3.2. Green Function of Harmonic Oscillator

From (3.22), we have

$$G_{\omega^2}(t, t') = \left(-\partial_t^2 - \omega^2\right)^{-1} \delta(t - t') \qquad t, t' \in [t_b, t_a] \qquad (3.26)$$

The ambiguity concerning the homogeneous solution is removed by imposing the Dirichlet B.C.

$$G_{\omega^2}(t_b, t) = G_{\omega^2}(t, t_a) = 0 \qquad (3.26a)$$

Interchanging $t \leftrightarrow t'$ in (3.26) gives

$$\begin{aligned} G_{\omega^2}(t', t) &= \left(-\partial_{t'}^2 - \omega^2\right)^{-1} \delta(t' - t) \\ &= \left(-\partial_t^2 - \omega^2\right)^{-1} \delta(t' - t) \\ &= G_{\omega^2}(t, t') \end{aligned} \qquad (3.26b)$$

We shall also consider the more general form

$$\left[-\partial_t^2 - \Omega^2(t)\right] G_{\Omega^2}(t, t') = \delta(t - t') \qquad (3.27)$$

with

$$G_{\Omega^2}(t_b, t) = G_{\Omega^2}(t, t_a) = 0 \qquad (3.27a)$$

3.2.1. Wronski Construction

Ref: G.Arfken, “Mathematical Methods for Physicists”, 3rd ed., §16.5, Elsevier (1985).

The Wronski construction begins by writing

$$G_{\Omega^2}(t, t') = \begin{cases} G_{\Omega^2}^{>}(t, t') & \text{for } t > t' \\ G_{\Omega^2}^{<}(t, t') & \text{for } t < t' \end{cases} \qquad (3.28a)$$

so that (3.27-a) give

$$\left[-\partial_t^2 - \Omega^2(t)\right] G_{\Omega^2}^{>}(t, t') = 0 \qquad \text{with} \qquad G_{\Omega^2}^{>}(t_b, t') = 0 \qquad (3.28b)$$

$$\left[-\partial_t^2 - \Omega^2(t)\right] G_{\Omega^2}^{<}(t, t') = 0 \qquad \text{with} \qquad G_{\Omega^2}^{<}(t_a, t') = 0 \qquad (3.28c)$$

Continuity at $t = t'$ gives

$$G_{\Omega^2}^{>}(t', t') = G_{\Omega^2}^{<}(t', t') \qquad (3.28d)$$

Integrating (3.27) then gives the discontinuity at $t = t'$:

$$-\partial_t G_{\Omega^2}^{>}(t', t') + \partial_t G_{\Omega^2}^{<}(t', t') = 1 \qquad (3.28e)$$

Let

$$\left[-\partial_t^2 - \Omega^2(t)\right] \xi(t) = 0 \qquad \text{with} \qquad \xi(t_b) = 0 \qquad (3.28f)$$

$$\left[-\partial_t^2 - \Omega^2(t)\right] \eta(t) = 0 \qquad \text{with} \qquad \eta(t_a) = 0 \qquad (3.28g)$$

then

$$G_{\Omega^2}^{>}(t, t') = \alpha\, \xi(t) \qquad G_{\Omega^2}^{<}(t, t') = \beta\, \eta(t) \qquad (3.28h)$$

where α, β are functions of t'.

(3.28d-e) become

$$\alpha\, \xi(t') - \beta\, \eta(t') = 0$$

$$\alpha\, \dot{\xi}(t') - \beta\, \dot{\eta}(t') = -1$$

where

$$\partial_t f(t) \equiv \dot{f}(t) \qquad \to \qquad \partial_{t'} f(t') = \dot{f}(t')$$

Thus,

$$\begin{pmatrix} \xi & \eta \\ \dot{\xi} & \dot{\eta} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad (3.28i)$$

where

$$W = \begin{vmatrix} \xi & \eta \\ \dot{\xi} & \dot{\eta} \end{vmatrix} = \xi\,\dot{\eta} - \eta\,\dot{\xi} \qquad (3.28j)$$

is the Wronskian of u & v. Note that u & v are linearly independent iff $W \neq 0$.

$\xi\,(3.28\,g) - \eta\,(3.28\,f)$ gives

$$-\xi\,\partial_t^2\,\eta + \eta\,\partial_t^2\,\xi = 0$$

$$= -\partial_t\left(\xi\,\dot{\eta} - \eta\,\dot{\xi}\right)$$

$$= -\partial_t\,W$$

$\rightarrow \qquad W = \text{const} \qquad (3.28k)$

Solving (3.28i) gives

$$\alpha = \frac{\eta(t')}{W} \qquad \beta = \frac{\xi(t')}{W} \qquad (3.28m)$$

so that (3.28h) gives

$$G^{>}_{\Omega^2}(t, t') = \frac{1}{W}\,\xi(t)\,\eta(t') \qquad G^{<}_{\Omega^2}(t, t') = \frac{1}{W}\,\eta(t)\,\xi(t') \qquad (3.28n)$$

Constant Frequency

For $\Omega^2(t) = \omega^2$, we have

$$\xi(t) = \sin\omega(t_b - t) \qquad \eta(t) = \sin\omega(t - t_a) \qquad (3.29a)$$

$$\dot{\xi}\,(t) = -\omega\cos\omega(t_b - t) \qquad \dot{\eta}\,(t) = \omega\cos\omega(t - t_a) \qquad (3.29b)$$

$$\therefore \qquad W = \begin{vmatrix} \sin\omega\,(t_b - t) & \sin\omega\,(t - t_a) \\ -\omega\cos\omega\,(t_b - t) & \omega\cos\omega\,(t - t_a) \end{vmatrix}$$

$$= \omega\left[\sin\omega(t_b - t)\cos\omega(t - t_a) + \sin\omega(t - t_a)\cos\omega(t_b - t)\right]$$

$$= \omega\sin\omega\,(t_b - t_a) \qquad (3.29c)$$

(3.28n) thus becomes

$$G^{>}_{\omega^2}(t, t') = \frac{1}{\omega\sin\omega\,(t_b - t_a)}\sin\omega(t_b - t)\sin\omega(t' - t_a) \qquad (3.29)$$

$$G^{<}_{\omega^2}(t, t') = \frac{1}{\omega\sin\omega\,(t_b - t_a)}\sin\omega(t - t_a)\sin\omega(t_b - t') \qquad (3.30)$$

which can be combined to give

$$G_{\omega^2}(t, t') = \frac{1}{\omega\sin\omega\,(t_b - t_a)}\sin\omega\,(t_b - t_>)\sin\omega\,(t_< - t_a) \qquad (3.36)$$

where $t_> = \max(t, t')$ and $t_< = \min(t, t')$.

Using

$$\sin\omega\,(t_b - t_>)\sin\omega\,(t_< - t_a) = \frac{1}{2}\left[\cos\omega\,(t_b - t_> - t_< + t_a) - \cos\omega\,(t_b - t_> + t_< - t_a)\right]$$

$$= \frac{1}{2}\left[\cos\omega\,(t_b + t_a - t - t') - \cos\omega\,(t_b - t_a - |\,t - t'\,|)\right]$$

(3.36) becomes

$$G_{\omega^2}(t, t') = \frac{\cos\omega\,(t_b + t_a - t - t') - \cos\omega\,(t_b - t_a - |\,t - t'\,|)}{2\,\omega\sin\omega\,(t_b - t_a)} \qquad (3.38)$$

For $\omega \to 0$, (3.36) becomes

$$G_0(t, t') = \frac{(t_b - t_>)\,(t_< - t_a)}{t_b - t_a} \qquad (3.39a)$$

while (3.38) gives

$$G_0(t,\, t') = \frac{-(t_b + t_a - t - t')^2 + (t_b - t_a - |\, t - t'\,|)^2}{4\,(t_b - t_a)}$$

$$= \frac{1}{4\,(t_b - t_a)}\left(-(t_b + t_a)^2 + 2\,(t_b + t_a)\,(t + t') - (t + t')^2 + (t_b - t_a)^2 - 2\,(t_b - t_a)\,\left|\, t - t'\,\right| + (t - t')^2\right)$$

$$= \frac{1}{4\,(t_b - t_a)}\left(-4\, t_b\, t_a + 2\,(t_b + t_a)\,(t + t') - 4\, t\, t' - 2\,(t_b - t_a)\,|\, t - t'\,|\,\right)$$

$$= \frac{1}{t_b - t_a}\left[-t\, t' - t_b\, t_a + \frac{1}{2}\,(t_b + t_a)\,(t + t') - \frac{1}{2}\,(t_b - t_a)\,\left|\, t - t'\,\right|\right] \qquad (3.39)$$

Time-Dependent Frequency

For a discussion of the various types of Green functions, see
E.N.Economou, “Green’s Functions in Quantum Physics”, 2nd ed., §2.3, Springer (1983).

Reminder: all types of Green functions satisfy the same differential eq., which for the present discussion, is

$$\left[-\partial_t^2 - \Omega^2\,(t)\right] G\,(t,\, t') = \delta\,(t - t') \qquad (3.40a)$$

They differ only in the B.C’s.

Note: we’ll adapt the convention that all derivatives operate only on the factor adjacent to them, i.e.,

$$\partial_t^n f\, g \equiv (\partial_t^n f)\, g \qquad (3.40b)$$

Consider the retarded Green function

$$G_{\Omega^2}(t,\, t') = \Theta(t - t')\,\Delta(t,\, t') \qquad (3.40)$$

where $\Delta(t,\, t')$ is an analytic function of t for any fixed t'. Hence, the eq. governing $\Delta(t,\, t')$ should not contain any singularities.

→ $$\partial_t\, G_{\Omega^2}(t,\, t') = \delta(t - t')\,\Delta(t,\, t') + \Theta(t - t')\,\partial_t \Delta(t,\, t')$$

$$\partial_t^2\, G_{\Omega^2}(t,\, t') = \partial_t\,\delta(t - t')\,\Delta(t,\, t') + 2\,\delta(t - t')\,\partial_t \Delta(t,\, t') + \Theta(t - t')\,\partial_t^2\,\Delta(t,\, t')$$

∴ $$\left[-\partial_t^2 - \Omega^2(t)\right] G_{\Omega^2}(t,\, t') = \Theta(t - t')\left[-\partial_t^2 - \Omega^2(t)\right]\Delta(t,\, t') - \partial_t\,\delta(t - t')\,\Delta(t,\, t') - 2\,\delta(t - t')\,\partial_t \Delta\,(t,\, t') \qquad (3.41)$$

Since $\Delta(t,\, t')$ is analytic, we can write

$$\Delta(t,\, t') = \Delta(t',\, t') + (t - t')\,[\,\partial_t \Delta(t,\, t')]_{t = t'} + \frac{1}{2}\,(t - t')^2\left[\,\partial_t^2\,\Delta(t,\, t')\right]_{t = t'} + \ldots \qquad (3.42)$$

From

$$\int d\, t\,(t - t')\,\delta(t - t')\, f(t) = 0 \qquad \forall\ f$$

we have

$$(t - t')\,\delta(t - t') = 0 \qquad (3.43a)$$

From

$$\int d\, t\,(t - t')\,\partial_t\,\delta(t - t')\, f(t) = -\int d\, t\ \delta(t - t')\,\partial_t[(t - t')\, f(t)]$$

$$= -\int d\, t\ \delta(t - t')\, f(t) - \int d\, t\,\delta\,(t - t')\,(t - t')\,\partial_t f(t)$$

$$= -\int d\, t\ \delta(t - t')\, f(t) \qquad \forall\ f$$

we get

$$(t - t')\,\partial_t \delta(t - t') = -\delta(t - t') \qquad (3.43b)$$

For $n > 1$,

$$(t - t')^n\,\partial_t \delta(t - t') = -(t - t')^{n-1}\,\delta(t - t') = 0 \qquad (3.43c)$$

(3.42) then gives

$$\begin{aligned}\partial_t \delta(t - t')\,\Delta(t, t') &= \partial_t \delta(t - t')\,\Delta(t', t') - \delta(t - t')\,[\,\partial_t \Delta(t, t')]_{t=t'} \\ &= \partial_t \delta(t - t')\,\Delta(t', t') - \delta(t - t')\,\partial_t \Delta(t, t') \qquad (3.43c)\end{aligned}$$

The last 2 terms in (3.41) become

$$\begin{aligned}&-\partial_t \delta(t - t')\,\Delta(t, t') - 2\,\delta(t - t')\,\partial_t \Delta\,(t, t') \\ &= -\partial_t \delta(t - t')\,\Delta(t', t') - \delta(t - t')\,\partial_t \Delta\,(t, t') \qquad (3.44)\end{aligned}$$

With the I.C.

$$\Delta(t', t') = 0 \qquad [\,\partial_t \Delta(t, t')]_{t=t'} = -1 \qquad (3.45)$$

we see that (3.44) evaluates to $\delta(t - t')$ so that (3.41) becomes

$$\left[-\partial_t^2 - \Omega^2(t)\right] G_{\Omega^2}(t, t') = \Theta(t - t')\left[-\partial_t^2 - \Omega^2(t)\right]\Delta(t, t') + \delta(t - t')$$

Comparing with (3.27), we have

$$\left[-\partial_t^2 - \Omega^2(t)\right]\Delta(t, t') = 0 \qquad \forall\;\; t > t' \qquad (3.46)$$

while $\Delta(t, t')$ can be anything for $t < t'$.

Let ξ & η be two linearly independent solutions to the homogeneous eq., i.e.,

$$\left[-\partial_t^2 - \Omega^2(t)\right]\xi(t) = 0 \qquad \left[-\partial_t^2 - \Omega^2(t)\right]\eta(t) = 0 \qquad (3.48)$$

with

$$W = \begin{vmatrix} \xi & \eta \\ \dot{\xi} & \dot{\eta} \end{vmatrix} = \xi\,\dot{\eta} - \eta\,\dot{\xi} \neq 0 \qquad (3.48a)$$

From (3.28k), we see that W is independent of time.

One can easily check by inspection that a solution to (3.46) that satisfies the I.C. (3.45) is

$$\Delta(t, t') = \frac{1}{W}\left[\,\xi(t)\,\eta(t') - \eta(t)\,\xi(t')\,\right] \qquad (3.49)$$

The quantity

$$\xi(t)\,\eta(t') - \eta(t)\,\xi(t')$$

is called the Jacobi commutator of $\xi(t)$ & $\eta(t)$.

From (3.49), we see that $\Delta(t, t')$ is antisymmetric:

$$\Delta(t', t) = \frac{1}{W}\left[\,\xi(t')\,\eta(t) - \eta(t')\,\xi(t)\,\right] = -\Delta(t, t') \qquad (3.50a)$$

which means the 2nd I.C. in (3.45) can be written as

$$[\,\partial_{t'} \Delta(t, t')]_{t=t'} = -[\,\partial_t \Delta(t, t')]_{t=t'} = 1 \qquad (3.50b)$$

Furthermore [see “3.02._Code.nb” for proof],

$$\Delta(t, t')\,\Delta(t_b, t_a) = \Delta(t_b, t')\,\Delta(t, t_a) - \Delta(t_b, t)\,\Delta(t', t_a) \qquad (3.50)$$

Caution: R.H.S. of (3.50) differs from Kleinert’s by a minus sign.

∂_t(3.50) gives

$$\partial_t \Delta(t, t')\,\Delta(t_b, t_a) = \Delta(t_b, t')\,\partial_t \Delta(t, t_a) - \partial_t \Delta(t_b, t)\,\Delta(t', t_a)$$

Setting $t = t_b$ and using (3.45) & (3.50b) gives

$$\partial_{t_b} \Delta\,(t_b, t')\,\Delta(t_b, t_a) = \Delta(t_b, t')\,\partial_{t_b} \Delta(t_b, t_a) - \Delta(t', t_a) \qquad (3.51)$$

$\partial_{t'}$(3.50) gives

$$\partial_{t'} \Delta(t, t')\,\Delta(t_b, t_a) = \partial_{t'} \Delta(t_b, t')\,\Delta(t, t_a) - \Delta(t_b, t)\,\partial_{t'} \Delta(t', t_a)$$

Setting $t' = t_a$ & using (3.45) gives

$$\partial_{t_a} \Delta(t, t_a)\, \Delta(t_b, t_a) = \partial_{t_a} \Delta(t_b, t_a)\, \Delta(t, t_a) + \Delta(t_b, t) \qquad (3.52)$$

Note that (3.52) can also be obtained by setting $b \leftrightarrow a$ & $t' \to t$ to (3.51).

Taking t' as a parameter, the retarded Green function (3.40) is just the particular solution to the differential eq (3.27). The general solution is

$$G_{\Omega^2}(t, t') = \Theta(t - t')\, \Delta(t, t') + a\, \xi(t) + b\, \eta(t) \qquad (3.53)$$

where a, b are functions of t' to be determined by some I.C. or B.C.

The Wronskian construction used the Dirichlet B.C. (3.28b-c)

$$G_{\Omega^2}(t_b, t') = 0 \qquad \forall\ t' \neq t_b$$
$$G_{\Omega^2}(t_a, t') = 0 \qquad \forall\ t \neq t_a \qquad (3.54)$$

(3.53) then gives

$$\Theta(t_b - t')\, \Delta(t_b, t') + a\, \xi(t_b) + b\, \eta(t_b) = 0$$
$$\Theta(t_a - t')\, \Delta(t_a, t') + a\, \xi(t_a) + b\, \eta(t_a) = 0$$

For $t' \in (t_b, t_a)$, these become

$$\Delta(t_b, t') + a\, \xi(t_b) + b\, \eta(t_b) = 0 \qquad (3.56)$$
$$a\, \xi(t_a) + b\, \eta(t_a) = 0 \qquad (3.55)$$

$$\rightarrow \qquad \begin{pmatrix} \xi(t_b) & \eta(t_b) \\ \xi(t_a) & \eta(t_a) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\Delta(t_b, t') \\ 0 \end{pmatrix} \qquad (3.56a)$$

Setting

$$\Lambda = \begin{pmatrix} \xi(t_b) & \eta(t_b) \\ \xi(t_a) & \eta(t_a) \end{pmatrix} \qquad (3.57)$$

we have

$$\det \Lambda = \xi(t_b)\, \eta(t_a) - \eta(t_b)\, \xi(t_a)$$
$$= W\, \Delta(t_b, t_a) \qquad [\text{ (3.49) used. }] \qquad (3.58)$$

Assuming $\det \Lambda \neq 0$, (3.56a) can be solved to give

$$a = -\frac{\eta(t_a)\, \Delta(t_b, t')}{W\, \Delta(t_b, t_a)} \qquad b = \frac{\xi(t_a)\, \Delta(t_b, t')}{W\, \Delta(t_b, t_a)} \qquad (3.58a)$$

(3.53) thus becomes

$$G_{\Omega^2}(t, t') = \Theta(t - t')\, \Delta(t, t') + \left[-\eta(t_a)\, \xi(t) + \xi(t_a)\, \eta(t) \right] \frac{\Delta(t_b, t')}{W\, \Delta(t_b, t_a)}$$
$$= \Theta(t - t')\, \Delta(t, t') - \frac{\Delta(t, t_a)\, \Delta(t_b, t')}{\Delta(t_b, t_a)}$$

Using (3.50), we have

$$G_{\Omega^2}(t, t') = \Theta(t - t')\, \frac{\Delta(t_b, t')\, \Delta(t, t_a) - \Delta(t_b, t)\, \Delta(t', t_a)}{\Delta(t_b, t_a)} - \frac{\Delta(t, t_a)\, \Delta(t_b, t')}{\Delta(t_b, t_a)}$$
$$= -\Theta(t - t')\, \frac{\Delta(t_b, t)\, \Delta(t', t_a)}{\Delta(t_b, t_a)} - \Theta(t' - t)\, \frac{\Delta(t, t_a)\, \Delta(t_b, t')}{\Delta(t_b, t_a)}$$
$$= -\frac{1}{\Delta(t_b, t_a)} \left(\Theta(t - t')\, \Delta(t_b, t)\, \Delta(t', t_a) + \Theta(t' - t)\, \Delta(t, t_a)\, \Delta(t_b, t') \right) \qquad (3.59)$$

which is the generalization of the Wronski construction (3.36) to time-dependent frequencies.

It is instructive to compare the present results with those of §2.4. From (3.45-6) & (2.226-7), we see that

$$D_b(t) = \Delta(t_b, t) \qquad D_a(t) = \Delta(t, t_a) \qquad (3.60)$$

so that (3.59) becomes

$$G_{\Omega^2}(t, t') = -\frac{\Theta(t-t')\, D_b(t)\, D_a(t') + \Theta(t'-t)\, D_a(t)\, D_b(t')}{D_a(t_b)} \qquad (3.61)$$

Note that if the homogeneous eq.

$$\left[-\partial_t^2 - \Omega^2(t)\right] y(t) = 0$$

admits the so-called zero-mode solutions that satisfy the Dirichlet B.C.

$$y(t_b) = y(t_a) = 0$$

then

$$\Lambda = \det\begin{pmatrix} 0 & \eta(t_b) \\ 0 & \eta(t_a) \end{pmatrix} = 0$$

and all that derived from (3.58a) is no longer vaild.

3.2.2. Spectral Representat ion

Constant Frequency

The fluctuation $\delta x(t)$ satisfies

$$\left(-\partial_t^2 - \omega^2\right) \delta x = 0 \qquad (3.62)$$

with B.C.

$$\delta x(t_b) = \delta x(t_a) = 0 \qquad (3.62a)$$

Any function satisfying the B.C. (3.62a) can be spanned by a set of orthonormal functions

$$x_n(t) = \sqrt{\frac{2}{t_b - t_a}} \sin \nu_n (t - t_a) \qquad (3.63)$$

where

$$\nu_n = \frac{n\pi}{t_b - t_a} \qquad n = 1, 2, 3, \ldots \qquad (3.64)$$

so that

$$\begin{aligned}
\int_{t_a}^{t_b} dt\, x_n(t)\, x_m(t) &= \frac{2}{t_b - t_a} \int_{t_a}^{t_b} dt \sin \nu_n (t - t_a) \sin \nu_m (t - t_a) \\
&= \frac{2}{t_b - t_a} \int_0^{t_b - t_a} dt \sin \nu_n t \sin \nu_m t \\
&= \frac{1}{t_b - t_a} \int_0^{t_b - t_a} dt \left[\cos(\nu_n - \nu_m) t - \cos(\nu_n + \nu_m) t\right] \\
&= \frac{1}{\pi} \int_0^{\pi} d\theta \left[\cos(n - m)\theta - \cos(n + m)\theta\right] \qquad \left[\ \theta = \frac{\pi t}{t_b - t_a}\ \right]
\end{aligned}$$

For $k \neq 0$ and integer,

$$\int_0^{\pi} d\theta \cos k\theta = \frac{1}{k} \sin k\theta \Big|_0^{\pi} = 0$$

$$\rightarrow \qquad \int_0^{\pi} d\theta \cos k\theta = \delta_{k0}\, \pi$$

By (3.64), $n, m > 0$. Therefore,

$$\int_{t_a}^{t_b} dt\, x_n(t)\, x_m(t) = \delta_{nm} \qquad (3.65)$$

Since x_n satisfies

$$\left(-\partial_t^2 - v_n^2\right) x_n = 0 \tag{3.65a}$$

the operator $-\partial_t^2 - \omega^2$ is diagonal with respect to the basis $\{x_n\}$.

The Green function

$$G_{\omega^2}(t, t') = \left(-\partial_t^2 - \omega^2\right)^{-1} \delta(t - t')$$

can be used as the kernel to construct an integral operator so that

$$\mathcal{I}[f](t) \equiv \int_{t_a}^{t_b} d\,t'\, G_{\omega^2}(t, t')\, f(t')$$

$$= \left(-\partial_t^2 - \omega^2\right)^{-1} f(t) \tag{3.65b}$$

Then $\mathcal{I}$ too is diagonal w.r.t. $\{x_n\}$. Let G_n be the eigenvalue associated with x_n, then

$$\mathcal{I}[x_n](t) = \int_{t_a}^{t_b} d\,t'\, G_{\omega^2}(t, t')\, x_n(t') = G_n\, x_n(t) \tag{3.66}$$

Using (3.65), we see that (3.66) is satisfied by

$$G_{\omega^2}(t, t') = \sum_{n=1}^{\infty} G_n\, x_n(t)\, x_n(t') \tag{3.67}$$

Using (3.65b) on (3.66) gives

$$\left(-\partial_t^2 - \omega^2\right)^{-1} x_n(t) = G_n\, x_n(t)$$

$\rightarrow$ $$\left(v_n^2 - \omega^2\right)^{-1} x_n(t) = G_n\, x_n(t) \qquad \text{[(3.65a) used.]}$$

$\therefore$ $$G_n = \frac{1}{v_n^2 - \omega^2} \tag{3.68}$$

(3.67) thus becomes

$$G_{\omega^2}(t, t') = \sum_{n=1}^{\infty} \frac{x_n(t)\, x_n(t')}{v_n^2 - \omega^2} \tag{3.69a}$$

$$= \frac{2}{t_b - t_a} \sum_{n=1}^{\infty} \frac{\sin v_n(t - t_a)\, \sin v_n(t' - t_a)}{v_n^2 - \omega^2} \qquad \text{[(3.63) used.] (3.69)}$$

Using

$$\sin v_n(t_b - t) = -\sin v_n \left[\, t - t_a - (t_b - t_a)\,\right]$$

$$= -\sin\left[\, v_n\,(t - t_a) - n\,\pi\,\right] \qquad \text{[(3.64) used.]}$$

$$= (-)^{n+1} \sin v_n\,(t - t_a)$$

we can write (3.69) as

$$G_{\omega^2}(t, t') = \frac{2}{t_b - t_a} \sum_{n=1}^{\infty} (-)^{n+1} \frac{\sin v_n(t_b - t)\, \sin v_n(t' - t_a)}{v_n^2 - \omega^2} \tag{3.70}$$

Note that (2.70) becomes singular if

$$\omega^2 = v_n^2 \text{ for some } n$$

$\rightarrow$ $$\omega = \frac{n\,\pi}{t_b - t_a}$$

in agreement with the conclusions drawn from (3.36).

Time-Dependent Frequency

The foregoing results can be directly generalized to the case of time-dependent frequency.

The basis $\{y_n\}$ is chosen as the set of orthonormal eigenfunctions of $-\partial_t^2 - \Omega^2(t)$. Thus,

$$\left[\, -\partial_t^2 - \Omega^2(t)\,\right] y_n(t) = \lambda_n\, y_n(t) \tag{3.71}$$

with

$$\int_{t_a}^{t_b} d\,t\, y_n(t)\, y_m(t) = \delta_{n\,m} \qquad \text{(orthonormality)}$$

$$\sum_n y_n(t)\, y_n(t') = \delta(t - t') \qquad \text{(completeness)}$$

(3.65b) & (3.66) become

$$\begin{aligned} \mathcal{I}[y_n]\,(t) &= \int_{t_a}^{t_b} d\,t'\, G_{\Omega^2}(t,\, t')\, y_n(t') \\ &= \left(-\partial_t^2 - \Omega^2\right)^{-1} y_n(t) \\ &= \frac{1}{\lambda_n}\, y_n(t) \end{aligned}$$

thus turning (3.69a) into

$$G_{\Omega^2}(t,\, t') = \sum_n \frac{y_n(t)\, y_n(t')}{\lambda_n} \tag{3.74}$$

provided

$$\lambda_n \neq 0 \qquad \forall\, n$$