} = g\,h\,g^{-1}\,g\,k\,g^{-1} = j_g(h)\,j_g(k)$$

it is also a homomorphism of G.

Since

$$j_{g\,h}(k) = g\,h\,k\,(g\,h)^{-1} = g\,h\,k\,h^{-1}\,g^{-1} = g\,j_h(k)\,g^{-1} = j_g(\,j_h(k)\,) = (j_g \circ j_h)\,(k) \qquad \forall\; k$$

$$\to \qquad j_{g\,h} = j_g \circ j_h$$

the map

$$\begin{aligned} & G \to \text{Aut}(G) \\ & g \mapsto j_g \end{aligned}$$

is a homomorphism of G.

The mapping

$$\begin{aligned} \text{Ad} :\quad & G \to \text{Aut}(\mathfrak{g}) \\ & g \mapsto \text{Ad}(g) \equiv \text{Ad}_g = d\,j_g = j_{g\,*} \end{aligned}$$

is called the adjoint representation of G. Note that it is a representation using $\mathfrak{g}$ as its vector space.

Ad_g is called the adjoint action. Using (a), we have

$$e^{\text{Ad}_g X} = e^{j_{g\,*} X} = j_g\left(e^X\right) = g\,e^X\,g^{-1} \qquad \text{(b)}$$

$$\begin{aligned} \to \qquad 1 + j_{g*} X + \frac{1}{2!}(j_{g*} X)^2 + \ldots &= g\left(1 + X + \frac{1}{2!} X^2 + \ldots\right) g^{-1} \\ &= 1 + g\,X\,g^{-1} + \frac{1}{2}\,g\,X^2\,g^{-1} + \ldots \\ &= 1 + g\,X\,g^{-1} + \frac{1}{2}(g\,X\,g^{-1})^2 + \ldots \end{aligned}$$

$$\therefore \qquad Ad_g\,X = j_{g*}\,X = g\,X\,g^{-1} \qquad \text{(c)}$$

Alternatively, setting $h = e^{t X}$ gives

$$j_g\,e^{t X} = g\,e^{t X}\,g^{-1}$$

Since X is the tangent vector of $e^{t X}$ at the identity $t = 0$, we have

$$j_{g*} X = d\, j_g(X) = \frac{d}{d t}\left(j_g\, e^{t X}\right)\Big|_{t=0} = g\, X\, e^{t X} g^{-1}\Big|_{t=0} = g\, X\, g^{-1}$$

in agreement with (c).

Thus, (b) can be written as

$$e^{g X g^{-1}} = g\, e^X g^{-1}$$

The mapping

$$\mathrm{ad} = \mathrm{Ad}_* = d(\mathrm{Ad}): \quad \mathfrak{g} \to \mathrm{Aut}(\mathfrak{g})$$
$$X \mapsto \mathrm{ad}(X) \equiv \mathrm{ad}_X$$

is called the adjoint representation of $\mathfrak{g}$. Note that it is a representation using $\mathfrak{g}$ as its vector space.

Setting $g(t) = e^{t X}$ turns (c) into

$$\mathrm{Ad}_{e^{tX}}\, Y = e^{t X}\, Y\, e^{-t X}$$

$$\to \quad \mathrm{ad}_X\, Y = \frac{d}{d t}\left(\mathrm{Ad}_{e^{tX}}\, Y\right)\Big|_{t=0}$$
$$= e^{t X}\left(X\, Y - Y\, X\right) e^{-t X}\Big|_{t=0}$$
$$= X\, Y - Y\, X$$
$$= [X, Y] \qquad \text{(d)}$$

It is easy to verify that $\mathrm{Ad}_{e^{tX}} = \mathrm{Ad}\left(e^{t X}\right)$ is a 1-parameter Lie group of transformations on $\mathfrak{g}$. As stated before, such groups can always be written as $g(t) = e^{t S}$ where $S = g'(0)$. Therefore, we have

$$\mathrm{Ad}_{e^{tX}} = e^{t\, \mathrm{ad}_X} \qquad \text{(e)}$$

$$\to \quad \mathrm{Ad}_{e^{tX}}\, Y = e^{t X}\, Y\, e^{-t X} = e^{t\, \mathrm{ad}_X}\, Y \qquad \text{(f)}$$

Note that in a n-D matrix representation, X, Y are arbitrary $n \times n$ matrices and g, h are invertible $n \times n$ matrices. In general, they can be taken as operators with the stated invertibility.

BCH Formula

Ref:

B.C.Hall, “Lie Groups, Lie Algebras, & Representaions”, 2nd ed., Springer (2003).

See in particular Chap 5.

Using capital letters to denote operators, we set

$$C(t) = \ln\left(e^{A t}\, e^B\right) \qquad \text{(2A.11)}$$

$$\to \quad e^{C(t)} = e^{A t}\, e^B \qquad \text{(2A.11a)}$$

(f) can be written as

$$e^{C(t)}\, M\, e^{-C(t)} = e^{\mathrm{ad}\, C(t)}\, M \qquad \text{(2A.12)}$$

Inserting (2A.11) into (2A.12), we have

$$L.H.S. = e^{A t}\, e^B\, M\, e^{-B}\, e^{-A t} = e^{A t}\left(e^{\mathrm{ad}\, B}\, M\right) e^{-A t} = e^{\mathrm{ad}\,(A t)}\, e^{\mathrm{ad}\, B}\, M$$

$$\to \quad e^{\mathrm{ad}\, C(t)} = e^{\mathrm{ad}\,(A t)}\, e^{\mathrm{ad}\, B} \qquad \text{(2A.13)}$$

Differentiating (2A.11) gives

$$e^{C(t)} \frac{d\, C(t)}{d t} = A\, e^{A t}\, e^B = A\, e^{C(t)}$$

$$\to \quad e^{C(t)} \frac{d\, C(t)}{d t}\, e^{-C(t)} = A$$

$$-e^{C(t)} \frac{d}{d t}\, e^{-C(t)} = A \qquad \text{(2A.14)}$$

Later on, we shall show that

$$-e^{C(t)} \frac{d}{d\,t} e^{-C(t)} = f(\text{ad}\, C(t)\,)\left[\,\dot{C}(t)\,\right] \tag{2A.15}$$

where

$$f(z) = \frac{e^z - 1}{z} \tag{2A.16}$$

(2A.14) then becomes

$$f(\text{ad}\, C(t)\,)\left[\,\dot{C}(t)\,\right] = A \tag{2A.17}$$

Setting

$$g(z) = \frac{\ln z}{z-1} = \sum_{n=0}^{\infty} \frac{(1-z)^n}{n+1} \tag{2A.3}$$

we have

$$g(e^z) = \frac{z}{e^z - 1}$$

Using (2A.16), we have

$$g(e^z)\, f(z) = 1 \tag{2A.18}$$

(2A.17) thus implies

$$\dot{C}(t) = g\left(\,e^{\text{ad}\, C(t)}\,\right) A$$

$$= g\left(e^{\text{ad}\,(A\,t)}\, e^{\text{ad}\, B}\right) A \qquad \text{[(2A.13) used.]} \tag{2A.20}$$

Using (2A.11a), we have

$$C \equiv C(1) = \ln\left(e^A\, e^B\right) \qquad\qquad C(0) = \ln e^B = B$$

(2A.20) can then be integrated to give

$$C = B + \int_0^1 d\,t\, g\left(e^{\text{ad}\,(A\,t)}\, e^{\text{ad}\, B}\right) A \tag{2A.2}$$

$$= \ln\left(\,e^A\, e^B\right)$$

which is known as the Baker-Campbell-Hausdorff (BCH) formula.

Using (2A.3). we have

$$g\left(e^{\text{ad}\,(A\,t)}\, e^{\text{ad}\, B}\right) A \;=\; \sum_{n=0}^{\infty} \frac{\left(\,1 - e^{\text{ad}\,(A\,t)}\, e^{\text{ad}\, B}\,\right)^n}{n+1} A$$

$$= A + \sum_{n=1}^{\infty} \frac{\left(\,1 - e^{\text{ad}\,(A\,t)}\, e^{\text{ad}\, B}\,\right)^n}{n+1} A$$

$$\left(\,1 - e^{\text{ad}\,(A\,t)}\, e^{\text{ad}\, B}\,\right)^n = \left[\,1 - \sum_{j,\,k=0}^{\infty} \frac{(\text{ad}\, A\, t)^j}{j!} \frac{(\text{ad}\, B)^k}{k!}\,\right]^n$$

$$= (-)^n \left[\,\sum_{k=1}^{\infty} \frac{(\text{ad}\, B)^k}{k!} + \sum_{j=1}^{\infty} \frac{(\text{ad}\, A\, t)^j}{j!} + \sum_{j,\,k=1}^{\infty} \frac{(\text{ad}\, A\, t)^j}{j!} \frac{(\text{ad}\, B)^k}{k!}\,\right]^n$$

The square bracket contains a sum of terms of the form

$$T_{p\,q} = t^p \frac{(\text{ad}\, A)^p}{p\,!} \frac{(\text{ad}\, B)^q}{q!} \qquad\qquad p,\, q = 0,\ 1,\ 2,\ \ldots \text{ but } p+q \geq 1.$$

Expanding the product of n of these brackets, we have a sum of terms each of which is a product of n factors of $T_{p\,q}$'s :

$$\left(1 - e^{\text{ad}(A\,t)}\, e^{\text{ad}\,B}\right)^n = (-)^n \sum_{\substack{p_i, q_i = 0 \\ p_i + q_i \geq 1}}^{\infty} T_{p_1 q_1} \ldots T_{p_n q_n}$$

$$= (-)^n \sum_{\substack{p_i, q_i = 0 \\ p_i + q_i \geq 1}}^{\infty} t^{\sum_{i=1}^n p_i} \frac{(\text{ad}\,A)^{p_1}}{p_1!} \frac{(\text{ad}\,B)^{q_1}}{q_1!} \ldots \frac{(\text{ad}\,A)^{p_n}}{p_n!} \frac{(\text{ad}\,B)^{q_n}}{q_n!}$$

Putting everything together and using

$$\int_0^1 d\,t\, t^a = \frac{1}{1+a}$$

(2A.2) takes the power series form

$$C = B + A + \sum_{n=1}^{\infty} \frac{(-)^n}{n+1} \sum_{\substack{p_i, q_i = 0 \\ p_i + q_i \geq 1}}^{\infty} \frac{1}{1 + \sum_{i=1}^n p_i} \frac{(\text{ad}\,A)^{p_1}}{p_1!} \frac{(\text{ad}\,B)^{q_1}}{q_1!} \ldots \frac{(\text{ad}\,A)^{p_n}}{p_n!} \frac{(\text{ad}\,B)^{q_n}}{q_n!} A \qquad \text{(2A.5)}$$

Let us calculate C up to $O(\text{ad}^3)$.

The $n = 1$ terms in (2A.5) are

$$\mathcal{S}_1 = -\frac{1}{2} \left(\sum_{\substack{p_1, q_1 = 0 \\ p_1 + q_1 \geq 1}} \frac{1}{1+p_1} \frac{(\text{ad}\,A)^{p_1}}{p_1!} \frac{(\text{ad}\,B)^{q_1}}{q_1!} \right) A$$

$$= -\frac{1}{2} \left(\sum_{p_1=1}^{\infty} \frac{1}{1+p_1} \frac{(\text{ad}\,A)^{p_1}}{p_1!} + \sum_{q_1=1}^{\infty} \frac{(\text{ad}\,B)^{q_1}}{q_1!} + \sum_{p_1, q_1 = 1} \frac{1}{1+p_1} \frac{(\text{ad}\,A)^{p_1}}{p_1!} \frac{(\text{ad}\,B)^{q_1}}{q_1!} \right) A$$

The values of the terms are

p_1	q_1	T
0	1	$\text{ad}\,B$
1	0	$\frac{1}{2}\text{ad}\,A$
1	1	$\frac{1}{2}\text{ad}\,A\,\text{ad}\,B$
0	2	$\frac{1}{2}(\text{ad}\,B)^2$
2	0	$\frac{1}{6}(\text{ad}\,A)^2$
1	2	$\frac{1}{4}\text{ad}\,A\,(\text{ad}\,B)^2$
2	1	$\frac{1}{6}(\text{ad}\,A)^2\,\text{ad}\,B$
0	3	$\frac{1}{6}(\text{ad}\,B)^3$
3	0	$\frac{1}{24}(\text{ad}\,A)^3$

$$\rightarrow \qquad \mathcal{S}_1 = -\frac{1}{2} \left(\text{ad}\,B + \frac{1}{2}\text{ad}\,A + \frac{1}{2}\text{ad}\,A\,\text{ad}\,B + \frac{1}{2}(\text{ad}\,B)^2 + \frac{1}{6}(\text{ad}\,A)^2 \right.$$

$$\left. + \frac{1}{4}\text{ad}\,A\,(\text{ad}\,B)^2 + \frac{1}{6}(\text{ad}\,A)^2\,\text{ad}\,B + \frac{1}{6}(\text{ad}\,B)^3 + \frac{1}{24}(\text{ad}\,A)^3 \right) A + O\left(\text{ad}^4\right)$$

Using

$$\text{ad}_A A = [A, A] = 0$$

we have

$$\mathcal{S}_1 = -\frac{1}{2}\left(\text{ad}\,B + \frac{1}{2}\,\text{ad}\,A\,\text{ad}\,B + \frac{1}{2}\,(\text{ad}\,B)^2 + \frac{1}{4}\,\text{ad}\,A\,(\text{ad}\,B)^2 + \frac{1}{6}\,(\text{ad}\,A)^2\,\text{ad}\,B + \frac{1}{6}\,(\text{ad}\,B)^3\right)A + O\left(\text{ad}^4\right)$$

The $n = 2$ terms in (2A.5) are

$$\mathcal{S}_2 = \frac{1}{3}\left(\sum_{\substack{p_i,\, q_i = 0\\ p_i + q_i \geq 1}} \frac{1}{1 + p_1 + p_2}\,\frac{(\text{ad}\,A)^{p_1}}{p_1\,!}\,\frac{(\text{ad}\,B)^{q_1}}{q_1\,!}\,\frac{(\text{ad}\,A)^{p_2}}{p_2\,!}\,\frac{(\text{ad}\,B)^{q_2}}{q_2\,!}\right)A$$

The values of the terms are

p_1	p_2	q_1	q_2	T
0	0	1	1	$(\text{ad}\,B)^2$
0	1	1	0	$\frac{1}{2}\,\text{ad}\,B\,\text{ad}\,A$
1	0	0	1	$\frac{1}{2}\,\text{ad}\,A\,\text{ad}\,B$
1	1	0	0	$\frac{1}{3}\,(\text{ad}\,A)^2$
0	1	1	1	$\frac{1}{2}\,\text{ad}\,B\,\text{ad}\,A\,\text{ad}\,B$
1	0	1	1	$\frac{1}{2}\,\text{ad}\,A\,(\text{ad}\,B)^2$
1	1	0	1	$\frac{1}{3}\,(\text{ad}\,A)^2\,\text{ad}\,B$
1	1	1	0	$\frac{1}{3}\,\text{ad}\,A\,\text{ad}\,B\,\text{ad}\,A$
0	0	1	2	$\frac{1}{2}\,(\text{ad}\,B)^3$
0	0	2	1	$\frac{1}{2}\,(\text{ad}\,B)^3$
0	1	2	0	$\frac{1}{4}\,(\text{ad}\,B)^2\,\text{ad}\,A$
1	0	0	2	$\frac{1}{4}\,\text{ad}\,A\,(\text{ad}\,B)^2$
0	2	1	0	$\frac{1}{6}\,\text{ad}\,B\,(\text{ad}\,A)^2$
2	0	0	1	$\frac{1}{6}\,(\text{ad}\,A)^2\,\text{ad}\,B$
1	2	0	0	$\frac{1}{8}\,(\text{ad}\,A)^3$
2	1	0	0	$\frac{1}{8}\,(\text{ad}\,A)^3$

Since all terms ending with ad A vanish, we have

$$\mathcal{S}_2 = \frac{1}{3}\Big((\text{ad } B)^2 + \frac{1}{2}\text{ad } A \,\text{ad } B + \frac{1}{2}\text{ad } B\,\text{ad } A\,\text{ad } B + \frac{3}{4}\text{ad } A\,(\text{ad } B)^2 + \frac{1}{2}(\text{ad } A)^2\,\text{ad } B + (\text{ad } B)^3 + \ldots\Big)A$$

The $n = 3$ terms in (2A.5) are

$$\mathcal{S}_3 = -\frac{1}{4}\left(\sum_{\substack{p_i,\, q_i = 0\\ p_i + q_i \geq 1}} \frac{1}{1+p_1+p_2+p_3}\,\frac{(\text{ad } A)^{p_1}}{p_1!}\,\frac{(\text{ad } B)^{q_1}}{q_1!}\,\frac{(\text{ad } A)^{p_2}}{p_2!}\,\frac{(\text{ad } B)^{q_2}}{q_2!}\,\frac{(\text{ad } A)^{p_3}}{p_3!}\,\frac{(\text{ad } B)^{q_3}}{q_3!}\right)A$$

The values of the terms are

p_1	p_2	p_3	q_1	q_2	q_3	T
0	0	0	1	1	1	$(\text{ad } B)^3$
0	0	1	1	1	0	$\frac{1}{2}(\text{ad } B)^2\,\text{ad } A$
0	1	0	1	0	1	$\frac{1}{2}\text{ad } B\,\text{ad } A\,\text{ad } B$
1	0	0	0	1	1	$\frac{1}{2}\text{ad } A\,(\text{ad } B)^2$
0	1	1	1	0	0	$\frac{1}{3}\text{ad } B\,(\text{ad } A)^2$
1	0	1	0	1	0	$\frac{1}{3}\text{ad } A\,\text{ad } B\,\text{ad } A$
1	1	0	0	0	1	$\frac{1}{3}(\text{ad } A)^2\,\text{ad } B$
1	1	1	0	0	0	$\frac{1}{4}(\text{ad } A)^3$

$$\therefore \qquad \mathcal{S}_3 = -\frac{1}{4}\Big((\text{ad } B)^3 + \frac{1}{2}\text{ad } B\,\text{ad } A\,\text{ad } B + \frac{1}{2}\text{ad } A\,(\text{ad } B)^2 + \frac{1}{3}(\text{ad } A)^2\,\text{ad } B + \ldots\Big)A$$

(2A.5) thus becomes

$$\begin{aligned} C &= B + A + \Big(-\frac{1}{2}\text{ad } B + \frac{1}{12}(\text{ad } B)^2 - \frac{1}{12}\text{ad } A\,\text{ad } B + \frac{1}{24}\text{ad } B\,\text{ad } A\,\text{ad } B\Big)A + O(\text{ad}^4) \\ &= B + A - \frac{1}{2}[B, A] + \frac{1}{12}[B, [B, A]] - \frac{1}{12}[A, [B, A]] + \frac{1}{24}[B, [A, [B, A]]] + O(\text{ad}^4) \\ &= A + B + \frac{1}{2}[A, B] + \frac{1}{12}\Big([A, [A, B]] - [B, [A, B]]\Big) + \frac{1}{24}[[A, [A, B]], B] + O(\text{ad}^4) \qquad (2A.6) \end{aligned}$$

A related expansion is the Zassenhaus formula,

$$e^{t(A+B)} = e^{tA}\, e^{tB}\, e^{t^2 Z_2}\, e^{t^3 Z_3}\, e^{t^4 Z_4} \ldots \qquad (2A.7)$$

By applying the BCH formula recursively to

$$e^{-tA}\, e^{t(A+B)} = \prod_{n=1}^{\infty} e^{t^n Z_n}$$

One obtains [see Casas, F.; Murua, A.; Nadinic, M. (2012). “Efficient computation of the Zassenhaus formula". Computer Physics Communications 183 (11): 2386.]

$$Z_2 = -\frac{1}{2}[A, B] \tag{2A.8}$$

$$Z_3 = \frac{1}{3!}\Big([A, [A, B]] + 2\, [B, [A, B]] \Big) \tag{2A.9}$$

$$Z_4 = -\frac{1}{4!}\Big([[[A, B], A], A] + 3[[[A, B], A], B] + 3[[[A, B], B], B] \Big) \tag{2A.10}$$

Finally, we need to verify (2A.15). To this end, consider the operator

$$O(s, t) = e^{C(t)\, s} \frac{d}{d\,t} e^{-C(t)\, s} \tag{2A.21}$$

$$\begin{aligned}
\to \qquad \partial_s O(s, t) &= e^{C(t)\, s}\, C(t) \frac{d}{d\,t} e^{-C(t)\, s} - e^{C(t)\, s} \frac{d}{d\,t}\big(C(t)\, e^{-C(t)\, s} \big) \\
&= -e^{C(t)\, s} \frac{d\, C(t)}{d\,t} e^{-C(t)\, s} \\
&= -e^{C(t)\, s}\, \dot{C}(t)\, e^{-C(t)\, s} \\
&= -e^{\text{ad}\, C(t)\, s}\, \dot{C}(t) \qquad\qquad [\text{ (f) used. }]
\end{aligned} \tag{2A.22}$$

Integrating, we have

$$\begin{aligned}
O(s, t) - O(0, t) &= -\int_0^s d\,s'\, e^{\text{ad}\, C(t)\, s'}\, \dot{C}(t) \\
&= -\sum_{n=0}^{\infty} \frac{1}{n!} \int_0^s d\,s'\, s'^{\,n} \big(\text{ad}\; C(t)\big)^n\, \dot{C}(t) \\
&= -\sum_{n=0}^{\infty} \frac{s^{n+1}}{(n+1)!} \big(\text{ad}\; C(t)\big)^n\, \dot{C}(t)
\end{aligned} \tag{2A.23}$$

(2A.16) gives

$$f(z) = \frac{e^z - 1}{z} = \frac{1}{z} \sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$$

(2A.23) thus becomes

$$O(s, t) - O(0, t) = -s\, f\big(s\, \text{ad}\; C(t)\big)\, \dot{C}(t) \tag{2A.23a}$$

(2A.21) gives

$$O(1, t) = e^{C(t)} \frac{d}{d\,t} e^{-C(t)} \qquad\qquad O(0, t) = 0$$

Setting $s = 1$ turns (2A.23a) into

$$e^{C(t)} \frac{d}{d\,t} e^{-C(t)} = -f\big(\text{ad}\; C(t)\big)\, \dot{C}(t) \tag{2A.24}$$

thus proving (2A.15).

The same mathematical technique can be used to derive a useful modification of the Neumann-Liouville expansion or Dyson series (1.239) and (1.251). This is the so-called Magnus expansion [see , A. Iserles, A. Marthinsen, and S.P. Norsett , "On the implementation of the method of Magnus series for linear differential equations", BIT 39, 281 (1999)] :

$$\hat{U}(t_b, t_a) = e^{\hat{E}}$$

with

$$\hat{E} = \frac{1}{i\hbar}\int_{t_a}^{t_b} d\,t_1\,\hat{H}(t_1) + \frac{1}{2\,(i\hbar)^2}\int_{t_a}^{t_b} d\,t_2 \int_{t_a}^{t_2} d\,t_1 \left[\hat{H}(t_2), \hat{H}(t_1)\right] \tag{2A.25}$$

$$+\frac{1}{4\,(i\hbar)^3}\left\{\int_{t_a}^{t_b} d\,t_3 \int_{t_a}^{t_3} d\,t_2 \int_{t_a}^{t_2} d\,t_1 \left[\hat{H}(t_3), \left[\hat{H}(t_2), \hat{H}(t_1)\right]\right]\right.$$

$$\left. + \frac{1}{3}\int_{t_a}^{t_b} d\,t_3 \int_{t_a}^{t_b} d\,t_2 \int_{t_a}^{t_b} d\,t_1 \left[\hat{H}(t_3), \left[\hat{H}(t_2), \hat{H}(t_1)\right]\right]\right\} + \ldots$$

which converges faster than the Neumann-Liouville expansion.