3.5. Wronski Construction for Periodic and Antiperiodic Green Functions

The Wronski Construction in §3.2.1 for

$$\left[-\partial_t^2 - \Omega^2(t)\right] G_{\Omega^2}(t, t') = \delta(t - t') \tag{3.158}$$

can be carried over to

$$\left[-\partial_t^2 - \Omega^2(t)\right] G_{\Omega^2}^{p,a}(t, t') = \delta^{p,a}(t - t') \tag{3.158a}$$

where

$$G_{\Omega^2}^{p,a}(t, t') = \sum_{n=-\infty}^{\infty} (\pm)^n G_{\Omega^2}(t - t' - n\beta\hbar) \tag{3.159a}$$

$$\delta^{p,a}(t - t') = \sum_{n=-\infty}^{\infty} (\pm)^n \delta(t - t' - n\beta\hbar) \tag{3.159}$$

with the B.C.

$$G_{\Omega^2}^{p,a}(t, t') = \pm G_{\Omega^2}^{p,a}(t + t_b - t_a, t') \tag{3.159b}$$

$$\partial_t G_{\Omega^2}^{p,a}(t, t') = \pm \partial_t G_{\Omega^2}^{p,a}(t + t_b - t_a, t') \tag{3.159c}$$

As in (3.53), we set

$$G_{\Omega^2}^{p,a}(t, t') = \overline{\Theta}(t - t')\,\Delta(t, t') + a(t')\,\xi(t) + b(t')\,\eta(t) \tag{3.159}$$

where ξ & η are independent solutions of the homogeneous solution and

$$\left[-\partial_t^2 - \Omega^2(t)\right]\Delta(t, t') = 0 \tag{3.161}$$

with I.C. [see (3.45-6)]

$$\Delta(t, t) = 0 \qquad\qquad \partial_t \Delta(t, t')\Big|_{t=t'} = -1 \tag{3.161a}$$

Imposing the B.C. (3.159b) on (3.159) with

$$t = t_a \qquad \& \qquad t' \in [t_a, t_b)$$

gives

$$G_{\Omega^2}^{p,a}(t_a, t') = a(t')\,\xi(t_a) + b(t')\,\eta(t_a)$$

$$= \pm G_{\Omega^2}^{p,a}(t_b, t') = \pm\left[\Delta(t_b, t') + a(t')\,\xi(t_b) + b(t')\,\eta(t_b)\right]$$

$$\rightarrow \quad \left[\xi(t_b) \mp \xi(t_a)\right] a(t') + \left[\eta(t_b) \mp \eta(t_a)\right] b(t') = -\Delta(t_b, t') \tag{3.162}$$

For the B.C. (3.159c),

$$\partial_t G_{\Omega^2}^{p,a}(t, t')\Big|_{t=t_a} = \delta(t_a - t')\,\Delta(t_a, t') + \overline{\Theta}(t_a - t')\,\partial_{t_a}\Delta(t_a, t')$$

$$+ a(t')\,\dot{\xi}(t_a) + b(t')\,\dot{\eta}(t_a)$$

$$= a(t')\,\dot{\xi}(t_a) + b(t')\,\dot{\eta}(t_a)$$

$$= \pm\partial_t G_{\Omega^2}^{p,a}(t, t')\Big|_{t=t_b} = \pm\left[\delta(t_b - t')\,\Delta(t_b, t') + \partial_{t_b}\Delta(t_b, t')\right.$$

$$\left. + a(t')\,\dot{\xi}(t_b) + b(t')\,\dot{\eta}(t_b)\right]$$

$$= \pm\left[\partial_{t_b}\Delta(t_b, t') + a(t')\,\dot{\xi}(t_b) + b(t')\,\dot{\eta}(t_b)\right]$$

$$\rightarrow \quad \left[\dot{\xi}(t_b) \mp \dot{\xi}(t_a)\right] a(t') + \left[\dot{\eta}(t_b) \mp \dot{\eta}(t_a)\right] b(t') = -\partial_{t_b}\Delta(t_b, t') \tag{3.162a}$$

(3.162-a) can be written in matrix form as

$$\begin{pmatrix} \xi(t_b) \mp \xi(t_a) & \eta(t_b) \mp \eta(t_a) \\ \dot{\xi}(t_b) \mp \dot{\xi}(t_a) & \dot{\eta}(t_b) \mp \dot{\eta}(t_a) \end{pmatrix} \begin{pmatrix} a(t') \\ b(t') \end{pmatrix} = \begin{pmatrix} -\Delta(t_b, t') \\ -\partial_{t_b}\Delta(t_b, t') \end{pmatrix} \tag{3.163a}$$

Let

$$\overline{\Lambda}^{p,a}(t_a, t_b) = \begin{pmatrix} \xi(t_b) \mp \xi(t_a) & \eta(t_b) \mp \eta(t_a) \\ \dot{\xi}(t_b) \mp \dot{\xi}(t_a) & \dot{\eta}(t_b) \mp \dot{\eta}(t_a) \end{pmatrix} \tag{3.163}$$

then the condition for (3.163a) to have a unique solution is

$$\det \overline{\Lambda}^{p,a}(t_a, t_b) \neq 0 \tag{3.164a}$$

whereupon

$$a(t') = \frac{-[\dot\eta(t_b) \mp \dot\eta(t_a)]\,\Delta(t_b, t') + [\eta(t_b) \mp \eta(t_a)]\,\partial_{t_b}\Delta(t_b, t')}{\det \overline{\Lambda}^{p,a}(t_a, t_b)} \tag{3.163b}$$

$$b(t') = \frac{\left[\dot\xi(t_b) \mp \dot\xi(t_a)\right]\Delta(t_b, t') - [\xi(t_b) \mp \xi(t_a)]\,\partial_{t_b}\Delta(t_b, t')}{\det \overline{\Lambda}^{p,a}(t_a, t_b)} \tag{3.163c}$$

From (3.163), we have

$$\begin{aligned}\det \overline{\Lambda}^{p,a}(t_a, t_b) = {} & [\xi(t_b) \mp \xi(t_a)][\dot\eta(t_b) \mp \dot\eta(t_a)] \\ & -\left[\dot\xi(t_b) \mp \dot\xi(t_a)\right][\eta(t_b) \mp \eta(t_a)] \\ = {} & \xi(t_b)\,\dot\eta(t_b) - \dot\xi(t_b)\,\eta(t_b) + \xi(t_a)\,\dot\eta(t_a) - \dot\xi(t_a)\,\eta(t_a) \\ & \mp\left[\xi(t_b)\,\dot\eta(t_a) + \xi(t_a)\,\dot\eta(t_b) - \dot\xi(t_b)\,\eta(t_a) - \dot\xi(t_a)\,\eta(t_b)\right]\end{aligned} \tag{3.164b}$$

From (3.48a), we see that

$$W = \begin{vmatrix} \xi & \eta \\ \dot\xi & \dot\eta \end{vmatrix} = \xi\,\dot\eta - \dot\xi\,\eta \tag{3.164c}$$

is independent of time. By (3.49), we have

$$\Delta(t, t') = \frac{1}{W}\left[\xi(t)\,\eta(t') - \xi(t')\,\eta(t)\right] \tag{3.164d}$$

$$\rightarrow \quad \partial_t\Delta(t, t') = \frac{1}{W}\left[\dot\xi(t)\,\eta(t') - \xi(t')\,\dot\eta(t)\right] \tag{3.164e}$$

$$\partial_{t'}\Delta(t, t') = \frac{1}{W}\left[\xi(t)\,\dot\eta(t') - \dot\xi(t')\,\eta(t)\right] \tag{3.164f}$$

(3.164b) thus simplifies to

$$\det \overline{\Lambda}^{p,a}(t_a, t_b) = 2\,W \mp W\left[-\partial_{t_b}\Delta(t_b, t_a) + \partial_{t_a}\Delta(t_b, t_a)\right] \tag{3.164g}$$

Setting

$$\begin{aligned}\overline{\Delta}^{p,a}(t_a, t_b) &= 2 \pm \left[\partial_{t_b}\Delta(t_b, t_a) - \partial_{t_a}\Delta(t_b, t_a)\right] \\ &= 2 \pm \left[\partial_{t_b}\Delta(t_b, t_a) + \partial_{t_a}\Delta(t_a, t_b)\right] \qquad [\,\Delta(t, t') = -\Delta(t', t)\,]\end{aligned} \tag{3.165}$$

(3.164g) & hence (3.164a) become

$$\det \overline{\Lambda}^{p,a}(t_a, t_b) = W\,\overline{\Delta}^{p,a}(t_a, t_b) \neq 0 \tag{3.164}$$

We now turn to the evaluation of (3.159). To begin, we write it as

$$G^{p,a}_{\Omega^2}(t, t') = \overline{\Theta}(t - t')\,\Delta(t, t') + \frac{\mathcal{I}}{W\,\overline{\Delta}^{p,a}(t_a, t_b)} \tag{a}$$

where [see (3.163b-c)]

$$\begin{aligned}\mathcal{I} &= [\,a(t')\,\xi(t) + b(t')\,\eta(t)\,]\det \overline{\Lambda}^{p,a}(t_a, t_b) \\ &= A\,\Delta(t_b, t') + B\,\partial_{t_b}\Delta(t_b, t')\end{aligned} \tag{b}$$

where

$$\begin{aligned}A &= -[\dot\eta(t_b) \mp \dot\eta(t_a)]\,\xi(t) + \left[\dot\xi(t_b) \mp \dot\xi(t_a)\right]\eta(t) \\ &= W\left[\partial_{t_b}\Delta(t_b, t) \pm \partial_{t_a}\Delta(t, t_a)\right]\end{aligned} \tag{c}$$

$$\begin{aligned}B &= [\eta(t_b) \mp \eta(t_a)]\,\xi(t) - [\xi(t_b) \mp \xi(t_a)]\,\eta(t) \\ &= W\left[-\Delta(t_b, t) \mp \Delta(t, t_a)\right]\end{aligned} \tag{d}$$

$$\rightarrow \quad \mathcal{I} = W\left\{\left[\partial_{t_b}\Delta(t_b, t) \pm \partial_{t_a}\Delta(t, t_a)\right]\Delta(t_b, t') + \left[-\Delta(t_b, t) \mp \Delta(t, t_a)\right]\partial_{t_b}\Delta(t_b, t')\right\}$$

Using [see (3.51-2)]

$$\partial_{t_b}\Delta(t_b, t') = \frac{\Delta(t_b, t')\,\partial_{t_b}\Delta(t_b, t_a) - \Delta(t', t_a)}{\Delta(t_b, t_a)}$$

$$\partial_{t_a}\Delta(t,\,t_a) = \frac{\partial_{t_a}\Delta(t_b,\,t_a)\,\Delta(t,\,t_a) + \Delta(t_b,\,t)}{\Delta(t_b,\,t_a)}$$

we have

$$\begin{aligned}
\mathcal{I} &= \frac{W}{\Delta(t_b,\,t_a)}\Big\{\Big[\Delta(t_b,\,t)\,\partial_{t_b}\Delta(t_b,\,t_a) - \Delta(t,\,t_a) \pm \partial_{t_a}\Delta(t_b,\,t_a)\,\Delta(t,\,t_a) \pm \Delta(t_b,\,t)\Big]\Delta(t_b,\,t') \\
&\qquad + \Big[-\Delta(t_b,\,t) \mp \Delta(t,\,t_a)\Big][\Delta(t_b,\,t')\,\partial_{t_b}\Delta(t_b,\,t_a) - \Delta(t',\,t_a)]\Big\} \\
&= \frac{W}{\Delta(t_b,\,t_a)}\Big\{\Big[-\Delta(t,\,t_a) \pm \Delta(t_b,\,t)\Big]\Delta(t_b,\,t') - \Big[-\Delta(t_b,\,t) \mp \Delta(t,\,t_a)\Big]\Delta(t',\,t_a) \\
&\qquad \pm \Delta(t,\,t_a)\,\Delta(t_b,\,t')\Big[\partial_{t_a}\Delta(t_b,\,t_a) - \partial_{t_b}\Delta(t_b,\,t_a)\Big]\Big\} \\
&= \frac{W}{\Delta(t_b,\,t_a)}\Big\{\Big[\Delta(t,\,t_a) \pm \Delta(t_b,\,t)\Big]\Big[\Delta(t_b,\,t') \pm \Delta(t',\,t_a)\Big] \\
&\qquad - 2\,\Delta(t,\,t_a)\,\Delta(t_b,\,t') \pm \Delta(t,\,t_a)\,\Delta(t_b,\,t')\Big[\partial_{t_a}\Delta(t_b,\,t_a) - \partial_{t_b}\Delta(t_b,\,t_a)\Big]\Big\} \\
&= \frac{W}{\Delta(t_b,\,t_a)}\Big\{\Big[\Delta(t,\,t_a) \pm \Delta(t_b,\,t)\Big]\Big[\Delta(t_b,\,t') \pm \Delta(t',\,t_a)\Big] \qquad \text{[(3.165) used.]} \\
&\qquad - \Delta(t,\,t_a)\,\Delta(t_b,\,t')\,\overline{\Delta}^{p,a}(t_a,\,t_b)\Big\}
\end{aligned}$$

(a) thus becomes

$$\begin{aligned}
G^{p,a}_{\Omega^2}(t,\,t') &= \overline{\Theta}(t-t')\,\Delta(t,\,t') - \frac{\Delta(t,\,t_a)\,\Delta(t_b,\,t')}{\Delta(t_b,\,t_a)} + \frac{\Big[\Delta(t,\,t_a) \pm \Delta(t_b,\,t)\Big]\Big[\Delta(t_b,\,t') \pm \Delta(t',\,t_a)\Big]}{\Delta(t_b,\,t_a)\,\overline{\Delta}^{p,a}(t_a,\,t_b)} \\
&= G_{\Omega^2}(t,\,t') + \frac{\Big[\Delta(t,\,t_a) \pm \Delta(t_b,\,t)\Big]\Big[\Delta(t_b,\,t') \pm \Delta(t',\,t_a)\Big]}{\Delta(t_b,\,t_a)\,\overline{\Delta}^{p,a}(t_a,\,t_b)} \qquad (3.166)
\end{aligned}$$

where the equation for $G_{\Omega^2}(t,\,t')$ just below (3.58a) in §3.2 was used.