Appendix 2C. Derivation of Mehler Formula

Let

$$M(x,\,x') = \frac{1}{\sqrt{1-a^2}}\exp\left\{-\frac{1}{2\,(1-a^2)}\left[\,(x^2+x'^2)\,(1+a^2)-4\,a\,x\,x'\,\right]\right\} \qquad \text{(2C.46a)}$$

$$N(x,\,x') = \exp\left[-\frac{1}{2}\,(x^2+x'^2)\right]\sum_{n=0}^{\infty}\frac{a^n}{2^n\,n!}\,H_n(x)\,H_n(x') \qquad \text{(2C.46b)}$$

We wish to verify the Mehler formula (2.295):

$$M(x,\,x') = N(x,\,x') \qquad \text{(2C.46c)}$$

To begin, consider

$$\tilde{F}(k,\,k') = \pi\exp\left[-\frac{1}{4}\,(\,k^2+k'^2+2\,a\,k\,k'\,)\right] \qquad \text{(2C.46)}$$

Its (inverse) Fourier transform is [see “A2C._Code.nb”]

$$F(x,\,x') = \int_{-\infty}^{\infty}\frac{d\,k}{2\,\pi}\int_{-\infty}^{\infty}\frac{d\,k'}{2\,\pi}\,e^{i\,k\,x+i\,k'\,x'}\,\tilde{F}(k,\,k') \qquad \text{(2C.47)}$$

$$= \frac{1}{\sqrt{1-a^2}}\exp\left[-\frac{1}{1-a^2}\,(x^2+x'^2-2\,a\,x\,x'\,)\right] \qquad (a<1) \qquad \text{(2C.47a)}$$

$$= M(x,\,x')\exp\left[-\frac{1}{2}\,(x^2+x'^2)\right] \qquad \text{(2C.47b)}$$

Assuming Mehler’s formula is correct, we have

$$F(x,\,x') = \exp\left[-(x^2+x'^2)\right]\sum_{n=0}^{\infty}\frac{a^n}{2^n\,n!}\,H_n(x)\,H_n(x') \qquad \text{(2C.47c)}$$

From the generating function of the Hermite polynomials [see Gradshteyn & Ryzhik, Formula 8.957.1]

$$e^{-t^2+2\,t\,x} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\,H_n(x) \rightarrow \qquad e^{\frac{k^2}{4}-i\,k\,x} = \sum_{n=0}^{\infty}\frac{\left(-i\,\frac{k}{2}\right)^n}{n!}\,H_n(x) \qquad \text{(2C.48)}$$

we have

$$e^{-i\,k\,x} = e^{-k^2/4}\sum_{n=0}^{\infty}\frac{\left(-i\,\frac{k}{2}\right)^n}{n!}\,H_n(x) \qquad \text{(2C.49)}$$

Therefore,

$$\tilde{F}(k,\,k') = \int_{-\infty}^{\infty}d\,x\int_{-\infty}^{\infty}d\,x'\,e^{-i\,k\,x-i\,k'\,x'}\,F(x,\,x')$$

$$= e^{-(k^2+k'^2)/4}\int_{-\infty}^{\infty}d\,x\int_{-\infty}^{\infty}d\,x'\,F(x,\,x')\sum_{n,\,m=0}^{\infty}\frac{\left(-i\,\frac{k}{2}\right)^n}{n!}\,\frac{\left(-i\,\frac{k'}{2}\right)^m}{m!}\,H_n(x)\,H_m(x') \qquad \text{(2C.49a)}$$

Assuming Mehler’s formula is correct, we can use (2C.47c) and get

$$\tilde{F}(k,\,k') = e^{-(k^2+k'^2)/4}\sum_{n,\,m=0}^{\infty}\frac{\left(-i\,\frac{k}{2}\right)^n}{n!}\,\frac{\left(-i\,\frac{k'}{2}\right)^m}{m!}\sum_{j=0}^{\infty}\frac{a^j}{2^j\,j!}\int_{-\infty}^{\infty}d\,x\int_{-\infty}^{\infty}d\,x' \qquad \text{(2C.49b)}$$

$$\exp\left[-(x^2+x'^2)\right]\,H_j(x)\,H_j(x')\,H_n(x)\,H_m(x')$$

Using [see Gradshteyn & Ryzhik, Formula 7.374.1]

$$\int_{-\infty}^{\infty}d\,x\,e^{-x^2}\,H_n(x)\,H_m(x) = 2^n\,n!\,\sqrt{\pi}\,\,\delta_{m\,n}$$

we have

$$\tilde{F}(k, k') = \sqrt{\pi}\, e^{-(k^2+k'^2)/4} \sum_{n,m=0}^{\infty} \frac{\left(-i\,\frac{k}{2}\right)^n}{n!} \frac{\left(-i\,\frac{k'}{2}\right)^m}{m!} a^m \int_{-\infty}^{\infty} d\,x\; e^{-x^2} H_m(x)\, H_n(x)$$

$$= \pi\, e^{-(k^2+k'^2)/4} \sum_{n=0}^{\infty} \frac{\left(-i\,\frac{k}{2}\right)^n}{n!} (-i\,k')^n\, a^n$$

$$= \pi\, e^{-(k^2+k'^2)/4} \exp\left(-\frac{1}{2}\, a\, k\, k'\right)$$

which agrees with (2C.46). Hence, the assumption that Mehler’s formula is correct is verified.