Appendix 2B. Direct Calculation of the Time-Sliced Oscillator Amplitude

After time-slicing, the amplitude (2.143) becomes a multiple integral over short-time amplitudes, which, for the action (2.190), take the form

$$
\begin{aligned}
&(x_n , n\epsilon \mid x_{n-1} , (n-1)\epsilon) \\
&= \sqrt{\frac{M}{2\pi\hbar i\epsilon}} \exp\left\{ \frac{i}{\hbar}\frac{1}{2} M\left[\frac{(x_n - x_{n-1})^2}{\epsilon} - \epsilon\,\omega^2 \frac{x_n^2 + x_{n-1}^2}{2} \right]\right\} \qquad (2B.26) \\
&= \sqrt{\frac{M}{2\pi\hbar i\epsilon}} \exp\left\{ \frac{i}{\hbar}\frac{1}{2} M\left[\left(\frac{1}{\epsilon} - \frac{\epsilon\,\omega^2}{2} \right)\left(x_n^2 + x_{n-1}^2 \right) - \frac{2}{\epsilon} x_n x_{n-1} \right]\right\} \\
&\equiv \mathcal{N}_1 \exp\left\{ \frac{i}{\hbar}\left[a_1\left(x_n^2 + x_{n-1}^2 \right) - 2\, b_1\, x_n\, x_{n-1} \right]\right\} \qquad (2B.27)
\end{aligned}
$$

where

$$
a_1 = \frac{1}{2\epsilon} M\left[1 - 2\left(\frac{\omega\,\epsilon}{2} \right)^2 \right] \qquad b_1 = \frac{1}{2\epsilon} M
$$

$$
\mathcal{N}_1 = \sqrt{\frac{M}{2\pi\hbar i\epsilon}} \qquad (2B.28)
$$

The path integral integrates the x_n's for $n = 1, \ldots, N$.

For $n = 1$, we have,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} d\,x_1\, (x_2,\ 2\epsilon \mid x_1\, \epsilon)(x_1\, \epsilon \mid x_0\, 0) \\
&= \mathcal{N}_1^2 \int_{-\infty}^{\infty} d\,x_1 \exp\left\{ \frac{i}{\hbar}\left[a_1\left(x_2^2 + x_1^2 \right) - 2\, b_1\, x_2\, x_1 + a_1\left(x_1^2 + x_0^2 \right) - 2\, b_1\, x_1\, x_0 \right]\right\} \\
&= \mathcal{N}_1^2 \exp\left[\frac{i}{\hbar} a_1\left(x_2^2 + x_0^2 \right) \right] \int_{-\infty}^{\infty} d\,x_1 \exp\left\{ \frac{i}{\hbar}\left[2\, a_1\, x_1^2 - 2\, b_1\, (x_2 + x_0)\, x_1 \right]\right\} \\
&= \mathcal{N}_1^2 \exp\left[\frac{i}{\hbar} a_1\left(x_2^2 + x_0^2 \right) \right] \sqrt{\frac{\pi\hbar i}{2\, a_1}} \exp\left\{ -\frac{i}{\hbar} \frac{b_1^2\, (x_2 + x_0)^2}{2\, a_1} \right\} \\
&\equiv \mathcal{N}_2 \exp\left[\frac{i}{\hbar}\left(a_2\, x_2^2 - 2\, b_2\, x_0\, x_2 \right) \right] \exp\left(\frac{i}{\hbar} c_2\, x_0^2 \right) \\
&\equiv (x_2,\ 2\epsilon \mid x_0\, 0)
\end{aligned}
$$

where

$$
\mathcal{N}_2 = \mathcal{N}_1^2 \sqrt{\frac{\pi\hbar i}{2\, a_1}}
$$

$$
a_2 = a_1 - \frac{b_1^2}{2\, a_1} \qquad b_2 = \frac{b_1^2}{2\, a_1} \qquad c_2 = a_2
$$

For $n = 2$, we have,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} d\,x_2\, (x_3,\ 3\epsilon \mid x_2 ,\, 2\epsilon)(x_2 ,\, 2\epsilon \mid x_0\, 0) \\
&= \mathcal{N}_1\, \mathcal{N}_2 \int_{-\infty}^{\infty} d\,x_2 \exp\left\{ \frac{i}{\hbar}\left[a_1\left(x_3^2 + x_2^2 \right) - 2\, b_1\, x_3\, x_2 + a_2\, x_2^2 - 2\, b_2\, x_2\, x_0 + c_2\, x_0^2 \right]\right\} \\
&= \mathcal{N}_1\, \mathcal{N}_2 \exp\left[\frac{i}{\hbar}\left(a_1\, x_3^2 + c_2\, x_0^2 \right) \right] \int_{-\infty}^{\infty} d\,x_2 \exp\left\{ \frac{i}{\hbar}\left[(a_1 + a_2)\, x_2^2 - 2\, (b_1\, x_3 + b_2\, x_0)\, x_2 \right]\right\}
\end{aligned}
$$

$$= \mathcal{N}_1 \, \mathcal{N}_2 \exp\left[\frac{i}{\hbar} \left(a_1 \, x_3^2 + c_2 \, x_0^2 \right) \right] \sqrt{\frac{\pi \hbar i}{a_1 + a_2}} \exp\left[-\frac{i}{\hbar} \frac{(b_1 \, x_3 + b_2 \, x_0)^2}{a_1 + a_2} \right]$$

$$\equiv \mathcal{N}_3 \exp\left[\frac{i}{\hbar} \left(a_3 \, x_3^2 - 2 \, b_3 \, x_0 \, x_3\right) \right] \exp\left(\frac{i}{\hbar} c_3 \, x_0^2 \right)$$

$$\equiv (x_3, \, 3 \epsilon \mid x_0 \, 0)$$

where

$$\mathcal{N}_3 = \mathcal{N}_1 \, \mathcal{N}_2 \sqrt{\frac{\pi \hbar i}{a_1 + a_2}}$$

$$a_3 = a_1 - \frac{b_1^2}{a_1 + a_2} \qquad b_3 = \frac{b_1 \, b_2}{a_1 + a_2} \qquad c_3 = c_2 - \frac{b_2^2}{a_1 + a_2}$$

For a general n, we have,

$$\int_{-\infty}^{\infty} d\, x_n \, (x_{n+1}, \, (n+1) \epsilon \mid x_n \, , \, n \epsilon) (x_n \, , \, n \epsilon \mid x_0 \, 0)$$

$$= \mathcal{N}_1 \, \mathcal{N}_n \int_{-\infty}^{\infty} d\, x_n \exp\left\{ \frac{i}{\hbar} \left[a_1 \left(x_{n+1}^2 + x_n^2\right) - 2 \, b_1 \, x_{n+1} \, x_n + a_n \, x_n^2 - 2 \, b_n \, x_n \, x_0 + c_n \, x_0^2 \right] \right\}$$

$$= \mathcal{N}_1 \, \mathcal{N}_n \exp\left[\frac{i}{\hbar} \left(a_1 \, x_{n+1}^2 + c_n \, x_0^2 \right) \right] \int_{-\infty}^{\infty} d\, x_n \exp\left\{ \frac{i}{\hbar} \left[(a_1 + a_n) \, x_n^2 - 2 \, (b_1 \, x_{n+1} + b_n \, x_0) \, x_n \right] \right\}$$

$$= \mathcal{N}_1 \, \mathcal{N}_n \exp\left[\frac{i}{\hbar} \left(a_1 \, x_{n+1}^2 + c_n \, x_0^2 \right) \right] \sqrt{\frac{\pi \hbar i}{a_1 + a_n}} \exp\left[-\frac{i}{\hbar} \frac{(b_1 \, x_{n+1} + b_n \, x_0)^2}{a_1 + a_n} \right]$$

$$\equiv \mathcal{N}_{n+1} \exp\left[\frac{i}{\hbar} \left(a_{n+1} \, x_{n+1}^2 - 2 \, b_{n+1} \, x_0 \, x_{n+1}\right) \right] \exp\left(\frac{i}{\hbar} c_{n+1} \, x_0^2 \right)$$

$$\equiv (x_{n+1}, \, (n+1) \epsilon \mid x_0 \, 0)$$

where, for $n \geq 1$,

$$\mathcal{N}_{n+1} = \mathcal{N}_1 \, \mathcal{N}_n \sqrt{\frac{\pi \hbar i}{a_1 + a_n}} \qquad (2B.30)$$

$$a_{n+1} = a_1 - \frac{b_1^2}{a_1 + a_n} = \frac{a_1^2 - b_1^2 + a_1 \, a_n}{a_1 + a_n} \qquad (2B.31)$$

$$b_{n+1} = \frac{b_1 \, b_n}{a_1 + a_n} \qquad (2B.32)$$

$$c_{n+1} = c_n - \frac{b_n^2}{a_1 + a_n} \qquad (2B.32a)$$

Following Kleinert, we first demand

$$c_n = a_n$$

so that the multi-time-step amplitude

$$(x_{n+1}, \, (n+1) \epsilon \mid x_0 \, 0) = \mathcal{N}_{n+1} \exp\left\{ \frac{i}{\hbar} \left[a_{n+1} \left(x_{n+1}^2 + x_0^2\right) - 2 \, b_{n+1} \, x_0 \, x_{n+1} \right] \right\} \qquad (2B.29)$$

keeps the same form as the 1-time-step amplitude (2B.2).

(2B.32a) then becomes

$$a_{n+1} = a_n - \frac{b_n^2}{a_1 + a_n} = a_1 - \frac{b_1^2}{a_1 + a_n} \qquad \text{[(2B.31) used.]}$$

$$\rightarrow \qquad a_n - a_1 = \frac{b_n^2 - b_1^2}{a_1 + a_n}$$

$$a_n^2 - a_1^2 = b_n^2 - b_1^2 \qquad (2B.33)$$

$$\rightarrow \qquad a_n = \sqrt{b_n^2 - b_1^2 + a_1^2} \qquad (2B.33a)$$

(2B.32) then becomes

$$b_{n+1} = \frac{b_1 \, b_n}{a_1 + \sqrt{b_n^2 - b_1^2 + a_1^2}} \qquad n \geq 1 \qquad (2B.34)$$

$$\rightarrow \qquad \frac{1}{b_{n+1}} = \frac{1}{b_1} \left(\frac{a_1}{b_n} + \sqrt{1 - \frac{b_1^2 - a_1^2}{b_n^2}} \right) \qquad (2B.35)$$

In terms of the auxiliary frequency [see (2.161)]

$$\sin \frac{\epsilon \, \tilde{\omega}}{2} \equiv \frac{\epsilon \, \omega}{2}$$

(2B.3) becomes

$$a_1 = \frac{1}{2 \, \epsilon} M \left(1 - 2 \sin^2 \frac{\epsilon \, \tilde{\omega}}{2} \right) = \frac{1}{2 \, \epsilon} M \cos \epsilon \, \tilde{\omega} \qquad (2B.36)$$

$$= b_1 \cos \epsilon \, \tilde{\omega}$$

so that (2B.35) reads

$$\frac{1}{b_{n+1}} = \frac{\cos \epsilon \, \tilde{\omega}}{b_n} + \frac{1}{b_1} \sqrt{1 - \frac{b_1^2 \sin^2 \epsilon \, \tilde{\omega}}{b_n^2}}$$

$$= \frac{\cos \epsilon \, \tilde{\omega}}{b_n} + \frac{2 \, \epsilon}{M} \sqrt{1 - \frac{M^2}{4 \, \epsilon^2} \frac{\sin^2 \epsilon \, \tilde{\omega}}{b_n^2}} \qquad (2B.37)$$

Introducing the dimensionless quantity

$$\beta_n \equiv \frac{b_n}{b_1} = \frac{2 \, \epsilon}{M} b_n \qquad (2B.38)$$

with

$$\beta_1 = 1 \qquad (2B.39)$$

turns (2B.37) into

$$\frac{1}{\beta_{n+1}} = \frac{\cos \epsilon \, \tilde{\omega}}{\beta_n} + \sqrt{1 - \frac{\sin^2 \epsilon \, \tilde{\omega}}{\beta_n^2}} \qquad n \geq 1 \qquad (2B.40)$$

$$\rightarrow \qquad \frac{1}{\beta_2} = \cos \epsilon \, \tilde{\omega} + \sqrt{1 - \sin^2 \epsilon \, \tilde{\omega}} = 2 \cos \epsilon \, \tilde{\omega} = \frac{\sin 2 \, \epsilon \, \tilde{\omega}}{\sin \epsilon \, \tilde{\omega}}$$

$$\frac{1}{\beta_3} = \cos \epsilon \, \tilde{\omega} \, \frac{\sin 2 \, \epsilon \, \tilde{\omega}}{\sin \epsilon \, \tilde{\omega}} + \sqrt{1 - \sin^2 2 \, \epsilon \, \tilde{\omega}}$$

$$= \cos \epsilon \, \tilde{\omega} \, \frac{\sin 2 \, \epsilon \, \tilde{\omega}}{\sin \epsilon \, \tilde{\omega}} + \cos 2 \, \epsilon \, \tilde{\omega}$$

$$= \frac{\sin 3 \, \epsilon \, \tilde{\omega}}{\sin \epsilon \, \tilde{\omega}}$$

Assuming

$$\frac{1}{\beta_n} = \frac{\sin n \, \epsilon \, \tilde{\omega}}{\sin \epsilon \, \tilde{\omega}} \qquad (2B.42a)$$

then (2B.37) becomes

$$\frac{1}{\beta_{n+1}} = \cos\epsilon\tilde{\omega}\,\frac{\sin n\epsilon\tilde{\omega}}{\sin\epsilon\tilde{\omega}} + \sqrt{1-\sin^2 n\epsilon\tilde{\omega}}$$
$$= \cos\epsilon\tilde{\omega}\,\frac{\sin n\epsilon\tilde{\omega}}{\sin\epsilon\tilde{\omega}} + \cos n\epsilon\tilde{\omega}$$
$$= \frac{\sin[(n+1)\epsilon\tilde{\omega}]}{\sin\epsilon\tilde{\omega}} \qquad (2B.42)$$

thus proving (2B.42a).

(2B.38) then gives

$$b_{n+1} = b_1\,\beta_{n+1} = \frac{M}{2\epsilon}\,\frac{\sin\epsilon\tilde{\omega}}{\sin[(n+1)\epsilon\tilde{\omega}]} \qquad (2B.43)$$

(2B.33a) becomes

$$a_{n+1} = \sqrt{b_{n+1}^2 - b_1^2 + a_1^2} = \sqrt{b_1^2\,\beta_{n+1}^2 - b_1^2\sin^2\epsilon\tilde{\omega}}$$
$$= b_1\,\frac{\sin\epsilon\tilde{\omega}}{\sin[(n+1)\epsilon\tilde{\omega}]}\sqrt{1-\sin^2[(n+1)\epsilon\tilde{\omega}]}$$
$$= \frac{M}{2\epsilon}\sin\epsilon\tilde{\omega}\,\frac{\cos[(n+1)\epsilon\tilde{\omega}]}{\sin[(n+1)\epsilon\tilde{\omega}]} \qquad (2B.44)$$

(2B.32) then gives

$$a_1 + a_n = \frac{b_1\,b_n}{b_{n+1}} = b_1\,\frac{\beta_n}{\beta_{n+1}} = b_1\,\frac{\sin[(n+1)\epsilon\tilde{\omega}]}{\sin(n\epsilon\tilde{\omega})} \qquad (2B.44a)$$

From (2B.30), we have

$$\mathcal{N}_{n+1} = \mathcal{N}_1\,\mathcal{N}_n\sqrt{\frac{\pi\hbar i}{a_1+a_n}}$$
$$= \mathcal{N}_1^2\,\mathcal{N}_{n-1}\sqrt{\frac{(\pi\hbar i)^2}{(a_1+a_n)(a_1+a_{n-1})}}$$
$$= \mathcal{N}_1^{n+1}\sqrt{\frac{(\pi\hbar i)^n}{(a_1+a_n)(a_1+a_{n-1})\ldots(a_1+a_2)(2a_1)}}$$
$$= \mathcal{N}_1\left(\frac{b_1}{\pi\hbar i}\right)^{n/2}\sqrt{\frac{(\pi\hbar i)^n\sin\epsilon\tilde{\omega}}{b_1^n\sin[(n+1)\epsilon\tilde{\omega}]}}$$

$$= \mathcal{N}_1\sqrt{\frac{\sin\epsilon\tilde{\omega}}{\sin[(n+1)\epsilon\tilde{\omega}]}} \qquad (2B.45)$$

Putting all these into (2B.29) turns it into the previous result (2.197).