3.4. Summing Spectral Representation of Green Function

Setting

$$t_2 = t_b - t \qquad\qquad t_1 = t' - t_a \tag{3.140a}$$

we can write (3.70) as

$$
\begin{aligned}
G_{\omega^2}(t,\, t') &= \frac{2}{t_b - t_a} \sum_{n=1}^{\infty} (-)^{n+1} \frac{\sin \nu_n t_2 \sin \nu_n t_1}{\nu_n^2 - \omega^2} \\
&= \frac{2}{t_b - t_a} \sum_{n=1}^{\infty} (-)^{n+1} \frac{\left(e^{i \nu_n t_2} - e^{-i \nu_n t_2}\right)\left(e^{i \nu_n t_1} - e^{-i \nu_n t_1}\right)}{(2\,i)^2 \left(\nu_n^2 - \omega^2\right)} \\
&= \frac{1}{2} \frac{1}{t_b - t_a} \sum_{n=1}^{\infty} (-)^{n} \frac{1}{\nu_n^2 - \omega^2}\left(e^{-i \nu_n (t_2+t_1)} - e^{-i \nu_n (t_2-t_1)} + e^{i \nu_n (t_2+t_1)} - e^{i \nu_n (t_2-t_1)}\right) \\
&= \frac{1}{2} \frac{1}{t_b - t_a} \sum_{n=-\infty}^{\infty} (-)^{n} \frac{e^{-i \nu_n (t_2+t_1)} - e^{-i \nu_n (t_2-t_1)}}{\nu_n^2 - \omega^2}
\end{aligned}
\tag{3.140}
$$

where we've used [see (3.64)]

$$\nu_n = \frac{n\,\pi}{t_b - t_a} = -\nu_{-n}$$

Setting [see (3.80) & (3.104)]

$$\omega_m = \nu_{2\,m} \qquad\qquad \omega_m^f = \nu_{2\,m+1}$$

(3.140) becomes

$$G_{\omega^2}(t,\, t') = \frac{1}{2} \frac{1}{t_b - t_a} \sum_{m=-\infty}^{\infty} \left(\frac{e^{-i \omega_m (t_2+t_1)} - e^{-i \omega_m (t_2-t_1)}}{\omega_m^2 - \omega^2} - \frac{e^{-i \omega_m^f (t_2+t_1)} - e^{-i \omega_m^f (t_2-t_1)}}{\omega_m^{f2} - \omega^2} \right) \tag{3.141}$$

Using (3.97) & its fermionic counterpart, this becomes

$$G_{\omega^2}(t,\, t') = \frac{1}{2}\left[G_{\omega^2}^p (t_2 + t_1) - G_{\omega^2}^p (t_2 - t_1) - G_{\omega^2}^a (t_2 + t_1) + G_{\omega^2}^a (t_2 - t_1) \right] \tag{3.142}$$

With the help of (3.99), we have

$$
\begin{aligned}
&G_{\omega^2}^p (t_2 + t_1) - G_{\omega^2}^p (t_2 - t_1) = \\
&-\frac{1}{2\,\omega}\left(\left(\cos \omega \left[t_2 + t_1 - \frac{1}{2}(t_b - t_a)\right] - \cos \omega \left[t_2 - t_1 - \frac{1}{2}(t_b - t_a)\right]\right) \Big/ \sin[\,\omega\,(t_b - t_a)\,/\,2\,] \right) \\
&\qquad = \frac{\sin \omega t_1 \;\sin \omega \left[\, t_2 - \frac{1}{2}(t_b - t_a)\,\right]}{\omega \sin[\,\omega\,(t_b - t_a)\,/\,2\,]}
\end{aligned}
\tag{3.143}
$$

Using & (3.113), we get

$$
\begin{aligned}
&G_{\omega^2}^a (t_2 + t_1) - G_{\omega^2}^a (t_2 - t_1) = \\
&-\frac{1}{2\,\omega}\left(\left(\sin \omega \left[t_2 + t_1 - \frac{1}{2}(t_b - t_a)\right] - \sin \omega \left[t_2 - t_1 - \frac{1}{2}(t_b - t_a)\right]\right) \Big/ \cos[\,\omega\,(t_b - t_a)\,/\,2\,] \right) \\
&\qquad = -\frac{\sin \omega t_1 \;\cos \omega \left[\, t_2 - \frac{1}{2}(t_b - t_a)\,\right]}{\omega \cos[\,\omega\,(t_b - t_a)\,/\,2\,]}
\end{aligned}
\tag{3.144}
$$

(3.142) thus becomes

$$G_{\omega^2}(t,\, t') = \frac{\sin \omega t_1}{2\,\omega}\left(\frac{\sin \omega \left[\, t_2 - \frac{1}{2}(t_b - t_a)\,\right]}{\sin[\,\omega\,(t_b - t_a)\,/\,2\,]} + \frac{\cos \omega \left[\, t_2 - \frac{1}{2}(t_b - t_a)\,\right]}{\cos[\,\omega\,(t_b - t_a)\,/\,2\,]} \right)$$

$$= \frac{\sin \omega t_1}{2\,\omega \sin[\,\omega\,(\,t_b - \,t_a\,)\,/\,2\,]\cos[\,\omega\,(\,t_b - \,t_a\,)\,/\,2\,]}$$
$$\times\left(\sin \omega\left[t_2 - \frac{1}{2}(t_b - t_a) \right]\cos[\,\omega\,(\,t_b - \,t_a\,)\,/\,2\,] \right.$$
$$\left. + \cos \omega\left[t_2 - \frac{1}{2}(t_b - t_a) \right]\sin[\,\omega\,(\,t_b - \,t_a\,)\,/\,2\,]\right)$$
$$= \frac{\sin \omega t_1 \sin \omega t_2}{\omega \sin \omega\,(\,t_b - \,t_a\,)} \tag{3.145}$$
$$= \frac{\sin \omega\,(t' - t_a) \sin \omega\,(t_b - t)}{\omega \sin \omega\,(\,t_b - \,t_a\,)}$$

in agreement with (3.36) for $t > t'$.

In conjunction with $G_{\omega^2}(t,\, t') = 0$ for $t < t'$, (3.145) then gives the retarded Green function.

Alternatively, by switching the role of t & t' for $t < t'$, we get the causal Green function of (3.36)

$$G_{\omega^2}(t,\, t') = (\theta(t - t')\sin \omega\,(t_b - t)\sin \omega\,(t' - t_a)) + \theta(t' - t)\;\sin \omega\,(t_b - t')\sin \omega\,(t - t_a)\,)/(\omega \sin \omega\,(\,t_b - \,t_a\,))$$
$$= \frac{\sin \omega\,(t_b - t_>)\sin \omega\,(t_< - t_a)}{\omega \sin \omega\,(\,t_b - \,t_a\,)} \tag{3.145a}$$

For the case

$$t_a \to -\infty \qquad t_b \to \infty \tag{3.146}$$

there are only two independent solutions, $e^{\pm i \omega t}$, to the homogeneous eq. Using the Wronskian construction [see §3.2.1], we get

$$G_{\omega^2}(t,\, t') = -\frac{i}{2\,\omega}\begin{cases} e^{-i\,\omega\,(t-t')} & \text{for } t > t' \\ e^{i\,\omega\,(t-t')} & \text{for } t < t' \end{cases}$$
$$= -\frac{i}{2\,\omega}\left[\, \Theta(t - t')\, e^{-i\,\omega\,(t-t')} + \Theta(t' - t)\, e^{i\,\omega\,(t-t')} \,\right]$$
$$= -\frac{i}{2\,\omega}\, e^{-i\,\omega\,|t-t'|} \tag{3.147}$$

$$\to \qquad \partial_t G_{\omega^2}(t,\, t') = -\frac{i}{2\,\omega}\left[\, \delta(t - t') - i\,\omega\,\Theta(t - t')\, e^{-i\,\omega\,(t-t')} - \delta(t' - t) + i\,\omega\,\Theta(t' - t)\, e^{i\,\omega\,(t-t')} \,\right]$$
$$= \frac{1}{2}\left[-\Theta(t - t')\, e^{-i\,\omega\,(t-t')} + \Theta(t' - t)\, e^{i\,\omega\,(t-t')} \right]$$
$$\partial_t^2 G_{\omega^2}(t,\, t') = \frac{1}{2}\left[-\delta(t - t') + i\,\omega\,\Theta(t - t')\, e^{-i\,\omega\,(t-t')} - \delta(t' - t) + i\,\omega\,\Theta(t' - t)\, e^{i\,\omega\,(t-t')} \right]$$
$$= -\delta(t - t') - \omega^2\, G_{\omega^2}(t,\, t')$$

Hence,

$$\left(-\partial_t^2 - \omega^2\right) G_{\omega^2}(t,\, t') = \delta(t - t') \tag{3.148}$$

On the other hand, the periodic & antiperiodic solutions can be constructed, as in (3.106), from the infinite ranged Green function (3.147) as

$$G^p_{\omega^2}(t,\, t') = \sum_{n=-\infty}^{\infty} G_{\omega^2}\left[\, t - t' - \,(t_b - \,t_a)\, n \,\right] \tag{3.149}$$
$$G^a_{\omega^2}(t,\, t') = \sum_{n=-\infty}^{\infty} (-)^n\, G_{\omega^2}\left[\, t - t' - \,(t_b - \,t_a)\, n \,\right] \tag{3.149a}$$

For completeness, we consider the spectral representation with repect to the basis functions [c.f. (3.63)],

$$x_0(t) = \frac{1}{\sqrt{t_b - t_a}} \qquad x_n(t) = \sqrt{\frac{2}{t_b - t_a}} \cos \nu_n (t - t_a) \tag{3.150}$$

with

$$\nu_n = \frac{n\pi}{t_b - t_a} \qquad n = 0, 1, 2, 3, \ldots \tag{3.150a}$$

that satisfy the Neumann B.C.

$$\partial_t x_n(t)\Big|_{t=t_a} = 0 \qquad \partial_t x_n(t)\Big|_{t=t_b} = 0 \tag{3.150b}$$

(3.69a) then gives

$$G^N_{\omega^2}(t, t') = \frac{2}{t_b - t_a} \sum_{n=0}^{\infty} \frac{x_n(t)\, x_n(t')}{\nu_n^2 - \omega^2}$$
$$= \frac{2}{t_b - t_a}\left[-\frac{1}{2\,\omega^2} + \sum_{n=1}^{\infty} \frac{\cos \nu_n (t - t_a) \cos \nu_n (t' - t_a)}{\nu_n^2 - \omega^2} \right] \tag{3.151}$$

Thus, $G^N_{\omega^2}(t, t')$ satisfies the Neumann B.C.

$$\partial_t\, G^N_{\omega^2}(t, t')\Big|_{t=t_b} = -\frac{2}{t_b - t_a} \sum_{n=1}^{\infty} \frac{\nu_n \sin \nu_n (t_b - t_a) \cos \nu_n (t' - t_a)}{\nu_n^2 - \omega^2} \Bigg] = 0 \qquad [\ (3.150a)\ \text{used.}\ ]$$
$$\partial_{t'}\, G^N_{\omega^2}(t, t')\Big|_{t'=t_a} = -\frac{2}{t_b - t_a} \sum_{n=1}^{\infty} \frac{\nu_n \cos \nu_n (t - t_a) \sin \nu_n (t_a - t_a)}{\nu_n^2 - \omega^2} \Bigg] = 0 \tag{3.152}$$

Following the derivations (3.140-2), we have

$$\cos \nu_n (t_b - t) = \cos \nu_n \left[t - t_a - (t_b - t_a) \right]$$
$$= \cos \left[\nu_n (t - t_a) - n\pi \right] \qquad [\ (3.64)\ \text{used.}\ ]$$
$$= (-)^n \cos \nu_n (t - t_a)$$

$$G^N_{\omega^2}(t, t') = \frac{2}{t_b - t_a}\left[-\frac{1}{2\,\omega^2} + \sum_{n=1}^{\infty} (-)^n \frac{\cos \nu_n t_2 \cos \nu_n t_1}{\nu_n^2 - \omega^2} \right]$$
$$= \frac{2}{t_b - t_a}\left[-\frac{1}{2\,\omega^2} + \sum_{n=1}^{\infty} (-)^n \frac{\left(e^{i\nu_n t_2} + e^{-i\nu_n t_2}\right)\left(e^{i\nu_n t_1} + e^{-i\nu_n t_1}\right)}{4\,(\nu_n^2 - \omega^2)} \right]$$
$$= \frac{1}{2} \frac{1}{t_b - t_a}\left[-\frac{2}{\omega^2} + \sum_{n=1}^{\infty} (-)^n \frac{1}{\nu_n^2 - \omega^2} \left(e^{-i\nu_n (t_2 + t_1)} + e^{-i\nu_n (t_2 - t_1)} + e^{i\nu_n (t_2 + t_1)} + e^{i\nu_n (t_2 - t_1)} \right) \right]$$
$$= \frac{1}{2} \frac{1}{t_b - t_a} \sum_{n=-\infty}^{\infty} (-)^n \frac{e^{-i\nu_n (t_2 + t_1)} + e^{-i\nu_n (t_2 - t_1)}}{\nu_n^2 - \omega^2} \tag{3.152a}$$
$$= \frac{1}{2}\left[G^p_{\omega^2}(t_2 + t_1) + G^p_{\omega^2}(t_2 - t_1) - G^a_{\omega^2}(t_2 + t_1) - G^a_{\omega^2}(t_2 - t_1) \right] \tag{3.153}$$

With the help of (3.99), we have [c.f. (3.143)]

$$G^p_{\omega^2}(t_2 + t_1) + G^p_{\omega^2}(t_2 - t_1) =$$
$$-\frac{1}{2\,\omega}\left(\left(\cos \omega \left[t_2 + t_1 - \frac{1}{2}(t_b - t_a) \right] + \cos \omega \left[t_2 - t_1 - \frac{1}{2}(t_b - t_a) \right] \right) \Big/ \sin[\,\omega\,(t_b - t_a)/2\,] \right)$$
$$= -\frac{\cos \omega t_1 \,\cos \omega \left[t_2 - \frac{1}{2}(t_b - t_a) \right]}{\omega \sin[\,\omega\,(t_b - t_a)/2\,]} \tag{3.154}$$

Using & (3.113), we get [c.f. (3.144)]

$$G^a_{\omega^2}(t_2+t_1)+G^a_{\omega^2}(t_2-t_1)=$$
$$-\frac{1}{2\,\omega}\left(\left(\sin\omega\left[t_2+t_1-\frac{1}{2}(t_b-t_a)\right]+\sin\omega\left[t_2-t_1-\frac{1}{2}(t_b-t_a)\right]\right)\Big/\cos[\,\omega\,(t_b-t_a)/2\,]\right)$$
$$=-\frac{\cos\omega t_1\ \sin\omega\left[t_2-\frac{1}{2}(t_b-t_a)\right]}{\omega\cos[\,\omega\,(t_b-t_a)/2\,]} \qquad (3.155)$$

(3.153) thus becomes [c.f. (3.145)]

$$G^N_{\omega^2}(t,\,t')=-\frac{\cos\omega t_1}{2\,\omega}\left(\frac{\cos\omega\left[t_2-\frac{1}{2}(t_b-t_a)\right]}{\sin[\,\omega\,(t_b-t_a)/2\,]}-\frac{\sin\omega\left[t_2-\frac{1}{2}(t_b-t_a)\right]}{\cos[\,\omega\,(t_b-t_a)/2\,]}\right)$$
$$=-\frac{\cos\omega t_1\cos\omega t_2}{\omega\sin[\,\omega\,(t_b-t_a)\,]}$$
$$=-\frac{\cos\omega\,(t'-t_a)\cos\omega\,(t_b-t)}{\omega\sin[\,\omega\,(t_b-t_a)\,]} \qquad (3.156a)$$

which is easily verified to satisfy the Neumann B.C. (3.152). As with (3.145), this is valid for $t>t'$. The causal Green function is therefore [see (3.145a)]

$$G^N_{\omega^2}(t,\,t')=-\frac{\cos\omega\,(t_b-t_>)\cos\omega\,(t_<-t_a)}{\omega\sin[\,\omega\,(t_b-t_a)\,]} \qquad (3.156)$$

The power expansion for $\omega\to 0$ can be obtained from the following *Mathematica* code

$$-\frac{\mathrm{Cos}[\omega\ (\mathrm{tp}-\mathrm{t_a})]\ \mathrm{Cos}[\omega\ (\mathrm{t_b}-\mathrm{t})]}{\omega\ \mathrm{Sin}[\omega\ (\mathrm{t_b}-\mathrm{t_a})]}+\mathrm{O}[\omega]^2$$

$$\frac{1}{(\mathrm{t_a}-\mathrm{t_b})\ \omega^2}+\left(\frac{1}{6}\ (\mathrm{t_a}-\mathrm{t_b})-\frac{-\frac{1}{2}\ (\mathrm{tp}-\mathrm{t_a})^2-\frac{1}{2}\ (-\mathrm{t}+\mathrm{t_b})^2}{-\mathrm{t_a}+\mathrm{t_b}}\right)+\mathrm{O}[\omega]^2$$

Thus,

$$G^N_{\omega^2}(t,\,t')=-\frac{1}{\omega^2\,(t_b-t_a)}-\frac{1}{6}(t_b-t_a)-\frac{1}{2\,(t_b-t_a)}\left[(t'-t_a)^2+(t_b-t)^2\right]+O\left[\omega^2\right] \qquad (3.157)$$