3.1. External Sources

Consider a harmonic oscillator with an action

$$\mathcal{A}_\omega = \int_{t_a}^{t_b} d\,t\, \frac{1}{2} M \left(\dot{x}^2 - \omega^2 x^2 \right) \tag{3.1}$$

and a perturbation due to the coupling to an external source or current $j(t)$ via the source action

$$\mathcal{A}_j = \int_{t_a}^{t_b} d\,t\, x(t)\, j(t) \tag{3.2}$$

The total action

$$\mathcal{A} = \mathcal{A}_\omega + \mathcal{A}_j \tag{3.3}$$

is still harmonic in x & $\dot{x}$, which makes the corresponding path integral easy to solve. In particular, the factorization (2.151) still holds so that

$$(x_b\, t_b \mid x_a\, t_a)_\omega^j = e^{\frac{i}{\hbar}\mathcal{A}_{\text{cl}}^j} F_\omega^j(t_b, t_a) \tag{3.4}$$

where $\mathcal{A}_{\text{cl}}^j$ is the classical action in the presence of source $j(t)$. The classical equation of motion is simply the Euler-Lagrange eq. for the classical Lagrangian

$$L = \frac{1}{2} M \left(\dot{x}^2 - \omega^2 x^2 \right) + j(t)\, x$$

i.e.,

$$\ddot{x}_{\text{cl}}^j + \omega^2 x_{\text{cl}}^j = \frac{1}{M} j(t) \tag{3.5}$$

Consider now the classical orbit $x_{\text{cl}}(t)$ without source [see (2.152)] :

$$x_{\text{cl}}(t) = \frac{x_b \sin\omega(t - t_a) + x_a \sin\omega(t_b - t)}{\sin\omega(t_b - t_a)} \tag{3.6}$$

A general path can be written as

$$x(t) = x_{\text{cl}}(t) + \delta x(t) \tag{3.7}$$

where $\delta x(t)$ is the fluctuation.

(3.3) becomes

$$\begin{aligned} \mathcal{A} &= \int_{t_a}^{t_b} d\,t \left\{ \frac{1}{2} M \left[(\dot{x}_{\text{cl}} + \delta\dot{x})^2 - \omega^2 (x_{\text{cl}} + \delta x)^2 \right] + (x_{\text{cl}} + \delta x)\, j(t) \right\} \\ &= \int_{t_a}^{t_b} d\,t \left\{ \frac{1}{2} M \left(\dot{x}_{\text{cl}}^2 - \omega^2 x_{\text{cl}}^2 \right) + \frac{1}{2} M \left(\delta\dot{x}^2 - \omega^2 \delta x^2 \right) \right. \\ &\qquad \left. + M \left(\dot{x}_{\text{cl}}\, \delta\dot{x} - \omega^2 x_{\text{cl}}\, \delta x \right) + (x_{\text{cl}} + \delta x)\, j(t) \right\} \end{aligned} \tag{3.7a}$$

Since $x_{\text{cl}}(t_b) = x_b$ & $x_{\text{cl}}(t_a) = x_a$, we have $\delta x = 0$ at the end points so that

$$\int_{t_a}^{t_b} d\,t\, \dot{x}_{\text{cl}}\, \delta\dot{x} = -\int_{t_a}^{t_b} d\,t\, \ddot{x}_{\text{cl}}\, \delta x$$

Hence,

$$\int_{t_a}^{t_b} d\,t \left(\dot{x}_{\text{cl}}\, \delta\dot{x} - \omega^2 x_{\text{cl}}\, \delta x \right) = -\int_{t_a}^{t_b} d\,t \left(\ddot{x}_{\text{cl}} + \omega^2 x_{\text{cl}} \right) \delta x = 0$$

(3.7a) thus becomes

$$\begin{aligned} \mathcal{A} &= \int_{t_a}^{t_b} d\,t \left\{ \frac{1}{2} M \left(\dot{x}_{\text{cl}}^2 - \omega^2 x_{\text{cl}}^2 \right) + j(t)\, x_{\text{cl}} + \frac{1}{2} M \left(\delta\dot{x}^2 - \omega^2 \delta x^2 \right) + j(t)\, \delta x \right\} \\ &\equiv \mathcal{A}_{\omega,\text{cl}} + \mathcal{A}_{j,\text{cl}} + \mathcal{A}_{\omega,\text{fl}} + \mathcal{A}_{j,\text{fl}} \end{aligned} \tag{3.8}$$

$$\equiv \mathcal{A}_{\text{cl}} + \mathcal{A}_{\text{fl}}$$

Note that $\mathcal{A}_{\text{cl}}$ is equal to the $\mathcal{A}_{\text{cl}}^{j}$ defined in (3.4).

(3.4) now takes the form

$$(x_b\, t_b \mid x_a\, t_a)_\omega^j = \exp\left(\frac{i}{\hbar}\mathcal{A}_{\text{cl}}\right)\int \mathcal{D}\,x \exp\left(\frac{i}{\hbar}\mathcal{A}_{\text{fl}}\right)$$

$$= \exp\left[\frac{i}{\hbar}\left(\mathcal{A}_{\omega,\text{cl}} + \mathcal{A}_{j,\text{cl}}\right)\right]\int \mathcal{D}\,x \exp\left[\frac{i}{\hbar}\left(\mathcal{A}_{\omega,\text{fl}} + \mathcal{A}_{j,\text{fl}}\right)\right] \tag{3.9}$$

From (2.157), we have

$$\mathcal{A}_{\omega,\text{cl}} = \frac{M\,\omega}{2\sin\omega(t_b - t_a)}\left[\left(x_b^2 + x_a^2\right)\cos\omega(t_b - t_a) - 2\,x_b\,x_a\right] \tag{3.10}$$

Using (3.6), we have

$$\mathcal{A}_{j,\text{cl}} = \int_{t_a}^{t_b} d\,t\, x_{\text{cl}}(t)\, j(t)$$

$$= \frac{1}{\sin\omega(t_b - t_a)}\int_{t_a}^{t_b} d\,t\left[x_b \sin\omega(t - t_a) + x_a \sin\omega(t_b - t)\right] j(t) \tag{3.11}$$

Using

$$\int_{t_a}^{t_b} d\,t\,\delta\,\dot{x}^2 = \int_{t_a}^{t_b} d\,t\,(\partial_t\,\delta\,x)^2 = -\int_{t_a}^{t_b} d\,t\left(\partial_t^2\,\delta\,x\right)\delta\,x$$

we have

$$\int_{t_a}^{t_b} d\,t\left(\delta\,\dot{x}^2 - \omega^2\,\delta\,x^2\right) = \int_{t_a}^{t_b} d\,t\;\delta\,x\left(-\partial_t^2 - \omega^2\right)\delta\,x$$

$$= \int_{t_a}^{t_b} d\,t\int_{t_a}^{t_b} d\,t'\;\delta(t - t')\,\delta\,x(t)\left(-\partial_{t'}^2 - \omega^2\right)\delta\,x(t')$$

$$= \int_{t_a}^{t_b} d\,t\int_{t_a}^{t_b} d\,t'\,\delta\,x(t)\left(-\partial_t^2 - \omega^2\right)\delta(t - t')\,\delta\,x(t')$$

$$\equiv \int_{t_a}^{t_b} d\,t\int_{t_a}^{t_b} d\,t'\;\delta\,x(t)\,D_{\omega^2}(t, t')\,\delta\,x(t') \tag{3.11a}$$

where

$$D_{\omega^2}(t, t') = \delta(t - t')\left(-\partial_{t'}^2 - \omega^2\right)$$

$$= \left(-\partial_t^2 - \omega^2\right)\delta(t - t') \tag{3.13}$$

Hence, (3.8) gives

$$\mathcal{A}_{\text{fl}} = \frac{1}{2}M\int_{t_a}^{t_b} d\,t\int_{t_a}^{t_b} d\,t'\;\delta\,x(t)\,D_{\omega^2}(t, t')\,\delta\,x(t') + \int_{t_a}^{t_b} d\,t\,j(t)\,\delta\,x(t) \tag{3.12}$$

where $D_{\omega^2}(t, t')$ can be taken as a functional matrix in the space $\mathcal{F}$ of t-dependent functions that vanish at the end points t_b & t_a.

Although the equivalence of the two forms of $D_{\omega^2}(t, t')$ in (3.13) were derived from (3.11a), it remains valid when applied the arbitrary functions in $\mathcal{F}$, i.e.,

$$\int_{t_a}^{t_b} d\,t\int_{t_a}^{t_b} d\,t'\,f(t)\,\delta(t - t')\left(-\partial_{t'}^2 - \omega^2\right)g(t')$$

$$= \int_{t_a}^{t_b} d\,t\int_{t_a}^{t_b} d\,t'\,f(t)\left(-\partial_t^2 - \omega^2\right)\delta(t - t')\,g(t') \tag{3.12a}$$

Proof of this is as follows.

(3.12a) is obviously an identity if we evaluate the δ-functions judiciously to get

$$\int_{t_a}^{t_b} d\,t'\,f(t')\left(-\partial_{t'}^2 - \omega^2\right)g(t') = \int_{t_a}^{t_b} d\,t\;f(t)\left(-\partial_t^2 - \omega^2\right)g(t) \tag{3.12b}$$

which is an identity since t & t' are dummy variables. This however is not a complete proof since we don't know if the order of integration can be interchanged.

Alternatively, we can integrate by parts twice to turn the R.H.S. of (3.12a) into

$$\int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\, g(t')\, \delta(t-t')\left(-\partial_t^2-\omega^2\right) f(t)$$

$$= \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\, g(t')\, D_{\omega^2}(t',\, t')\, f(t)$$

$$= \int_{t_a}^{t_b} d\,t\; g(t)\left(-\partial_t^2-\omega^2\right) f(t)$$

Comparing with (3.12b), we have

$$\int_{t_a}^{t_b} d\,t\; f(t)\, \partial_t^2\, g(t) = \int_{t_a}^{t_b} d\,t\; g(t)\, \partial_t^2\, f(t) \qquad (3.14)$$

which is true for $f,\, g \in \mathcal{F}$, thus proving (3.12a).

The inverse $D_{\omega^2}^{-1}(t,\, t')$ is formally defined by

$$\int_{t_a}^{t_b} d\,t'\, D_{\omega^2}(t'',\, t')\, D_{\omega^2}^{-1}(t',\, t) = \delta(t''-t) \qquad \forall\; t'',\, t \in [t_b,\, t_a] \qquad (3.15)$$

$$= \int_{t_a}^{t_b} d\,t'\, \delta(t''-t')\left(-\partial_{t'}^2-\omega^2\right) D_{\omega^2}^{-1}(t',\, t) \qquad \text{[(3.13) used.]}$$

$$= \int_{t_a}^{t_b} d\,t'\, \delta(t''-t')\, \delta(t'-t) \qquad [\; \delta(t''-t) \text{ expressed as an integral.]}$$

$$\rightarrow \qquad \left(-\partial_{t'}^2-\omega^2\right) D_{\omega^2}^{-1}(t',\, t) = \delta(t'-t)$$

Comparing with the green function eq.

$$\left(\partial_t^2+\omega^2\right) G_{\omega^2}(t,\, t') = -\delta(t-t') \qquad (3.15a)$$

for the harmonic oscillator, we see that

$$G_{\omega^2}(t,\, t') = D_{\omega^2}^{-1}(t,\, t') = \left(-\partial_t^2-\omega^2\right)^{-1} \delta(t-t') \qquad (3.16)$$

$$= \delta(t-t')\left(-\partial_{t'}^2-\omega^2\right)^{-1}$$

Note however that (3.15) only determine $D_{\omega^2}^{-1}(t,\, t')$ up to the addition of the solution $H(t,\, t')$ of the homogenous eq.

$$\int_{t_a}^{t_b} d\,t'\, D_{\omega^2}(t'',\, t')\, H(t',\, t) = 0$$

This arbitrariness will be removed by imposing appropriate B.C.'s.

(3.12) can be quadratic completed by changing to the variable

$$\delta\,\tilde{x}(t) = \delta\,x(t) + \frac{1}{M}\int_{t_a}^{t_b} d\,t'\, G_{\omega^2}(t,\, t')\, j(t') \qquad (3.17)$$

Hence

$$\mathcal{I} = \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\;\; \delta\,x(t)\, D_{\omega^2}(t,\, t')\, \delta\,x(t')$$

$$= \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t' \left[\delta\,\tilde{x}(t) - \frac{1}{M}\int_{t_a}^{t_b} d\,t_1\, G_{\omega^2}(t,\, t_1)\, j(t_1)\right] D_{\omega^2}(t,\, t')$$

$$\times \left[\delta\,\tilde{x}(t') - \frac{1}{M}\int_{t_a}^{t_b} d\,t_2\, G_{\omega^2}(t',\, t_2)\, j(t_2)\right]$$

$$= \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t' \left\{ \delta\,\tilde{x}(t)\, D_{\omega^2}(t,\,t')\,\delta\,\tilde{x}(t') \right.$$
$$- \frac{1}{M} \int_{t_a}^{t_b} d\,t_2\, \delta\,\tilde{x}(t)\, D_{\omega^2}(t,\,t')\, G_{\omega^2}(t',\,t_2)\, j(t_2)$$
$$- \frac{1}{M} \int_{t_a}^{t_b} d\,t_1\, G_{\omega^2}(t,\,t_1)\, j(t_1)\, D_{\omega^2}(t,\,t')\,\delta\,\tilde{x}(t')$$
$$\left. + \frac{1}{M^2} \int_{t_a}^{t_b} d\,t_1 \int_{t_a}^{t_b} d\,t_2\, G_{\omega^2}(t,\,t_1)\, j(t_1)\, D_{\omega^2}(t,\,t')\, G_{\omega^2}(t',\,t_2)\, j(t_2) \right\}$$
$$= \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\; \delta\,\tilde{x}(t)\, D_{\omega^2}(t,\,t')\,\delta\,\tilde{x}(t')$$
$$- \frac{1}{M} \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t_2\, \delta\,\tilde{x}(t)\, \delta(t - t_2)\, j(t_2)$$
$$- \frac{1}{M} \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t' \int_{t_a}^{t_b} d\,t_1\, G_{\omega^2}(t,\,t_1)\, j(t_1)\, D_{\omega^2}(t,\,t')\,\delta\,\tilde{x}(t')$$
$$+ \frac{1}{M^2} \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t_1 \int_{t_a}^{t_b} d\,t_2\, G_{\omega^2}(t,\,t_1)\, j(t_1)\, \delta\,(t - t_2)\, j(t_2)$$

Now,

$$\mathcal{I}_1 = \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t' \int_{t_a}^{t_b} d\,t_1\, G_{\omega^2}(t,\,t_1)\, j(t_1)\, D_{\omega^2}(t,\,t')\,\delta\,\tilde{x}(t')$$
$$= \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t' \int_{t_a}^{t_b} d\,t_1\, \delta\,(t - t_1) \left(-\partial_{t_1}^2 - \omega^2 \right)^{-1} j(t_1)\, \delta(t - t') \left(-\partial_{t'}^2 - \omega^2 \right) \delta\,\tilde{x}(t')$$

Integrating by part twice, we have

$$\mathcal{I}_1 = \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t' \int_{t_a}^{t_b} d\,t_1\, \delta\,\tilde{x}(t') \left(-\partial_{t'}^2 - \omega^2 \right) \delta(t' - t)\, \delta(t - t_1) \left(-\partial_{t_1}^2 - \omega^2 \right)^{-1} j(t_1)$$
$$= \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t' \int_{t_a}^{t_b} d\,t_1\, \delta\,\tilde{x}(t')\, D_{\omega^2}(t',\,t)\, D_{\omega^2}^{-1}\left(t,\,t_1\right) j(t_1)$$
$$= \int_{t_a}^{t_b} d\,t' \int_{t_a}^{t_b} d\,t_1\, \delta\,\tilde{x}(t')\, \delta\,(t',\,t_1)\, j(t_1)$$
$$= \int_{t_a}^{t_b} d\,t'\, \delta\,\tilde{x}(t')\, j(t')$$

Thus,

$$\mathcal{I} = \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\; \delta\,\tilde{x}(t)\, D_{\omega^2}(t,\,t')\,\delta\,\tilde{x}(t')$$
$$- \frac{2}{M} \int_{t_a}^{t_b} d\,t\, \delta\,\tilde{x}(t)\, j(t') + \frac{1}{M^2} \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t_1\, G_{\omega^2}(t,\,t_1)\, j(t_1)\, j(t)$$

Together with

$$\int_{t_a}^{t_b} d\,t\, j(t)\, \delta\,x(t) = \int_{t_a}^{t_b} d\,t\, j(t)\, \delta\,\tilde{x}(t) - \frac{1}{M} \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t_1\, G_{\omega^2}(t,\,t_1)\, j(t_1)\, j(t)$$

(3.12) becomes

$$\mathcal{A}_{\text{fl}}[\delta\,\tilde{x}] = \tilde{\mathcal{A}}_{\omega,\,\text{fl}}[\delta\,\tilde{x}] + \tilde{\mathcal{A}}_{j,\,\text{fl}} \qquad (3.18)$$

where

$$\tilde{\mathcal{A}}_{\omega,\,\text{fl}}[\delta\,\tilde{x}] = \frac{1}{2} M \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\, \delta\,\tilde{x}(t)\, D_{\omega^2}(t,\,t')\,\delta\,\tilde{x}(t') \qquad (3.18a)$$

$$\tilde{\mathcal{A}}_{j,\text{fl}} = -\frac{1}{2M}\int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\ j(t)\, G_{\omega^2}(t, t')\, j(t') \qquad (3.22)$$

This tedious “completing the square” procedure to obtain (3.22) can be side-stepped by the observation that (3.22) is just one-half the value of [see (3.2)]

$$\int_{t_a}^{t_b} dt\, j(t)\, \delta x(t)$$

using the classical relation [see (3.5)]

$$x(t) = -\frac{1}{M}\int_{t_a}^{t_b} dt'\, G_{\omega^2}(t, t')\, j(t') \qquad (3.22a)$$

Since $G_{\omega^2} = D_{\omega^2}^{-1}$ are green functions on $\mathcal{F}$, they should obey the B.C.

$$G_{\omega^2}(t_b, t) = 0 = G_{\omega^2}(t, t_a) \qquad \forall\ t \in [t_b, t_a] \qquad (3.19)$$

Together with the symmetric property

$$G_{\omega^2}(t, t') = G_{\omega^2}(t', t) \qquad (3.19a)$$

we see from (3.17) that

$$\delta\tilde{x}(t_b) = \delta\tilde{x}(t_a) = 0 \qquad (3.19b)$$

so that $\delta\tilde{x} \in \mathcal{F}$.

Since the integral in (3.17) is independent of δx, we have

$$\int \mathcal{D}\,\delta x(t) = \int \mathcal{D}\,\delta\tilde{x}(t)$$

(3.4) now becomes

$$(x_b t_b \mid x_a t_a)_\omega^j = \exp\left[\frac{i}{\hbar}(\mathcal{A}_{\omega,\text{cl}} + \mathcal{A}_{j,\text{cl}})\right]\int \mathcal{D}\,\delta\tilde{x}(t)\exp\left[\frac{i}{\hbar}(\tilde{\mathcal{A}}_{\omega,\text{fl}} + \tilde{\mathcal{A}}_{j,\text{fl}})\right]$$

$$= \exp\left[\frac{i}{\hbar}(\mathcal{A}_{\omega,\text{cl}} + \mathcal{A}_{j,\text{cl}})\right] F_\omega(t_b - t_a)\,\tilde{F}_j \qquad (3.20a)$$

where

$$F_\omega(t_b - t_a) = \int \mathcal{D}\,\delta\tilde{x}(t)\exp\left(\frac{i}{\hbar}\tilde{\mathcal{A}}_{\omega,\text{fl}}[\delta\tilde{x}]\right)$$

$$= \sqrt{\frac{M}{2\pi i\hbar}}\sqrt{\frac{\omega}{\sin\omega(t_b - t_a)}} \qquad \text{[(2.169) used.]} \qquad (3.20)$$

and

$$\tilde{F}_j = \int \mathcal{D}\,\delta\tilde{x}(t)\exp\left(\frac{i}{\hbar}\tilde{\mathcal{A}}_{j,\text{fl}}\right)$$

$$= \exp\left(\frac{i}{\hbar}\tilde{\mathcal{A}}_{j,\text{fl}}\right) \qquad (3.21)$$

since $\tilde{\mathcal{A}}_{j,\text{fl}}$ is independent of $\delta\tilde{x}$.

The source free amplitude is

$$(x_b t_b \mid x_a t_a)_\omega = \exp\left(\frac{i}{\hbar}\mathcal{A}_{\omega,\text{cl}}\right) F_\omega(t_b - t_a) \qquad (3.20c)$$

$$= \sqrt{\frac{M}{2\pi i\hbar}}\sqrt{\frac{\omega}{\sin\omega(t_b - t_a)}} \qquad (3.24)$$

$$\times\exp\left\{\frac{M\omega}{2\sin\omega(t_b - t_a)}\left[(x_b^2 + x_a^2)\cos\omega(t_b - t_a) - 2x_b x_a\right]\right\}$$

where (3.10) & (3.20) were used.

(3.20a) thus becomes

$$(x_b\, t_b \mid x_a\, t_a)^j_\omega = (x_b\, t_b \mid x_a\, t_a)_\omega\, F_{j,\mathrm{cl}}\, \tilde{F}_j \qquad (3.23)$$

where

$$F_{j,\mathrm{cl}} \equiv \exp\left(\frac{i}{\hbar}\mathcal{A}_{j,\,\mathrm{cl}}\right) \qquad (3.25a)$$

Using (3.11) we have

$$F_{j,\mathrm{cl}} = \exp\left\{\frac{i}{\hbar}\,\frac{1}{\sin\omega(t_b - t_a)}\int_{t_a}^{t_b} d\,t\left[x_b \sin\omega(t - t_a) + x_a \sin\omega(t_b - t)\right] j(t)\right\} \qquad (3.25)$$