3.3. Green Functions of 1st-Order Differential Eq.

We now consider

$$\left[i\,\partial_t - \Omega(t) \right] G_\Omega(t,\, t') = i\,\delta(t - t') \qquad t,\, t' \in [\, t_a,\, t_b\,) \tag{3.75}$$

and its Euclidean counterpart, which is obtained using the analytic continuation

$$t \to -i\,\tau \qquad t_b - t_a \to -i\,\beta\,\hbar$$

so that (3.75) becomes

$$\left[-\partial_\tau - \Omega(\tau) \right] G_{\Omega,e}(\tau,\, \tau') = \frac{i}{-i}\,\delta(\tau - \tau') = -\delta(\tau - \tau')$$

or

$$\left[\partial_\tau + \Omega(\tau) \right] G_{\Omega,e}(\tau,\, \tau') = \delta(\tau - \tau') \qquad \tau,\, \tau' \in [\, 0,\, \beta\,\hbar\,) \tag{3.76}$$

3.3.1. Time-Independent Frequency

G^p_ω

For $\Omega = \omega$, (3.75) simplifies to

$$(\, i\,\partial_t - \omega\,)\, G_\omega(t,\, t') = i\,\delta(t - t') \qquad t,\, t' \in [\, t_a,\, t_b\,) \tag{3.77}$$

and we shall impose the periodic B.C.

$$G^p_\omega(t + t_b - t_a,\, t') = G^p_\omega(t,\, t' + t_b - t_a) = G^p_\omega(t,\, t') \tag{3.77a}$$

where the superscript p stands for periodic.

For the periodic basis, we choose the orthonormal solutions to

$$(\, i\,\partial_t - \omega_m\,)\, \phi_m(t) = 0 \tag{3.77b}$$

i.e.,

$$\phi_m(t) = \frac{1}{\sqrt{t_b - t_a}}\, e^{-i\,\omega_m\,t} \tag{3.77c}$$

with

$$\omega_m = \frac{2\,\pi\,m}{t_b - t_a} \qquad m = 0,\, \pm 1,\, \pm 2,\, \ldots \tag{3.80}$$

so that

$$\phi_m(t + t_b - t_a) = e^{-i\,2\,\pi\,m}\,\phi_m(t) = \phi_m(t)$$

$$\begin{aligned}
\int_{t_a}^{t_b} d\,t\,\phi^*_m(t)\,\phi_n(t) &= \frac{1}{t_b - t_a}\int_{t_a}^{t_b} d\,t\,e^{i\,(\omega_m - \omega_n)\,t} \\
&= \frac{e^{i\,(\omega_m - \omega_n)\,t_a}}{t_b - t_a}\int_0^{t_b - t_a} d\,t\,e^{i\,(\omega_m - \omega_n)\,t} \\
&= \frac{e^{i\,(\omega_m - \omega_n)\,t_a}}{t_b - t_a}\begin{cases} \dfrac{1}{i\,(\,\omega_m - \omega_n)}\left(e^{i\,2\,\pi\,(m-n)} - 1 \right) = 0 & \text{for } m \neq n \\ t_b - t_a & \text{for } m = n \end{cases} \\
&= \delta_{m\,n} \qquad \text{(orthonormality)}
\end{aligned} \tag{3.77e}$$

and

$$\sum_m \phi_m(t)\,\phi^*_m(t') = \frac{1}{t_b - t_a}\sum_{m=-\infty}^{\infty} e^{-i\,\omega_m\,(\,t - t'\,)}$$

$$= \frac{1}{t_b - t_a} \sum_{m=-\infty}^{\infty} e^{-i 2\pi m \left(\frac{t-t'}{t_b - t_a}\right)}$$

$$= \frac{1}{t_b - t_a} \delta\left(\frac{t - t'}{t_b - t_a}\right) \qquad \text{for} \qquad 0 \le t - t' \le t_b - t_a$$

$$= \delta(t - t') \qquad \text{(completeness)} \qquad (3.77f)$$

The spectral representation of $G^p_\omega(t, t')$ is therefore

$$G^p_\omega(t, t') = \frac{i}{i\partial_t - \omega} \delta(t - t')$$

$$= \frac{i}{i\partial_t - \omega} \sum_m \phi_m(t)\, \phi^*_m(t')$$

$$= \frac{1}{t_b - t_a} \sum_{m=-\infty}^{\infty} \frac{i}{\omega_m - \omega} e^{-i\omega_m (t-t')} \qquad (3.79)$$

$$= G^p_\omega(t - t') \qquad (3.79a)$$

Treated as a time evolution amplitude, it is obvious that for any system with a time-independent Hamiltonian,

$$G(t, t') = G(t - t')$$

The periodic B.C. (3.77a) thus simplifies to

$$G^p_\omega(t - t' + t_b - t_a) = G^p_\omega(t - t') \qquad (3.78)$$

Using the Poisson summation formula (1.197)

$$\sum_{m=-\infty}^{\infty} f(m) = \int_{-\infty}^{\infty} d\mu \sum_{n=-\infty}^{\infty} e^{2\pi i \mu n} f(\mu) \qquad (3.81)$$

we write (3.79) as

$$G^p_\omega(t - t') = \frac{1}{t_b - t_a} \int_{-\infty}^{\infty} d\mu \sum_{n=-\infty}^{\infty} e^{2\pi i \mu n} \frac{i}{\frac{2\pi\mu}{t_b - t_a} - \omega} e^{-i\frac{2\pi\mu}{t_b - t_a}(t-t')}$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{i\omega'(t_b - t_a) n} \frac{i}{\omega' - \omega} e^{-i\omega'(t-t')} \qquad \omega' = \frac{2\pi\mu}{t_b - t_a}$$

$$\to \qquad G^p_\omega(t) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i\omega'[t - (t_b - t_a) n]} \frac{i}{\omega' - \omega} \qquad (3.82)$$

The $n = 0$ term in (3.82)

$$G_\omega(t) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i\omega' t} \frac{i}{\omega' - \omega} \qquad (3.83)$$

can be taken as the periodic Green function for an infinite range $t_b - t_a \to \infty$.

(3.82) can then be written as

$$G^p_\omega(t) = \sum_{n=-\infty}^{\infty} G_\omega[t - (t_b - t_a) n] \qquad (3.84)$$

$$\to \qquad G^p_\omega(t + t_b - t_a) = \sum_{n=-\infty}^{\infty} G_\omega[t - (t_b - t_a)(n - 1)]$$

$$= \sum_{m=-\infty}^{\infty} G_\omega[t - (t_b - t_a) m] \qquad m = n - 1$$

$$= G^p_\omega(t)$$

as required for the periodic B.C. (3.78).

The integral in (3.83) can be evaluated by means of a contour integral. Since the pole at ω lies on the real-axis portion of the contour, we must displace slightly either the contour or the pole to obtain a meaningful result. Various types of Green functions are obtained depending on the displacement chosen [see Economou].

The retarded Green function is obtained by either lowering the pole ($\omega \to \omega - i\eta$) or raising the path ($\omega' \to \omega' + i\eta$) slightly. For the former, (3.83) becomes

$$G^R_\omega(t) = \lim_{\eta\to 0} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i\omega' t} \frac{i}{\omega' - \omega + i\eta} \tag{3.85a}$$

and constitues the real-axis part of the contour. For the latter, we have

$$G^R_\omega(t) = \lim_{\eta\to 0} \int_{-\infty+i\eta}^{\infty+i\eta} \frac{d\omega'}{2\pi} e^{-i\omega' t} \frac{i}{\omega' - \omega} \tag{3.85b}$$

Let

$$\omega' = \omega'_R + i\,\omega'_I$$

$$\to \qquad e^{-i\omega' t} = \exp(-i\,\omega'_R\, t + t\,\omega'_I)$$

For $t > 0$, the contour must close in the lower half-plane ($\omega'_I < 0$) so that the contribution from the half circle vanishes. The pole is therefore inside the contour. Therefore,

$$\oint \frac{d\omega'}{2\pi} e^{-i\omega' t} \frac{i}{\omega' - \omega + i\eta} = G^R_\omega(t)$$

$$= 2\pi i \left[-\frac{i}{2\pi} \text{Res}\left(\frac{e^{-i\omega t}}{\omega' - \omega + i\eta};\ \omega' = \omega - i\eta \right) \right] \qquad \text{Res} = \text{residue}$$

$$= 2\pi i \left(-\frac{i}{2\pi} e^{-i\omega t} \right)$$

where the overall minus sign comes from the contour being clockwise.

$$\to \qquad G^R_\omega(t) = e^{-i\omega t} \qquad \text{for} \qquad t > 0$$

For $t < 0$, the contour must close in the upper half-plane ($\omega'_I > 0$) so that the contribution from the half circle vanishes. The pole is therefore outside the contour. Therefore,

$$\oint \frac{d\omega'}{2\pi} e^{-i\omega' t} \frac{i}{\omega' - \omega + i\eta} = G^R_\omega(t) \;\; = 0$$

where the minus sign comes from the contour being clockwise.

$$\to \qquad G^R_\omega(t) = 0 \qquad \text{for} \qquad t < 0$$

For $t = 0$, the contribution from the half circle vanishes irregardless how the contour is closed. $G^R_\omega(0)$ can either be 1 or 0. We can either leave it as undefined or set it to some reasonable value, say, $\frac{1}{2}$.

In summary, we have

$$G^R_\omega(t) = \begin{cases} e^{-i\omega t} & \text{for} \;\; t > 0 \\ \frac{1}{2} & \text{for} \;\; t = 0 \\ 0 & \text{for} \;\; t < 0 \end{cases} \tag{3.85}$$

Thus, the retardedness of the Green function, i.e., $G^R = 0$ for $t < 0$, comes from its Fourier transform being analytic in the upper complex plane.

(3.85) can be written succinctly using the Heaviside function $\Theta(t)$ of (1.309) as

$$G^R_\omega(t) = e^{-i\omega t}\,\overline{\Theta}(t) \tag{3.86}$$

Similarly, the advanced Green function is given by

$$G^A_\omega(t) = \oint \frac{d\omega'}{2\pi} e^{-i\omega' t} \frac{i}{\omega' - \omega - i\eta}$$

$$= \begin{cases} 0 & \text{for } t > 0 \\ -\frac{1}{2} & \text{for } t = 0 \\ -e^{-i\omega t} & \text{for } t < 0 \end{cases}$$

$$= -e^{-i\omega t}\,\overline{\Theta}(-t) \tag{3.86a}$$

$$\rightarrow \qquad G^A_{-\omega}(-t) = -e^{-i\omega t}\,\overline{\Theta}(t) \;= -G^R_\omega(t) \tag{3.86b}$$

Using (3.86), we can write (3.84) as

$$G^{pR}_\omega(t) = \sum_{n=-\infty}^{\infty} G^R_\omega\Big[\, t - (t_b - t_a)\, n \Big]$$

$$= \sum_{n=-\infty}^{\infty} \exp\Big\{ -i\,\omega\Big[\, t - (t_b - t_a)\, n \Big]\Big\}\,\overline{\Theta}\Big[\, t - (t_b - t_a)\, n \Big] \tag{3.87}$$

$$= -\sum_{n=-\infty}^{\infty} G^A_{-\omega}\Big[-t + (t_b - t_a)\, n \Big] = -G^{pA}_{-\omega}(-t) \qquad \text{[(3.86b) used.]} \tag{3.87a}$$

Being periodic with period $t_b - t_a$, the values of $G^{pR}_\omega(t)$ are fullly represented within any interval of length $t_b - t_a$. For convenience, we'll use the so called the primary interval

$$t \in [\, 0,\; t_b - t_a\,) \tag{3.88}$$

Neglecting the end point $t = 0$ for the moment, (3.87) becomes

$$G^{pR}_\omega(t) = \sum_{n=-\infty}^{0} \exp\Big\{ -i\,\omega\Big[\, t - (t_b - t_a)\, n \Big]\Big\} \qquad \forall\;\; t \in (\, 0,\;\, t_b - t_a\,),$$

$$= e^{-i\omega t} \sum_{n=0}^{\infty} \exp\Big[-i\,\omega\,(t_b - t_a)\, n \Big]$$

$$= e^{-i\omega t}\, \frac{1}{1 - e^{-i\omega(t_b - t_a)}}$$

$$= e^{-i\omega t}\, \frac{e^{i\omega(t_b - t_a)/2}}{e^{i\omega(t_b - t_a)/2} - e^{-i\omega(t_b - t_a)/2}}$$

$$= e^{-i\omega t}\, \frac{e^{i\omega(t_b - t_a)/2}}{2\, i \sin[\, \omega\,(\, t_b - \; t_a\,)\, /\, 2\,]} \qquad \forall\;\; t \in (\, 0,\;\, t_b - t_a\,) \tag{3.89}$$

Note that (3.89) is not a periodic function of t. Therefore, periodicity must be enforced by hand if it is to be applied outside the specified interval.

As an example, consider the interval $\Big(-(t_b - t_a),\, 0 \Big)$. Using (3.87), we have

$$G^{pR}_\omega(t) = \sum_{n=-\infty}^{-1} \exp\Big\{ -i\,\omega\Big[\, t - (t_b - t_a)\, n \Big]\Big\} \qquad \forall\;\; t \in \Big(-(t_b - t_a),\, 0 \Big)$$

$$= \sum_{n=1}^{\infty} \exp\Big\{ -i\,\omega\Big[\, t + (t_b - t_a)\, n \Big]\Big\}$$

$$= e^{-i\omega t}\, \frac{e^{-i\omega(t_b - t_a)}}{1 - e^{-i\omega(t_b - t_a)}}$$

$$= e^{-i\omega t}\, \frac{e^{-i\omega(t_b - t_a)/2}}{e^{i\omega(t_b - t_a)/2} - e^{-i\omega(t_b - t_a)/2}}$$

$$= e^{-i\omega t}\, \frac{e^{-i\omega(t_b - t_a)/2}}{2\, i \sin[\, \omega\,(\, t_b - \; t_a\,)\, /\, 2\,]} \tag{3.91}$$

The same result can be obtained from (3.89) by setting

$$t_1 = t + (t_b - t_a) \in [\, 0,\; t_b - t_a)$$

$$\to \quad G^{pR}_\omega(t) = G^{pR}_\omega(t_1)$$
$$= e^{-i\omega[t+(t_b-t_a)]}\,\frac{e^{i\omega(t_b-t_a)/2}}{2\,i\sin[\,\omega\,(t_b-\,t_a)\,/\,2\,]}$$
$$= e^{-i\omega t}\,\frac{e^{-i\omega(t_b-t_a)/2}}{2\,i\sin[\,\omega\,(t_b-\,t_a)\,/\,2\,]}$$

in agreement with (3.91).

The end points of the primary interval require special attention. For example, (3.91) gives

$$G^{pR}_\omega(0_-) = \frac{e^{-i\omega(t_b-t_a)/2}}{2\,i\sin[\,\omega\,(t_b-\,t_a)\,/\,2\,]}$$

while, from (3.90),

$$G^{pR}_\omega(0_+) = \sum_{n=-\infty}^{0} \exp\Big[\,i\,\omega\,(t_b-t_a)\,n\Big]$$
$$= \frac{e^{i\omega(t_b-t_a)/2}}{2\,i\sin[\,\omega\,(t_b-\,t_a)\,/\,2\,]} \qquad \text{[(3.89) used.]}$$
$$\to \quad G^{pR}_\omega(0_+) - G^{pR}_\omega(0_-) = \frac{e^{i\omega(t_b-t_a)/2} - e^{-i\omega(t_b-t_a)/2}}{2\,i\sin[\,\omega\,(t_b-\,t_a)\,/\,2\,]} = 1 \qquad (3.90a)$$

On the other hand, (3.87) gives

$$G^{pR}_\omega(0) = \sum_{n=-\infty}^{1} \exp\Big[\,i\,\omega\,(t_b-t_a)\,n\Big] + \overline{\Theta}(0)$$
$$= \sum_{n=-\infty}^{0} \exp\Big[\,i\,\omega\,(t_b-t_a)\,n\Big] - 1 + \overline{\Theta}(0)$$
$$= G^{pR}_\omega\big(0_+\big) - 1 + \overline{\Theta}(0) \qquad \text{[(3.88) used.]}$$
$$= G^{pR}_\omega(0_+) - \frac{1}{2} \qquad (3.90)$$

Using (3.90), we have

$$G^{pR}_\omega(0) = G^{pR}_\omega(0_+) - \frac{1}{2} = G^{pR}_\omega(0_-) + \frac{1}{2} \qquad (3.91a)$$

Owing to the periodicity, similar relations hold at the lower end of the general interval

$$\Big[\,n\,(t_b-t_a),\;(n+1)\,(t_b-t_a)\Big)$$

We can also define the “advanced” version by

$$G^{pA}_\omega(t) = \sum_{n=-\infty}^{\infty} G^{A}_\omega\Big[\,t-(t_b-t_a)\,n\Big]$$
$$= -\sum_{n=-\infty}^{\infty} G^{R}_{-\omega}\Big[-t+(t_b-t_a)\,n\Big] \qquad \text{[(3.86b) used.]}$$
$$= -\sum_{n=-\infty}^{\infty} G^{R}_{-\omega}\Big[-t-(t_b-t_a)\,n\Big] \qquad [\,n\to -n\,]$$
$$= -G^{pR}_{-\omega}(-t) \qquad (3.91b)$$

(3.89) is applicable here for $-t \in (0,\, t_b-t_a)$. Hence,

$$G^{pA}_\omega(t) = e^{-i\omega t}\,\frac{e^{-i\omega(t_b-t_a)/2}}{2\,i\sin[\,\omega\,(t_b-\,t_a)\,/\,2\,]} \qquad \forall\ t\in\Big(-(t_b-t_a),\,0\Big) \qquad (3.91c)$$
$$= G^{pR}_\omega(t)\,\text{[See (3.91).]} \qquad (3.91d)$$

$\equiv G^p_\omega(t)$ [To simplify the notations.]

Quantum statistics for Bose particles (see Chap.7) are described by the Euclidean version of the periodic Green function. By means of the analytic continuation

$$t \to -i\,\tau \qquad t_b - t_a \to -i\,\beta\,\hbar$$

(3.89) becomes

$$G^p_{\omega,e}(\tau) = e^{-\omega\tau}\frac{1}{1 - e^{-\beta\hbar\omega}} \qquad \tau \in [0,\ \beta\hbar) \tag{3.92}$$

$$= e^{-\omega\tau}\frac{e^{\beta\hbar\omega}}{e^{\beta\hbar\omega} - 1}$$

$$= e^{-\omega\tau}\left(\frac{1}{e^{\beta\hbar\omega} - 1} + 1\right)$$

$$= e^{-\omega\tau}\left(n^b_\omega + 1\right) \tag{3.94}$$

where

$$n^b_\omega = \frac{1}{e^{\beta\hbar\omega} - 1} \tag{3.93}$$

is the Bose-Einstein distribution function.

Using the dimensionless quantities

$$\tilde{\tau} = \frac{\tau}{\beta\hbar} \qquad \tilde{\omega} = \beta\,\hbar\,\omega$$

we write (3.94) as

$$G^p_{\tilde{\omega},e}(\tilde{\tau}) = \frac{e^{-\tilde{\omega}\,\tilde{\tau}}}{1 - e^{-\tilde{\omega}}} \qquad \tilde{\tau} \in [0,\ 1) \tag{3.94a}$$

Note that

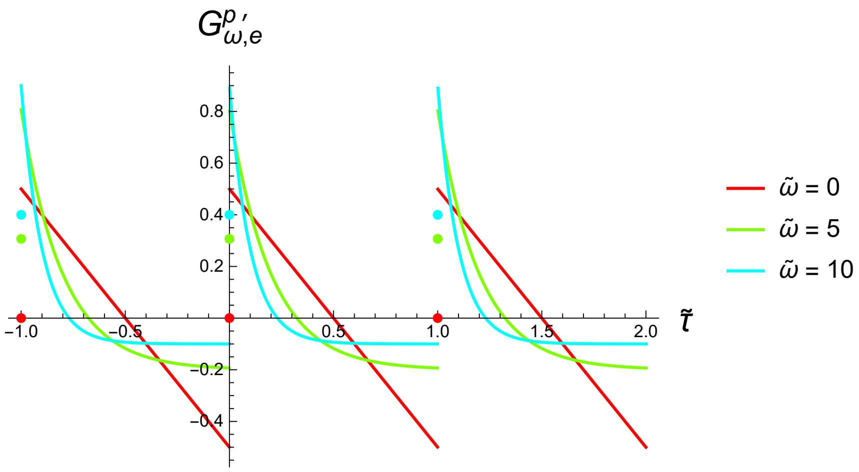
$$G^p_{0,e}(\tilde{\tau}) = \lim_{\tilde{\omega}\to 0}\frac{1}{\tilde{\omega}}$$

is not defined. Therefore, we remove the possible pole by hand and set

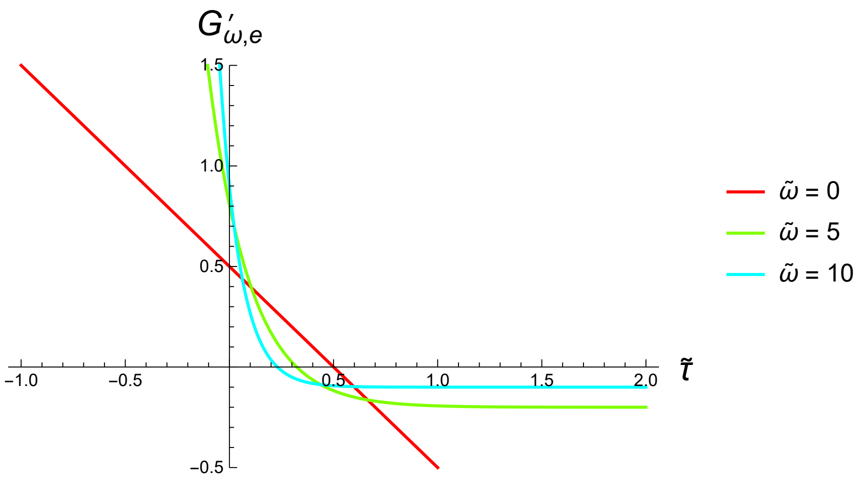
$$G^{p\,\prime}_{\omega,e}(\tau) = \frac{e^{-\omega\tau}}{1 - e^{-\beta\hbar\omega}} - \frac{1}{\beta\hbar\omega}$$

$$G^{p\,\prime}_{\tilde{\omega},e}(\tilde{\tau}) = \frac{e^{-\tilde{\omega}\,\tilde{\tau}}}{1 - e^{-\tilde{\omega}}} - \frac{1}{\tilde{\omega}} \tag{3.94b}$$

Plots of $G^{p\,\prime}_{\tilde{\omega},e}(\tilde{\tau})$ for $\tilde{\omega} = \{0,\ 5,\ 10\}$ are shown in the following figure, where the dots indicate $G^{p\,\prime}_{\tilde{\omega},e}(n) = G^{p\,\prime}_{\tilde{\omega},e}(0)$ as given by (3.91a). [See “3.03._Code.nb”.]



Note that the periodicity must be imposed by hand. Otherwise, the plot becomes



$G^p_{\omega^2}$

We now return to the familiar

$$G^p_{\omega^2}(t,\, t') = \left(-\partial_t^2 - \omega^2\right)^{-1}\, \delta(t-t') \qquad\qquad t,\, t' \in [\, t_a,\, t_b\,) \tag{3.95}$$

with the periodic B.C.

$$G^p_{\omega^2}(t + t_b - t_a,\, t') = G^p_{\omega^2}(t,\, t' + t_b - t_a) = G^p_{\omega^2}(t,\, t') \tag{3.96a}$$

In terms of the periodic basis [see (3.77c)]

$$\left\{ \phi_m = \frac{1}{\sqrt{t_b - t_a}}\, e^{-i\,\omega_m (t - t_a)}\,;\; m = 0,\, \pm 1,\; \ldots \right\} \qquad\qquad \omega_m = \frac{2\,\pi\, m}{t_b - t_a}$$

the spectral representation is

$$\begin{aligned} G^p_{\omega^2}(t,\, t') &= \left(-\partial_t^2 - \omega^2\right)^{-1} \sum_{m=-\infty}^{\infty} \phi_m(t)\, \phi_m^*(t') \\ &= \frac{1}{t_b - t_a} \left(-\partial_t^2 - \omega^2\right)^{-1} \sum_{m=-\infty}^{\infty} e^{-i\,\omega_m (t - t')} \\ &= \frac{1}{t_b - t_a} \sum_{m=-\infty}^{\infty} \frac{e^{-i\,\omega_m (t-t')}}{\omega_m^2 - \omega^2} \\ &= G^p_{\omega^2}(t - t') \end{aligned} \tag{3.97}$$

Thus, the periodic B.C. (3.96a) simplifies to

$$G^{p}_{\omega^2}(t-t'+t_b-t_a)=G^{p}_{\omega^2}(t-t') \qquad (3.96)$$

as befit a system with a time-independent Hamitonian.

The different ways to handle the poles at $\omega=\pm\omega_m$ again result in various types of Green functions [see Economou].

The (causal) Green function is defined as

$$G^{pR}_{\omega^2}(t-t')=\frac{1}{t_b-t_a}\sum_{m=-\infty}^{\infty}\frac{e^{-i\omega_m(t-t')}}{\omega_m^2-\omega^2+i\eta} \qquad (3.98a)$$

Using

$$\frac{1}{\omega_m^2-\omega^2+i\eta}=\frac{1}{2i\omega}\left(\frac{i}{\omega_m-\omega+i\eta}-\frac{i}{\omega_m+\omega-i\eta}\right) \qquad (3.98)$$

(3.98a) becomes

$$G^{pR}_{\omega^2}(t-t')=\frac{1}{2i\omega}\frac{1}{t_b-t_a}\sum_{m=-\infty}^{\infty}\left(\frac{i}{\omega_m-\omega+i\eta}-\frac{i}{\omega_m+\omega-i\eta}\right)e^{-i\omega_m(t-t')} \qquad (3.98b)$$

$$=\frac{1}{2i\omega}\left[G^{pR}_{\omega}(t-t')-G^{pA}_{-\omega}(t-t')\right] \qquad \text{[(3.79) used.]}$$

Using (3.89), we have, for $t-t'\in[0,\ t_b-t_a)$,

$$G^{pR}_{\omega^2}(t-t')=\frac{1}{2i\omega}\left[e^{-i\omega(t-t')}\frac{e^{i\omega(t_b-t_a)/2}}{2i\sin[\omega(t_b-t_a)/2]}+e^{i\omega(t-t')}\frac{e^{-i\omega(t_b-t_a)/2}}{2i\sin[\omega(t_b-t_a)/2]}\right]$$

$$=-\frac{1}{2\omega}\frac{\cos\omega\left[t-t'-\frac{1}{2}(t_b-t_a)\right]}{\sin[\omega(t_b-t_a)/2]} \qquad t-t'\in[0,\ t_b-t_a) \qquad (3.99)$$

Note that (3.99) is valid for $t-t'=0$ because the factor $\frac{1}{2}$ in (3.91a) is cancelled out. Thus, although $G^{pR}_{\omega}(t-t')$ & $G^{pA}_{-\omega}(t-t')$ are both discontinuous at $t-t'=0$ so that their derivatives diverge as $\delta(t-t')$, $G^{pR}_{\omega^2}(t-t')$ is continuous there and its derivative suffers only a discontinuity given by (3.28e).

Outside the primary interval, $G^{pR}_{\omega^2}(t-t')$ is determined by the periodicity.

For example,

$$\partial_t G^{pR}_{\omega^2}(t-t')=\frac{1}{2}\frac{\sin\omega\left[t-t'-\frac{1}{2}(t_b-t_a)\right]}{\sin[\omega(t_b-t_a)/2]} \qquad (3.99a)$$

$$\rightarrow \qquad \partial_t G^{pR}_{\omega^2}(0_+)=\frac{1}{2}\frac{\sin\omega\left[-\frac{1}{2}(t_b-t_a)\right]}{\sin[\omega(t_b-t_a)/2]}=-\frac{1}{2}$$

$$\partial_t G^{pR}_{\omega^2}(0_-)=\frac{1}{2}\frac{\sin\omega\left[t_b-t_a-\frac{1}{2}(t_b-t_a)\right]}{\sin[\omega(t_b-t_a)/2]}=\frac{1}{2} \qquad \text{[Periodic B.C. used.]}$$

$$\therefore \qquad -\partial_t G^{pR}_{\omega^2}(0_+)+\partial_t G^{pR}_{\omega^2}(0_-)=1$$

as required by (3.28e).

From (3.86a), we have

$$G^{A}_{-\omega}(t)=-e^{i\omega t}\,\overline{\Theta}(-t) \qquad (3.100)$$

G^{a}_{ω}

Next we consider the anti-periodic Green function $G^{a}_{\omega}(t,\ t')$ defined by

$$(i\partial_t-\omega)\,G^{a}_{\omega}(t,\ t')=i\,\delta(t-t') \qquad t-t'\in[0,\ t_b-t_a) \qquad (3.101)$$

with the anti-periodic B.C.

$$G^a_\omega\,(t-t'+t_b-t_a) = -G^a_\omega(t-t') = -G^a_\omega(t,\,t') \tag{3.102}$$

For the anti-periodic basis, we choose the orthonormal solutions to

$$(\,i\,\partial_t - \omega_m\,)\,\phi_m\,(t) = 0 \tag{3.102a}$$

i.e.,

$$\psi_m(t) = \frac{1}{\sqrt{t_b - t_a}}\,e^{-i\,\omega^f_m\,t} \tag{3.102b}$$

with

$$\omega^f_m = \frac{\pi\,(2\,m+1)}{t_b - t_a} \qquad\qquad m = 0,\ \pm 1,\ \pm 2,\ \ldots \tag{3.104}$$

where the superscript f stands for fermionic and hence the ensuing anti-symmetry.

Hence

$$\psi_m(t + t_b - t_a) = e^{-i\,2\,\pi\,(\,m+1)}\,\psi_m(t) = -\psi_m(t)$$

$$\begin{aligned}\int_{t_a}^{t_b} d\,t\,\psi^*_m(t)\,\psi_n(t) &= \frac{1}{t_b - t_a}\int_{t_a}^{t_b} d\,t\,e^{i\left(\omega^f_m - \omega^f_n\right)t} \\ &= \frac{1}{t_b - t_a}\int_{t_a}^{t_b} d\,t\,e^{i\,(\omega_m - \omega_n)\,t} \qquad\qquad \omega_m = \frac{2\,\pi\,m}{t_b - t_a} \\ &= \delta_{m\,n} \qquad\qquad [(3.77e)\text{ used.}]\end{aligned} \tag{3.104a}$$

and

$$\begin{aligned}\sum_m \psi_m(t)\,\psi^*_m(t') &= \frac{1}{t_b - t_a}\sum_{m=-\infty}^{\infty} e^{-i\,\omega^f_m\,(\,t-t'\,)} \\ &= \frac{e^{-i\,\pi\left(\frac{t-t'}{t_b - t_a}\right)}}{t_b - t_a}\sum_{m=-\infty}^{\infty} e^{-i\,2\,\pi\,m\left(\frac{t-t'}{t_b - t_a}\right)} \\ &= \frac{e^{-i\,\pi\left(\frac{t-t'}{t_b - t_a}\right)}}{t_b - t_a}\,\delta\!\left(\frac{t\;-\;t'}{t_b\;-\;t_a}\right) \qquad \text{for} \qquad 0 \le t - t' \le t_b - t_a \\ &= \delta(t - t') \qquad\qquad \text{(completeness)}\end{aligned} \tag{3.104b}$$

The spectral representation is therefore

$$\begin{aligned}G^a_\omega\,(t,\,t') &= \frac{i}{i\,\partial_t - \omega}\sum_m \psi_m(t)\,\psi^*_m(t') \\ &= \frac{1}{t_b - t_a}\sum_{m=-\infty}^{\infty}\frac{i}{\omega^f_m - \omega}\,e^{-i\,\omega^f_m\,(\,t-t'\,)} \\ &= G^a_\omega(t - t')\end{aligned} \tag{3.103}$$

The Poisson summation formula (3.81) may be adapted to the fermionic case as follows:

$$\begin{aligned}\sum_{m=-\infty}^{\infty} f\left(m + \frac{1}{2}\right) &= \int_{-\infty}^{\infty} d\,\mu\sum_{n=-\infty}^{\infty} e^{2\,\pi\,i\,\mu\,n}\,f\left(\mu + \frac{1}{2}\right) \\ &= \int_{-\infty}^{\infty} d\,\mu\sum_{n=-\infty}^{\infty} e^{2\,\pi\,i\,(\mu\,-\,1/2\,)\,n}\,f(\mu) \\ &= \int_{-\infty}^{\infty} d\,\mu\sum_{n=-\infty}^{\infty} (-)^n\,e^{2\,\pi\,i\,\mu\,n}\,f(\mu)\end{aligned} \tag{3.105}$$

(3.103) thus becomes

$$G_\omega^a(t,t') = \frac{1}{t_b - t_a}\int_{-\infty}^{\infty} d\mu \sum_{n=-\infty}^{\infty} (-)^n e^{2\pi i \mu n} \frac{i}{\frac{2\pi\mu}{t_b-t_a} - \omega} e^{-i\frac{2\pi\mu}{t_b-t_a}(t-t')}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \sum_{n=-\infty}^{\infty} (-)^n e^{i\omega'(t_b-t_a)n} \frac{i}{\omega'-\omega} e^{-i\omega'(t-t')} \qquad \omega' = \frac{2\pi\mu}{t_b - t_a} \qquad (3.105a)$$

Replacing $\frac{i}{\omega'-\omega}$ with $\frac{i}{\omega'-\omega+i\eta}$ then gives

$$G_\omega^{aR}(t,t') = \sum_{n=-\infty}^{\infty} (-)^n G_\omega^R\left[t - t' - (t_b - t_a)n\right] \qquad (3.106)$$

$$= G_\omega^{aR}(t-t')$$

Using (3.86), this becomes

$$G_\omega^{aR}(t-t') = \sum_{n=-\infty}^{\infty} (-)^n \exp\left\{-i\omega\left[t - t' - (t_b - t_a)n\right]\right\} \overline{\Theta}\left[t - t' - (t_b - t_a)n\right] \qquad (3.107)$$

For the primary interval,

$$G_\omega^{aR}(t) = \sum_{n=-\infty}^{0} (-)^n \exp\left\{-i\omega\left[t - (t_b - t_a)n\right]\right\} \qquad t \in (0,\ t_b - t_a)$$

$$= e^{-i\omega t} \sum_{n=0}^{\infty} (-)^n \exp\left[-i\omega(t_b - t_a)n\right]$$

$$= e^{-i\omega t} \sum_{n=0}^{\infty} \left\{-\exp\left[-i\omega(t_b - t_a)\right]\right\}^n$$

$$= e^{-i\omega t} \frac{1}{1 + \exp\left[-i\omega(t_b - t_a)\right]}$$

$$= e^{-i\omega t} \frac{\exp\left[i\frac{\omega}{2}(t_b - t_a)\right]}{\exp\left[i\frac{\omega}{2}(t_b - t_a)\right] + \exp\left[-i\frac{\omega}{2}(t_b - t_a)\right]}$$

$$= \frac{\exp\left\{-i\omega\left[t - \frac{1}{2}(t_b - t_a)\right]\right\}}{2\cos\left[\frac{\omega}{2}(t_b - t_a)\right]} \qquad t \in (0,\ t_b - t_a) \quad (3.108)$$

The lower end of the interval, $t = 0$, requires special attention:

$$G_\omega^{aR}(0) = \sum_{n=-\infty}^{1} (-)^n \exp\left[i\omega(t_b - t_a)n\right] + \overline{\Theta}(0) = G_\omega^{aR}(0_-) + \frac{1}{2}$$

$$= \sum_{n=-\infty}^{0} (-)^n \exp\left[i\omega(t_b - t_a)n\right] - 1 + \overline{\Theta}(0) = G_\omega^{aR}(0_+) - \frac{1}{2}$$

$$= \frac{1}{1 + \exp\left[-i\omega(t_b - t_a)\right]} - \frac{1}{2}$$

$$= \frac{\exp\left[i\frac{\omega}{2}(t_b - t_a)\right]}{2\cos\left[\frac{\omega}{2}(t_b - t_a)\right]} - \frac{1}{2} \qquad (3.108a)$$

Outside the primary interval, $G_\omega^{aR}(t)$ is determined by the anti-periodicity.

In the limit $\omega \to 0$, (3.108) gives

$$G_0^{aR}(t) = \frac{1}{2} \qquad \text{for } t \in \left(0,\ t_b - t_a\right)$$

while (3.108a) gives

$$G_0^{aR}(0) = 0$$

Anti-periodicity gives

$$G_0^{aR}(t) = -\frac{1}{2} \qquad \text{for } t \in \left(-(t_b - t_a),\ 0\right)$$

Hence,

$$G_0^{aR}(t) = \frac{1}{2}\,\epsilon(t) \qquad \text{for } t \in \left(-(t_b - t_a),\ t_b - t_a\right) \tag{3.109}$$

For the "advanced" version [c.f. (3.91b)],

$$G_\omega^{aA}(t) = \sum_{n=-\infty}^{\infty} (-)^n\, G_\omega^A\left[t - (t_b - t_a)\,n\right]$$

$$= -\sum_{n=-\infty}^{\infty} (-)^n\, G_{-\omega}^R\left[-t + (t_b - t_a)\,n\right] \qquad \text{[(3.86b) used.]}$$

$$= -\sum_{n=-\infty}^{\infty} (-)^n\, G_{-\omega}^R\left[-t - (t_b - t_a)\,n\right] \qquad [\, n \to -n \,]$$

$$= -G_{-\omega}^{aR}(-t) \tag{3.109a}$$

$$= -\frac{\exp\left\{i\,\omega\left[-t - \frac{1}{2}(t_b - t_a)\right]\right\}}{2\cos\left[\frac{\omega}{2}(t_b - t_a)\right]} \qquad t \in \left(-(t_b - t_a),\, 0\right) \tag{3.109b}$$

For $t \in (0,\ t_b - t_a)$, anti-periodicity gives

$$G_\omega^{aA}(t) = \frac{\exp\left\{i\,\omega\left[-(t - t_b + t_a) - \frac{1}{2}(t_b - t_a)\right]\right\}}{2\cos\left[\frac{\omega}{2}(t_b - t_a)\right]}$$

$$= \frac{\exp\left\{i\,\omega\left[-t + \frac{1}{2}(t_b - t_a)\right]\right\}}{2\cos\left[\frac{\omega}{2}(t_b - t_a)\right]} \tag{3.109c}$$

$$= G_\omega^{aR}(t) \equiv G_\omega^a(t)$$

$G_{\omega,\mathrm{e}}^a$

The Euclidean version of G_ω^a is used to described the quantum statistics of fermions. With

$$t \to -i\,\tau \qquad t_b - t_a \to -i\,\beta\,\hbar$$

(3.108) becomes

$$G_{\omega,e}^a(\tau) = e^{-\omega\tau}\,\frac{1}{1 + e^{-\beta\hbar\omega}} \qquad \tau \in (0,\ \beta\hbar) \tag{3.110}$$

$$= e^{-\omega\tau}\,\frac{e^{\beta\hbar\omega}}{e^{\beta\hbar\omega} + 1}$$

$$= e^{-\omega\tau}\left(1 - \frac{1}{e^{\beta\hbar\omega} + 1}\right)$$

$$= e^{-\omega\tau}\left(1 - n_\omega^f\right) \qquad \tau \in (0,\ \beta\hbar) \tag{3.112}$$

where

$$n_\omega^f = \frac{1}{e^{\beta\hbar\omega} + 1} \tag{3.111}$$

is the Fermi-Dirac distribution function (or occupation number).

$G_{\omega^2}^a$

The anti-periodic Green functions can be obtained from their periodic counterparts by replacing the bosonic Matsubara frequences ω_m with the fermionic ones ω_m^f. Relations between the various types of Green functions will survive such a replacement.

Replacing ω_m with ω_m^f turns (3.98a) into

$$G_{\omega^2}^{aR}(t) = \frac{1}{t_b - t_a} \sum_{m=-\infty}^{\infty} \frac{e^{-i\omega_m^f t}}{\omega_m^{f2} - \omega^2 + i\eta}$$
$$= \frac{1}{2i\omega} \frac{1}{t_b - t_a} \sum_{m=-\infty}^{\infty} \left(\frac{i}{\omega_m^f - \omega + i\eta} - \frac{i}{\omega_m^f + \omega - i\eta} \right) e^{-i\omega_m^f t}$$
$$= \frac{1}{2i\omega} \left[G_\omega^{aR}(t) - G_{-\omega}^{aA}(t) \right] \qquad (3.113a)$$

Using (3.108) & (3.109c), we have

$$G_{\omega^2}^{aR}(t) = \frac{1}{2i\omega} \left(\frac{\exp\left\{-i\omega\left[t - \frac{1}{2}(t_b - t_a)\right]\right\}}{2\cos\left[\frac{\omega}{2}(t_b - t_a)\right]} - \frac{\exp\left\{i\omega\left[t - \frac{1}{2}(t_b - t_a)\right]\right\}}{2\cos\left[\frac{\omega}{2}(t_b - t_a)\right]} \right)$$
$$= -\frac{\sin\omega\left[t - \frac{1}{2}(t_b - t_a)\right]}{2\omega\cos\left[\frac{\omega}{2}(t_b - t_a)\right]} \qquad t \in [0,\ t_b - t_a) \qquad (3.113)$$

Note that (3.113) applies to the end point $t = 0$ since, by (3.113a), the $\frac{1}{2}$ correction term from each Green function cancels each other.

If (3.113) is to be applied outside the primary interval, anti-periodicity must be enforced by hand. For example,

$$G_{\omega^2}^{aR}(t) = \frac{\sin\omega\left[t + t_b - t_a - \frac{1}{2}(t_b - t_a)\right]}{2\omega\cos\left[\frac{\omega}{2}(t_b - t_a)\right]} \qquad t \in \left[-(t_b - t_a),\ 0\right)$$
$$= \frac{\sin\omega\left[t + \frac{1}{2}(t_b - t_a)\right]}{2\omega\cos\left[\frac{\omega}{2}(t_b - t_a)\right]}$$
$$= \frac{\sin\omega\left[-\left|t\right| + \frac{1}{2}(t_b - t_a)\right]}{2\omega\cos\left[\frac{\omega}{2}(t_b - t_a)\right]}$$
$$= -\frac{\sin\omega\left[\left|t\right| - \frac{1}{2}(t_b - t_a)\right]}{2\omega\cos\left[\frac{\omega}{2}(t_b - t_a)\right]} \qquad (3.113a)$$

Comparing with (3.113) shows that (3.113a) is actually valid for $t \in \left[-(t_b - t_a),\ t_b - t_a\right]$.

$G_{\omega^2,e}^p$ & $G_{\omega^2,e}^a$

With

$$t \to -i\tau \qquad t_b - t_a \to -i\beta\hbar$$
$$\omega_m = \frac{2\pi m}{t_b - t_a} \to i\frac{2\pi m}{\beta\hbar} = i\nu_m \qquad \nu_m = \frac{2\pi m}{\beta\hbar}$$

(3.95) becomes

$$\left(\partial_\tau^2 - \omega^2\right) G_{\omega^2}^p(-i\tau, -i\tau') = \frac{1}{|-1|\,i}\delta(\tau - \tau') = -i\,\delta(\tau - \tau') \qquad (3.114a)$$

It is customary to deal with a real equation by defining the Euclidean Green function as

$$G_{\omega^2,e}^p(\tau, \tau') = iG_{\omega^2}^p(-i\tau, -i\tau') \qquad (3.114b)$$

so that (3.114a) becomes

$$\left(\partial_\tau^2 - \omega^2\right) G^p_{\omega^2,e}(\tau, \tau') = -\delta(\tau - \tau') \qquad (3.114c)$$

Analytic continuing (3.98a) gives

$$G^{pR}_{\omega^2,e}(\tau) = \frac{i}{-i\beta\hbar} \sum_{m=-\infty}^{\infty} \frac{e^{-i(i\nu_m)(-i\tau)}}{(i\nu_m)^2 - \omega^2 + i\eta} = \frac{1}{\beta\hbar} \sum_{m=-\infty}^{\infty} \frac{e^{-i\nu_m \tau}}{\nu_m^2 + \omega^2 - i\eta} \qquad (3.114d)$$

and its fermionic cousin

$$G^{aR}_{\omega^2,e}(\tau) = \frac{1}{\beta\hbar} \sum_{m=-\infty}^{\infty} \frac{e^{-i\nu_m^f \tau}}{\nu_m^{f2} + \omega^2 - i\eta} \qquad \nu_m^f = \frac{2\pi(m+1)}{\beta\hbar} \qquad (3.114e)$$

At $\tau = 0$, we have

$$G^{pR}_{\omega^2,e}(0) = \frac{1}{\beta\hbar} \sum_{m=-\infty}^{\infty} \frac{1}{\nu_m^2 + \omega^2 - i\eta} \qquad G^{aR}_{\omega^2,e}(0) = \frac{1}{\beta\hbar} \sum_{m=-\infty}^{\infty} \frac{1}{\nu_m^{f2} + \omega^2 - i\eta} \qquad (3.114)$$

Analytic continuing (3.79) gives

$$\begin{aligned} G^{pR}_{\omega,e}(\tau) &= \frac{1}{-i\beta\hbar} \sum_{m=-\infty}^{\infty} \frac{i}{i\nu_m - \omega + i\eta} e^{-i(i\nu_m)(-i\tau)} \\ &= \frac{1}{\beta\hbar} \sum_{m=-\infty}^{\infty} \frac{1}{-i\nu_m + \omega - i\eta} e^{-i\nu_m \tau} \qquad (3.115a) \\ &= \frac{1}{\beta\hbar} (\partial_\tau + \omega)^{-1} \sum_{m=-\infty}^{\infty} e^{-i\nu_m \tau} \\ &= (\partial_\tau + \omega)^{-1} \delta(\tau) \end{aligned}$$

in agreement with (3.76).

The advanced version is obtained by changing the sign of η :

$$G^{pA}_{\omega,e}(\tau) = \frac{1}{\beta\hbar} \sum_{m=-\infty}^{\infty} \frac{1}{-i\nu_m + \omega + i\eta} e^{-i\nu_m \tau} \qquad (3.115b)$$

From (3.91d), we have

$$\begin{aligned} G^{pR}_{\omega,e}(\tau) &= G^{pA}_{\omega,e}(\tau) \equiv G^{p}_{\omega,e}(\tau) \\ &= e^{-\omega\tau} \left(\frac{1}{e^{\beta\hbar\omega} - 1} + 1 \right) \\ &= e^{-\omega\tau} \left(n^b_\omega + 1 \right) \qquad \text{[see (3.94)]} \qquad (3.115c) \end{aligned}$$

The fermionic versions of (3.115a-b) are

$$G^{aR}_{\omega,e}(\tau) = \frac{1}{\beta\hbar} \sum_{m=-\infty}^{\infty} \frac{1}{-i\nu_m^f + \omega - i\eta} e^{-i\nu_m^f \tau} \qquad (3.115d)$$

$$G^{aA}_{\omega,e}(\tau) = \frac{1}{\beta\hbar} \sum_{m=-\infty}^{\infty} \frac{1}{-i\nu_m^f + \omega + i\eta} e^{-i\nu_m^f \tau} \qquad (3.115e)$$

$$\begin{aligned} G^{aR}_{\omega,e}(\tau) &= G^{aA}_{\omega,e}(\tau) \equiv G^{a}_{\omega,e}(\tau) \\ &= e^{-\omega\tau} \left(1 - \frac{1}{e^{\beta\hbar\omega} + 1} \right) \text{[see (3.112)]} \\ &= e^{-\omega\tau} \left(1 - n^f_\omega \right) \qquad (3.115f) \end{aligned}$$

(3.114) then gives

$$G^{pR}_{\omega^2,e}(0) = \frac{1}{\beta\hbar} \sum_{m=-\infty}^{\infty} \frac{1}{2\omega} \left(\frac{1}{-i\nu_m + \omega - i\eta} + \frac{1}{i\nu_m + \omega - i\eta} \right)$$

$$= \frac{1}{2\omega}\left(G^{pR}_{\omega,e}(0) - G^{pA}_{-\omega,e}(0)\right) \qquad \text{[(3.115a-b) used.]}$$

$$= \frac{1}{2\omega}\left(G^{p}_{\omega,e}(0) - G^{p}_{-\omega,e}(0)\right)$$

$$= \frac{1}{2\omega}\left[n^b_\omega + 1 - \left(n^b_{-\omega} + 1\right)\right] \qquad \text{[(3.115c) used.]}$$

$$= \frac{1}{2\omega}\left(\frac{e^{\beta\hbar\omega}}{e^{\beta\hbar\omega} - 1} - \frac{e^{-\beta\hbar\omega}}{e^{-\beta\hbar\omega} - 1}\right)$$

$$= \frac{1}{2\omega}\left(\frac{e^{\beta\hbar\omega}}{e^{\beta\hbar\omega} - 1} + \frac{1}{e^{\beta\hbar\omega} - 1}\right) = \frac{1}{2\omega}\left(n^b_\omega + 1\right)\left(1 + e^{-\beta\hbar\omega}\right)$$

$$= \frac{1}{2\omega}\left(\frac{e^{\beta\hbar\omega} + 1}{e^{\beta\hbar\omega} - 1}\right) = \frac{1}{2\omega}\left(\frac{e^{\beta\hbar\omega/2} + e^{-\beta\hbar\omega/2}}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}}\right) = \frac{1}{2\omega}\coth\frac{\beta\hbar\omega}{2} \qquad (3.115)$$

The fermionic version is easily obtained

$$G^{aR}_{\omega^2,e}(0) = \frac{1}{\beta\hbar}\sum_{m=-\infty}^{\infty}\frac{1}{2\omega}\left(\frac{1}{-i\nu^f_m + \omega - i\eta} + \frac{1}{i\nu^f_m + \omega - i\eta}\right)$$

$$= \frac{1}{2\omega}\left(G^{aR}_{\omega,e}(0) - G^{aA}_{-\omega,e}(0)\right) \qquad \text{[(3.115d-e) used.]}$$

$$= \frac{1}{2\omega}\left(G^{a}_{\omega,e}(0) - G^{a}_{-\omega,e}(0)\right)$$

$$= \frac{1}{2\omega}\left[1 - n^f_\omega - \left(1 - n^f_{-\omega}\right)\right] \qquad \text{[(3.115f) used.]}$$

$$= \frac{1}{2\omega}\left(\frac{e^{\beta\hbar\omega}}{e^{\beta\hbar\omega} + 1} - \frac{e^{-\beta\hbar\omega}}{e^{-\beta\hbar\omega} + 1}\right)$$

$$= \frac{1}{2\omega}\left(\frac{e^{\beta\hbar\omega}}{e^{\beta\hbar\omega} + 1} - \frac{1}{e^{\beta\hbar\omega} + 1}\right) = \frac{1}{2\omega}\left(1 - n^f_\omega\right)\left(1 - e^{-\beta\hbar\omega}\right)$$

$$= \frac{1}{2\omega}\left(\frac{e^{\beta\hbar\omega} - 1}{e^{\beta\hbar\omega} + 1}\right) = \frac{1}{2\omega}\left(\frac{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}}{e^{\beta\hbar\omega/2} + e^{-\beta\hbar\omega/2}}\right) = \frac{1}{2\omega}\tanh\frac{\beta\hbar\omega}{2} \qquad (3.116)$$

3.3.2. Time-Dependent Frequency

Setting

$$G_\Omega(t, t') = \overline{\Theta}(t - t')\, g(t, t') \qquad (3.117)$$

$$\rightarrow \quad i\,\partial_t G_\Omega(t, t') = i\,\delta(t - t')\, g(t, t') + i\,\overline{\Theta}(t - t')\,\partial_t g(t, t')$$

(3.75) becomes

$$\overline{\Theta}(t - t')\left[i\,\partial_t - \Omega(t)\right] g(t, t') + i\,\delta(t - t')\, g(t, t') = i\,\delta(t - t') \qquad (3.117a)$$

With the I.C.,

$$g(t', t') = 1 \qquad (3.117b)$$

(3.117a) becomes

$$\overline{\Theta}(t - t')\left[i\,\partial_t - \Omega(t)\right] g(t, t') = 0$$

or simply

$$\left[i\,\partial_t - \Omega(t)\right] g(t, t') = 0 \qquad \text{for} \qquad t \geq t' \qquad (3.118)$$

The solution can be obtained using the method of integrating factor, which gives

$$g(t, t') = K(t')\exp\left[-i\int_{t_0}^{t} d\,t''\,\Omega(t'')\right] \qquad (3.119)$$

With the I.C. (3.117b), (3.119) simplifies to

$$g(t,\,t') = \exp\left[-i\int_{t'}^{t} d\,t''\,\Omega(t'')\right] \tag{3.119a}$$

so that

$$G_\Omega(t,\,t') = \overline{\Theta}(t-t')\exp\left[-i\int_{t'}^{t} d\,t''\,\Omega(t'')\right] \tag{3.120}$$

Treating t' as a parameter, (3.120) is a particular solution to (3.75). Since there is only one independent homogeneous solution for a 1st order eq., the general solution is

$$\begin{aligned} G_\Omega(t,\,t') &= \overline{\Theta}(t-t')\exp\left[-i\int_{t'}^{t} d\,t''\,\Omega(t'')\right] + C(t')\,g(t,\,t') \\ &= \left[\overline{\Theta}(t-t') + C(t')\right]\exp\left[-i\int_{t'}^{t} d\,t''\,\Omega\,(t'')\right] \end{aligned} \tag{3.121}$$

For a periodic $\Omega(t)$, we expect $G_\Omega(t,\,t')$ to be also periodic, e.g.,

$$G_\Omega^p(t_a,\,t') = G_\Omega^p(t_b,\,t') \tag{3.121a}$$

In terms of (3.121), this means

$$C(t')\exp\left[-i\int_{t'}^{t_a} d\,t''\,\Omega\,(t'')\right] = \left[1 + C(t')\right]\exp\left[-i\int_{t'}^{t_b} d\,t''\,\Omega(t'')\right] \tag{3.122}$$

$$\begin{aligned} \rightarrow \qquad C(t') &= \frac{\exp\left[-i\int_{t'}^{t_b} d\,t''\,\Omega(t'')\right]}{\exp\left[-i\int_{t'}^{t_a} d\,t''\,\Omega(t'')\right] - \exp\left[-i\int_{t'}^{t_b} d\,t''\,\Omega(t'')\right]} \\ &= \frac{1}{\exp\left[i\left\{-\int_{t'}^{t_a}+\int_{t'}^{t_b}\right\} d\,t''\,\Omega(t'')\right] - 1} \\ &= \frac{1}{\exp\left[i\int_{t_a}^{t_b} d\,t''\,\Omega(t'')\right] - 1} \equiv n_\Omega^p \end{aligned} \tag{3.123}$$

(3.121) thus becomes

$$G_\Omega^p(t,\,t') = \left[\overline{\Theta}(t-t') + n_\Omega^p\right]\exp\left[-i\int_{t'}^{t} d\,t''\,\Omega(t'')\right] \tag{3.124}$$

For an anti-periodic B.C.,

$$G_\Omega^a(t_a,\,t') = -G_\Omega^a(t_b,\,t') \tag{3.124a}$$

we have

$$\begin{aligned} C(t') &= \frac{-\exp\left[-i\int_{t'}^{t_b} d\,t''\,\Omega(t'')\right]}{\exp\left[-i\int_{t'}^{t_a} d\,t''\,\Omega(t'')\right] + \exp\left[-i\int_{t'}^{t_b} d\,t''\,\Omega(t'')\right]} \\ &= \frac{-1}{\exp\left[i\int_{t_a}^{t_b} d\,t''\,\Omega(t'')\right] + 1} \equiv -n_\Omega^a \end{aligned} \tag{3.125}$$

so that

$$G_\Omega^a(t,\,t') = \left[\overline{\Theta}(t-t') - n_\Omega^a\right]\exp\left[-i\int_{t'}^{t} d\,t''\,\Omega(t'')\right] \tag{3.125a}$$

Note that

$$\begin{aligned} \tilde{G}_\Omega(t,\,t') &= G_\Omega(t',\,t) \\ &= \overline{\Theta}(t'-t)\exp\left[i\int_{t'}^{t} d\,t''\,\Omega(t'')\right] \qquad \text{[(3.120) used.]} \end{aligned} \tag{3.126a}$$

is a solution of the eq.

$$\left[-i\,\partial_t - \Omega(t)\right]\tilde{G}_\Omega(t,\,t') = i\,\delta(t-t') \tag{3.126}$$

Proof:

$-i\,\partial_t$(3.126 a) gives

$$-i\,\partial_t \tilde{G}_\Omega(t,\,t') = i\,\delta(t-t')\exp\left[i\int_{t'}^{t} d\,t''\,\Omega(t'')\right] + \overline{\Theta}(t'-t)\,\Omega(t)\exp\left[i\int_{t'}^{t} d\,t''\,\Omega(t'')\right]$$
$$= i\,\delta(t-t') + \Omega(t)\,\tilde{G}_\Omega(t,\,t')$$

If $\Omega(t)$ is a matrix, then

$$[\,\Omega(t),\,\Omega(t')\,] \neq 0 \qquad \text{for } t \neq t'$$

unless $\Omega(t)$ & $\Omega(t')$ are diagonal. As in (1.252), the problem is solved by the replacement

$$\exp\left[-i\int_{t_a}^{t_b} d\,t\,\Omega(t)\right] \rightarrow \hat{T}\exp\left[-i\int_{t_a}^{t_b} d\,t\,\Omega(t)\right] \qquad (3.127)$$

For imaginary times $\tau = i\,t$, (3.120), (3.124) & (3.125a) become

$$G_\Omega(\tau,\,\tau') = \overline{\Theta}(\tau-\tau')\exp\left[-\int_{\tau'}^{\tau} d\,\tau''\,\Omega(\tau'')\right] \qquad (3.128)$$

$$G_\Omega^p(\tau,\,\tau') = \left[\overline{\Theta}(\tau-\tau') + n^b\right]\exp\left[-\int_{\tau'}^{\tau} d\,\tau''\,\Omega(\tau'')\right] \qquad (3.129)$$

$$G_\Omega^a(\tau,\,\tau') = \left[\overline{\Theta}(\tau-\tau') - n^f\right]\exp\left[-\int_{\tau'}^{\tau} d\,\tau''\,\Omega(\tau'')\right] \qquad (3.129a)$$

where, by (3.123) & (3.125), we have

$$n^b = \frac{1}{\exp\left[\int_0^{\beta\hbar} d\,\tau''\,\Omega(\tau'')\right] - 1} \qquad (3.130)$$

$$n^f = \frac{1}{\exp\left[\int_0^{\beta\hbar} d\,\tau''\,\Omega(\tau'')\right] + 1} \qquad (3.131)$$

TraceLog

Using the Green functions (3.124) & (3.128), we can calculate the tracelog of the operators $-i\,\partial_t + \Omega(t)$ & $\partial_\tau + \Omega(\tau)$, respectively.

By definition

$$\mathrm{Tr}\ln[\,\partial_\tau + g\,\Omega(\tau)\,] = \int d\,\tau \sum_n \phi_n^*(\tau)\ln[\,\partial_\tau + g\,\Omega(\tau)\,]\,\phi_n(\tau) \qquad (132a)$$

where $\{\,\phi_n\,\}$ is any orthonormal, complete set of basis functions.

Let $\phi_n(\tau) = e^{-i\lambda_n\tau}$, then

$$\mathrm{Tr}\ln[\,\partial_\tau + g\,\Omega(\tau)\,] = \int d\,\tau \sum_{n=-\infty}^{\infty} \phi_n^*(\tau)\ln[\,-i\,\lambda_n + g\,\Omega(\tau)\,]\,\phi_n(\tau) \qquad (1.132b)$$

$$\rightarrow \qquad \frac{\partial}{\partial g}\mathrm{Tr}\ln[\,\partial_\tau + g\,\Omega(\tau)\,] = \int d\,\tau \sum_{n=-\infty}^{\infty} \phi_n^*(\tau)\,\frac{\Omega(\tau)}{-i\,\lambda_n + g\,\Omega(\tau)}\,\phi_n(\tau)$$
$$= \int d\,\tau\,\Omega(\tau)\,G_{g\,\Omega}^s(\tau,\,\tau) \qquad (1.132c)$$

where

$$s = \begin{cases} p & \text{if } \lambda_n = \nu_n \\ a & \text{if } \lambda_n = \nu_n^f \end{cases}$$

$$\therefore \qquad \mathrm{Tr}\ln[\,\partial_\tau + g\,\Omega(\tau)\,] = \int d\,\tau\,\Omega(\tau)\int^g d\,g'\,G_{g'\,\Omega}^s(\tau,\,\tau) + C \qquad (1.132)$$

(3.129-30) give

$$G^p_{g\Omega}(\tau,\ \tau) = \overline{\Theta}(0) + n^b_{g\Omega}$$
$$= \frac{1}{2} + \frac{1}{\exp\left[\int_0^{\beta\hbar} d\tau''\, g\,\Omega(\tau'')\right] - 1}$$

Using

$$\int d g\, \frac{1}{e^{gX} - 1} = -g + \frac{1}{X} \ln\left(1 - e^{gX}\right)$$

we have

$$\int_0^g d g'\, G^s_{g'\Omega}(\tau,\ \tau) = \frac{1}{2} g - g + \frac{\ln\left\{1 - \exp\left[\int_0^{\beta\hbar} d\tau''\, g\,\Omega(\tau'')\right]\right\}}{\int_0^{\beta\hbar} d\tau''\, \Omega(\tau'')}$$

$$= \left(\ln\left\{\exp\left[-\frac{1}{2} g \int_0^{\beta\hbar} d\tau''\, \Omega(\tau'')\right] - \exp\left[\frac{1}{2} g \int_0^{\beta\hbar} d\tau''\, \Omega(\tau'')\right]\right\}\right) \Big/ \left(\int_0^{\beta\hbar} d\tau''\, \Omega(\tau'')\right)$$
$$= \frac{\ln\left\{-2\sinh\left[\frac{1}{2} g \int_0^{\beta\hbar} d\tau''\, \Omega(\tau'')\right]\right\}}{\int_0^{\beta\hbar} d\tau''\, \Omega(\tau'')}$$

(1.132) thus becomes

$$\mathrm{Tr}\ln[\partial_\tau + g\,\Omega(\tau)] = \ln\left\{-2\sinh\left[\frac{1}{2} g \int_0^{\beta\hbar} d\tau''\, \Omega(\tau'')\right]\right\} + C$$

Comparing with the time-independent frequency result [see (2.536)],

$$\mathrm{Tr}\ln[\pm\partial_\tau + \omega] = \ln\left[2\sinh\frac{\beta\hbar\omega}{2}\right] \tag{3.133a}$$

we see that

$$C = -\ln(-1) = -i\pi$$

so that

$$\mathrm{Tr}\ln[\partial_\tau + g\,\Omega(\tau)] = \ln\left(2\sinh\left[\frac{1}{2} g \int_0^{\beta\hbar} d\tau''\, \Omega(\tau'')\right]\right)$$
$$= \ln\left(\exp\left[\frac{1}{2} g \int_0^{\beta\hbar} d\tau''\, \Omega(\tau'')\right] - \exp\left[-\frac{1}{2} g \int_0^{\beta\hbar} d\tau''\, \Omega(\tau'')\right]\right)$$
$$= \frac{1}{2} g \int_0^{\beta\hbar} d\tau''\, \Omega(\tau'') + \ln\left(1 - \exp\left[-g \int_0^{\beta\hbar} d\tau''\, \Omega(\tau'')\right]\right) \tag{3.133}$$

At low temperatures, $\beta\hbar >> 1$, the exponential term approches zero and we have

$$\mathrm{Tr}\ln[\partial_\tau + g\,\Omega(\tau)] \approx \frac{1}{2} g \int_0^{\beta\hbar} d\tau''\, \Omega(\tau'') \tag{3.134}$$

Owing to the infinite sum over n in (1.132c), it is easy to see that

$$\mathrm{Tr}\ln[\partial_\tau + \Omega(\tau)] = \mathrm{Tr}\ln[-\partial_\tau + \Omega(\tau)] \tag{3.135a}$$

which may also be expected from (3.133a).

For $\Omega(t) > 0$, we can write

$$\mathrm{Tr}\ln\left[\pm\partial_\tau + \Omega(\tau)\right] = \mathrm{Tr}\ln\left\{(\pm\partial_\tau + \eta)\left[1 + (\pm\partial_\tau + \eta)^{-1}\right]\Omega(\tau)\right\} \qquad \eta \to 0_+$$
$$= \mathrm{Tr}\ln(\pm\partial_\tau + \eta) + \mathrm{Tr}\ln\left\{\left[1 + (\pm\partial_\tau + \eta)^{-1}\right]\Omega(\tau)\right\} \tag{3.135}$$

Using (1.132b), the 1st term becomes

$$\mathrm{Tr}\ln(\pm\partial_\tau+\eta) = \sum_n \ln(\mp i\,\lambda_n) \;\propto\; \int_{-\infty}^{\infty} d\,\omega \ln\omega$$
$$= 0 \qquad\qquad [\text{ Veltman's rule (2.506) used. }]$$

The logarithm in the 2nd term can be expanded as

$$\ln\left\{\left[1+(\pm\partial_\tau+\eta)^{-1}\right]\Omega(\tau)\right\} = \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\left[\left(\pm\partial_\tau+\eta\right)^{-1}\Omega(\tau)\right]^n$$
$$= \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\int d\,\tau_1 \ldots \int d\,\tau_n \left(\pm\partial_\tau+\eta\right)^{-1}\delta(\tau-\tau_1)\,\Omega(\tau_1)$$
$$\times\left(\pm\partial_{\tau_1}+\eta\right)^{-1}\delta(\tau_1-\tau_2)\,\Omega(\tau_2)\times\ldots\times\left(\pm\partial_{\tau n-1}+\eta\right)^{-1}\delta(\tau_{n-1}-\tau_n)\,\Omega(\tau_n)$$

Using

$$\left(\pm\partial_\tau+\eta\right)\overline{\Theta}\left[\pm(\tau-\tau')\right] = \delta(\tau-\tau')$$

we have

$$\ln\left\{\left[1+(\pm\partial_\tau+\eta)^{-1}\right]\Omega(\tau)\right\} = \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\int d\,\tau_1 \ldots \int d\,\tau_n\,\overline{\Theta}\left[\pm(\tau-\tau_1)\right]\Omega(\tau_1)$$
$$\times\,\overline{\Theta}\left[\pm(\tau_1-\tau_2)\right]\Omega(\tau_2)\times\ldots\times\overline{\Theta}\left[\pm(\tau_{n-1}-\tau_n)\right]\Omega(\tau_n)$$

$$\therefore \qquad \mathrm{Tr}\ln\left\{\left[1+(\pm\partial_\tau+\eta)^{-1}\right]\Omega(\tau)\right\} = \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\int d\,\tau\,\delta(\tau-\tau_n)\int d\,\tau_1\ldots\int d\,\tau_n$$
$$\times\,\overline{\Theta}\left[\pm(\tau-\tau_1)\right]\Omega(\tau_1)\,\overline{\Theta}\left[\pm(\tau_1-\tau_2)\right]\Omega(\tau_2)\times\ldots\times\overline{\Theta}\left[\pm(\tau_{n-1}-\tau_n)\right]\Omega(\tau_n)$$
$$= \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\int d\,\tau_1\ldots\int d\,\tau_n\,\overline{\Theta}\left[\pm(\tau_n-\tau_1)\right]\Omega(\tau_1)\,\overline{\Theta}\left[\pm(\tau_1-\tau_2)\right]\Omega(\tau_2) \qquad (3.137)$$
$$\times\ldots\times\overline{\Theta}\left[\pm(\tau_{n-1}-\tau_n)\right]\Omega(\tau_n)$$

The cyclic product of Heaviside functions evaluates to zero, i.e.,

$$\overline{\Theta}\left[\pm(\tau_n-\tau_1)\right]\overline{\Theta}\left[\pm(\tau_1-\tau_2)\right]\times\ldots\times\overline{\Theta}\left[\pm(\tau_{n-1}-\tau_n)\right] = 0$$

since the arguments of the functions cannot all be of the same sign for any given set of values of $\{\tau_1,\ \ldots,\ \tau_n\}$ for $n \ge 2$. Hence, only the $n = 1$ term in (3.137) survives:

$$\mathrm{Tr}\ln\left\{\left[1+(\pm\partial_\tau+\eta)^{-1}\right]\Omega(\tau)\right\} = \int d\,\tau_1\,\overline{\Theta}(0)\,\Omega(\tau_1)$$
$$= \frac{1}{2}\int d\,\tau\,\Omega(\tau) \qquad (3.138)$$

Hence, (3.135) becomes

$$\mathrm{Tr}\ln\left[\pm\partial_\tau+\Omega(\tau)\right] = \frac{1}{2}\int d\,\tau\,\Omega(\tau)$$

in agreement with (3.134).

If we replace $\Omega(\tau)$ with a matrix or operator $\hat{H}(\tau)$, we must introduce the time-order operator $\hat{T}$ to handle the possibility that $H(\tau)$ & $H(\tau')$ may not commute. Furthermore, we need to start afresh from (3.133), which is expanded as

$$\mathrm{Tr}\ln\left[\hbar\,\partial_\tau+\hat{H}(\tau)\right] = \frac{1}{2\,\hbar}\mathrm{Tr}\left[\int_0^{\beta\hbar} d\,\tau\,\hat{H}(\tau)\right] - \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\left\{\hat{T}\exp\left[-\frac{1}{\hbar}n\int_0^{\beta\hbar} d\,\tau\,\hat{H}(\tau)\right]\right\} \qquad (3.139)$$

where $\mathrm{Tr} \ln \hbar = 0$ (by Veltman’s rule) was used.