3.6. Time Evolution Amplitude in Presence of Source Term

Using the causal version of (3.145) on (3.21), we get the fluctuation action

$$\tilde{\mathcal{A}}_{j,\,\mathrm{fl}} = -\frac{1}{2M}\int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\, j(t)\, G_{\omega^2}(t,\, t')\, j(t')$$
$$= -\frac{1}{2M\omega \sin\omega\,(t_b - t_a)}\int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\, j(t) \sin\omega\,(t_b - t_>) \sin\omega\,(t_< - t_a)\, j(t')$$
$$= -\frac{1}{2M\omega \sin\omega\,(t_b - t_a)}\left\{\int_{t_a}^{t_b} dt \int_{t_a}^{t} dt'\, j(t) \sin\omega\,(t_b - t) \sin\omega\,(t' - t_a)\, j(t')\right. \tag{3.167a}$$
$$\left. + \int_{t_a}^{t_b} dt \int_{t}^{t_b} dt'\, j(t) \sin\omega\,(t_b - t') \sin\omega\,(t - t_a)\, j(t')\right\}$$

Applying $t \leftrightarrow t'$, the 2nd integral can be written as

$$\int_{t_a}^{t_b} dt' \int_{t'}^{t_b} dt\, j(t) \sin\omega\,(t_b - t) \sin\omega\,(t' - t_a)\, j(t')$$

Since both $\int_{t_a}^{t_b} dt \int_{t_a}^{t} dt'$ & $\int_{t_a}^{t_b} dt' \int_{t'}^{t_b} dt$ represent an integration over the triangular area below the diagonal line $t = t'$ in the 1st quadrant of the t-t' plane with the origin placed at $(t_a,\, t_a)$, the two integrals in (3.167a) are the same. Therefore,

$$\tilde{\mathcal{A}}_{j,\,\mathrm{fl}} = -\frac{1}{M\omega \sin\omega\,(t_b - t_a)}\int_{t_a}^{t_b} dt \int_{t_a}^{t} dt'\, j(t) \sin\omega\,(t_b - t) \sin\omega\,(t' - t_a)\, j(t') \tag{3.167}$$

The path integral for the evolution amplitude in the presence of an external source $j(t)$ is [see (3.4)]

$$(x_b\, t_b \mid x_a\, t_a)^j_\omega = \int \mathcal{D}x \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt \left[\frac{1}{2} M\left(\dot{x}^2 - \omega^2 x^2\right) + j\, x\right]\right\}$$
$$= e^{\frac{i}{\hbar}\mathcal{A}^j_{\mathrm{cl}}}\, F^j_\omega(t_b,\, t_a) \tag{3.168}$$

where [see (3.10-11)]

$$\mathcal{A}^j_{\mathrm{cl}} = \mathcal{A}_{\omega,\mathrm{cl}} + \mathcal{A}_{j,\mathrm{cl}}$$
$$= \frac{M\omega}{2 \sin\omega(t_b - t_a)}\left[\left(x_b^2 + x_a^2\right)\cos\omega(t_b - t_a) - 2\, x_b\, x_a\right] \tag{3.169}$$
$$+ \frac{1}{\sin\omega(t_b - t_a)}\int_{t_a}^{t_b} dt \left[x_b \sin\omega(t - t_a) + x_a \sin\omega(t_b - t)\right] j(t)$$

and [see (3.20-a) & (3.21)],

$$F^j_\omega(t_b,\, t_a) = F_\omega(t_b - t_a)\, \tilde{F}_j$$
$$= \sqrt{\frac{M}{2\pi i\hbar}}\sqrt{\frac{\omega}{\sin\omega(t_b - t_a)}}\exp\left(\frac{i}{\hbar}\tilde{\mathcal{A}}_{j,\,\mathrm{fl}}\right)$$
$$= \sqrt{\frac{M}{2\pi i\hbar}}\sqrt{\frac{\omega}{\sin\omega(t_b - t_a)}}\exp\left(-\frac{i}{\hbar}\frac{1}{M\omega \sin\omega\,(t_b - t_a)}\right. \tag{3.170}$$
$$\left. \times \int_{t_a}^{t_b} dt \int_{t_a}^{t} dt'\, j(t) \sin\omega\,(t_b - t) \sin\omega\,(t' - t_a)\, j(t')\right)$$

According to (3.59), generalization to the time-dependent frequency case is easily accomplished by replacing the wave functions $\dfrac{\sin\omega\,(t - t')}{\omega}$ with $\Delta(t,\, t')$ [see (3.161)]. Alternatively, one can use [see

(3.60)]

$$D_b(t) = \Delta(t_b, t) \qquad D_a(t) = \Delta(t, t_a)$$

to do so in terms of $D_a(t)$ & $D_b(t)$. Note that

$$D_b(t_a) = D_a(t_b) = \Delta(t_b, t_a)$$

For example, (3.169) generalizes to

$$\mathcal{A}^j_{\text{cl}} = \frac{M}{2\,\Delta(t_b, t_a)}\Big[x_b^2\,\partial_{t_b}\Delta(t_b, t_a) - x_a^2\,\partial_{t_a}\Delta(t_b, t_a) - 2\,x_b\,x_a \Big] + \frac{1}{\Delta(t_b, t_a)}\int_{t_a}^{t_b} d\,t\Big[x_b\,\Delta(t, t_a) + x_a\,\Delta(t_b, t)\Big]\,j(t)$$

where $\cos\omega(t_b - t_a)$ was expressed in two equivalent ways to make things more symmetrical.

Using (3.60), we have

$$\mathcal{A}^j_{\text{cl}} = \frac{M}{2\,D_b(t_a)}\Big[x_b^2\,\dot{D}_a(t_b) - x_a^2\,\dot{D}_b(t_a) - 2\,x_b\,x_a \Big] + \frac{1}{D_b(t_a)}\int_{t_a}^{t_b} d\,t\Big[x_b\,D_a(t) + x_a\,D_b(t)\Big]\,j(t) \qquad (3.171)$$

Similarly, using the form (3.167a), (3.170) is promoted to

$$F^j_\Omega(t_b, t_a) = \sqrt{\frac{M}{2\,\pi\,i\,\hbar}}\;\frac{1}{\sqrt{\Delta(t_b, t_a)}}\exp\!\Bigg(-\frac{i}{\hbar}\,\frac{1}{2\,M\,\Delta(t_b, t_a)}\int_{t_a}^{t_b} d\,t\int_{t_a}^{t_b} d\,t' \times j(t)\Big[\,\overline{\Theta}(t - t')\,\Delta(t_b, t)\,\Delta(t', t_a) + \overline{\Theta}(t' - t)\,\Delta(t_b, t')\,\Delta(t, t_a)\Big]\,j(t')\Bigg)$$

$$= \sqrt{\frac{M}{2\,\pi\,i\,\hbar}}\;\frac{1}{\sqrt{D_b(t_a)}}\exp\!\Bigg(-\frac{i}{\hbar}\,\frac{1}{2\,M\,D_b(t_a)}\int_{t_a}^{t_b} d\,t\int_{t_a}^{t_b} d\,t' \times j(t)\Big[\,\overline{\Theta}(t - t')\,D_b(t)\,D_a(t',) + \overline{\Theta}(t' - t)\,D_b(t')\,D_a(t)\Big]\,j(t')\Bigg) \qquad (3.172)$$

For applications to statistical mechanics, it is easier to work with the Fourier transforms

$$A(\omega) = \frac{1}{M\,\omega}\int_{t_a}^{t_b} d\,t\,e^{-i\,\omega\,(t - t_a)}\,j(t) \qquad (3.173)$$

$$B(\omega) = \frac{1}{M\,\omega}\int_{t_a}^{t_b} d\,t\,e^{-i\,\omega\,(t_b - t)}\,j(t) = \frac{1}{M\,\omega}\int_{t_a}^{t_b} d\,t\,e^{-i\,\omega\,(t_b - t_a)}\,e^{i\,\omega\,(t - t_a)}\,j(t) = -e^{-i\,\omega\,(t_b - t_a)}\,A(-\omega) \qquad (3.174)$$

The source term $\mathcal{A}_{j,\text{cl}}$ in (3.169) then becomes

$$\mathcal{A}_{j,\text{cl}} = \frac{1}{\sin\omega(t_b - t_a)}\int_{t_a}^{t_b} d\,t\Big[x_b\sin\omega(t - t_a) + x_a\sin\omega(t_b - t)\Big]\,j(t)$$

$$= \frac{1}{\sin\omega(t_b - t_a)}\int_{t_a}^{t_b} d\,t\Bigg[x_b\,\frac{e^{i\,\omega\,(t - t_a)} - e^{-i\,\omega\,(t - t_a)}}{2\,i} + x_a\,\frac{e^{i\,\omega\,(t_b - t)} - e^{-i\,\omega\,(t_b - t)}}{2\,i}\Bigg]\,j(t)$$

$$= \frac{M\,\omega}{2\,i\sin\omega(t_b - t_a)}\Big\{x_b\Big[-A(-\omega) - A(\omega)\Big] + x_a\Big[-B(-\omega) - B(\omega)\Big]\Big\}$$

$$= \frac{M\,\omega}{2\,i\sin\omega(t_b - t_a)}\Big\{x_b\Big[e^{i\,\omega\,(t_b - t_a)}\,B(\omega) - A(\omega)\Big] + x_a\Big[e^{i\,\omega\,(t_b - t_a)}\,A(\omega) - B(\omega)\Big]\Big\} \qquad (3.175)$$

Similarly, the Green's function used in (3.167a) can be written as

$$\mathcal{G} = G_{\omega^2}(t,\, t')\, \omega \sin\omega\,(t_b - t_a)$$

$$= \overline{\Theta}(t - t') \sin\omega\,(t_b - t) \sin\omega\,(t' - t_a) + \overline{\Theta}(t' - t) \sin\omega\,(t_b - t') \sin\omega\,(t - t_a)$$

$$= \overline{\Theta}(t - t') \frac{e^{i\omega(t_b - t)} - e^{-i\omega(t_b - t)}}{2i} \frac{e^{i\omega(t' - t_a)} - e^{-i\omega(t' - t_a)}}{2i}$$

$$+ \overline{\Theta}(t' - t) \frac{e^{i\omega(t_b - t')} - e^{-i\omega(t_b - t')}}{2i} \frac{e^{i\omega(t - t_a)} - e^{-i\omega(t - t_a)}}{2i}$$

$$= -\frac{1}{4}\Big\{ \overline{\Theta}(t - t') \big[e^{i\omega(t_b - t_a - t + t')} + e^{-i\omega(t_b - t_a - t + t')} - e^{i\omega(t_b + t_a - t - t')} - e^{-i\omega(t_b + t_a - t - t')} \big]$$

$$+ \overline{\Theta}(t' - t) \big[e^{i\omega(t_b - t_a - t' + t)} + e^{-i\omega(t_b - t_a - t' + t)} - e^{i\omega(t_b + t_a - t - t')} - e^{-i\omega(t_b + t_a - t - t')} \big] \Big\}$$

$$= -\frac{1}{4}\Big\{ \overline{\Theta}(t - t') \Big[\big(e^{i\omega(t_b - t_a)} e^{-i\omega(t - t')} + c.c.\big) - \big(e^{i\omega(t_b + t_a)} e^{-i\omega(t + t')} + c.c.\big) \Big] \qquad (3.176)$$

$$+ (t \leftrightarrow t') \Big\}$$

Using

$$\overline{\Theta}(t - t') + \overline{\Theta}(t' - t) = 1$$

the $t + t'$ terms can be combined so that (1.176) becomes

$$\mathcal{G} = \frac{1}{4}\big(e^{i\omega(t_b + t_a)} e^{-i\omega(t + t')} + c.c.\big)$$

$$-\frac{1}{4}\Big\{ \overline{\Theta}(t - t') \Big(e^{i\omega(t_b - t_a)} e^{-i\omega(t - t')} + c.c.\Big) + (t \leftrightarrow t') \Big\}$$

$$= \frac{1}{4}\big(e^{i\omega(t_b + t_a)} e^{-i\omega(t + t')} + c.c.\big)$$

$$-\frac{1}{4}\Big\{ e^{i\omega(t_b - t_a)} \Big[\overline{\Theta}(t - t')\, e^{-i\omega(t - t')} + \overline{\Theta}(t' - t)\, e^{-i\omega(t' - t)} \Big]$$

$$+ e^{-i\omega(t_b - t_a)} \Big[\overline{\Theta}(t - t')\, e^{i\omega(t - t')} + \overline{\Theta}(t' - t)\, e^{i\omega(t' - t)} \Big] \Big\}$$

$$= \frac{1}{4}\big(e^{i\omega(t_b + t_a)} e^{-i\omega(t + t')} + c.c.\big) \qquad (3.178)$$

$$-\frac{1}{4}\Big\{ e^{i\omega(t_b - t_a)} \Big[\overline{\Theta}(t - t')\, e^{-i\omega(t - t')} + \overline{\Theta}(t' - t)\, e^{-i\omega(t' - t)} \Big]$$

$$+ e^{-i\omega(t_b - t_a)} \Big[\big[1 - \overline{\Theta}(t' - t)\big] e^{i\omega(t - t')} + \big[1 - \overline{\Theta}(t - t')\big] e^{i\omega(t' - t)} \Big] \Big\}$$

$$= \frac{1}{4}\Big(e^{i\omega(t_b + t_a)} e^{-i\omega(t + t')} + c.c.\Big) - \frac{1}{4} e^{-i\omega(t_b - t_a)} \Big(e^{i\omega(t - t')} + e^{i\omega(t' - t)}\Big) \qquad (3.178a)$$

$$-\frac{1}{4}\big(e^{i\omega(t_b - t_a)} e^{-i\omega|t - t'|} - e^{-i\omega(t_b - t_a)} e^{i\omega|t - t'|} \big)$$

where we've used

$$\overline{\Theta}(t - t')\, e^{-i\omega(t - t')} + \overline{\Theta}(t' - t)\, e^{i\omega(t - t')} = e^{-i\omega|t - t'|}$$

Putting (3.178a) into (3.167a) , we have

$$\tilde{\mathcal{A}}_{j,\,\mathrm{fl}} = -\frac{1}{8\,M\,\omega \sin\omega\,(t_b - t_a)} \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\, j(t)$$
$$\times\Big\{ e^{i\,\omega\,(t_b + t_a)}\, e^{-i\,\omega\,(t + t')} + e^{-i\,\omega\,(t_b + t_a)}\, e^{i\,\omega\,(t + t')} - e^{-i\,\omega\,(t_b - t_a)} \left(e^{i\,\omega\,(t - t')} + e^{i\,\omega\,(t' - t)}\right)$$
$$-\left(e^{i\,\omega\,(t_b - t_a)}\, e^{-i\,\omega\,|\,t - t'\,|} - e^{-i\,\omega\,(t_b - t_a)}\, e^{i\,\omega\,|\,t - t'\,|} \right)\Big\}\, j(t')$$

With the help of (1.173-4), this becomes

$$\tilde{\mathcal{A}}_{j,\,\mathrm{fl}} = -\frac{M\,\omega}{8 \sin\omega\,(t_b - t_a)} \Big\{ -B(-\omega)\,A(\omega) - B(\omega)\,A(-\omega) - 2\,B(\omega)\,A(\omega) \Big]$$
$$+\frac{1}{M^2\,\omega^2} \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\, j(t) \left(e^{i\,\omega\,(t_b - t_a)}\, e^{-i\,\omega\,|\,t - t'\,|} - e^{-i\,\omega\,(t_b - t_a)}\, e^{i\,\omega\,|\,t - t'\,|} \right) j(t') \Big\}$$
$$= -\frac{M\,\omega}{8 \sin\omega\,(t_b - t_a)} \Big\{ e^{i\,\omega\,(t_b - t_a)} \left[A(\omega)^2 + B(\omega)^2 \right] - 2\,B(\omega)\,A(\omega) \qquad (3.179)$$
$$+\frac{1}{M^2\,\omega^2} \left(e^{i\,\omega\,(t_b - t_a)} - e^{-i\,\omega\,(t_b - t_a)} \right) \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\, j(t)\, e^{-i\,\omega\,|\,t - t'\,|}\, j(t') \Big\}$$
$$= -\frac{M\,\omega}{8 \sin\omega\,(t_b - t_a)} \Big\{ e^{i\,\omega\,(t_b - t_a)} \left[A(\omega)^2 + B(\omega)^2 \right] - 2\,B(\omega)\,A(\omega) \Big\} \qquad (3.176)$$
$$+\frac{i}{4\,M\,\omega} \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\, j(t)\, e^{-i\,\omega\,|\,t - t'\,|}\, j(t')$$

Time-Independent j

For a time-independent source $j(t) = j$, (3.173-4) can be evaluated immediately:

$$A(\omega) = \frac{j}{M\,\omega} \int_{t_a}^{t_b} d\,t\, e^{-i\,\omega\,(t - t_a)} = i\,\frac{j}{M\,\omega^2} \left(e^{-i\,\omega\,(t_b - t_a)} - 1\right) \qquad (3.180a)$$
$$B(\omega) = \frac{j}{M\,\omega} \int_{t_a}^{t_b} d\,t\, e^{-i\,\omega\,(t_b - t)} = i\,\frac{j}{M\,\omega^2} \left(e^{-i\,\omega\,(t_b - t_a)} - 1\right) = A(\omega) \qquad (3.180b)$$

(3.175) thus simplifies to

$$\mathcal{A}_{j,\mathrm{cl}} = j\,\frac{e^{-i\,\omega\,(t_b - t_a)} - 1}{2\,\omega \sin\omega(t_b - t_a)} \Big\{ x_b \left[e^{i\,\omega\,(t_b - t_a)} - 1 \right] + x_a \left[e^{i\,\omega\,(t_b - t_a)} - 1 \right] \Big\}$$
$$= j\,\frac{1 - \cos\omega(t_b - t_a)}{\omega \sin\omega(t_b - t_a)} (x_b + x_a) \qquad (3.180c)$$
$$= j\,\frac{2 \sin^2[\omega\,(t_b - t_a)/2]}{\omega \sin\omega(t_b - t_a)} (x_b + x_a) = j\,\frac{\sin[\omega\,(t_b - t_a)/2]}{\omega \cos[\omega\,(t_b - t_a)/2]} (x_b + x_a)$$
$$= \frac{j}{\omega} \tan\left[\frac{\omega}{2}\,(t_b - t_a)\right] (x_b + x_a) \qquad (3.180d)$$

Similarly, (3.176) becomes

$$\tilde{\mathcal{A}}_{j,\,\mathrm{fl}} = j^2\,\frac{\left(e^{-i\,\omega\,(t_b - t_a)} - 1\right)^2}{4\,M\,\omega^3 \sin\omega\,(t_b - t_a)} \left(e^{i\,\omega\,(t_b - t_a)} - 1\right) + \frac{i}{4\,M\,\omega} \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\, j(t)\, e^{-i\,\omega\,|\,t - t'\,|}\, j(t')$$
$$= j^2\,\frac{e^{-i\,\omega\,(t_b - t_a)} - 1}{2\,M\,\omega^3} \tan\left[\frac{\omega}{2}\,(t_b - t_a)\right] + \frac{i}{4\,M\,\omega} \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\, j(t)\, e^{-i\,\omega\,|\,t - t'\,|}\, j(t')$$

The integrals can be evaluated immediately:

$$\int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\, j(t)\, e^{-i\,\omega\,|\,t - t'\,|}\, j(t') = j^2 \int_{t_a}^{t_b} d\,t \int_{t_a}^{t_b} d\,t'\, e^{-i\,\omega\,|\,t - t'\,|}$$

$$= j^2 \int_{t_a}^{t_b} d\,t \left[\int_{t_a}^{t} d\,t'\, e^{-i\,\omega\,(t-t')} + \int_{t}^{t_b} d\,t'\, e^{i\,\omega\,(t-t')} \right]$$
$$= j^2 \int_{t_a}^{t_b} d\,t \left[-\frac{i}{\omega} \left(1 - e^{-i\,\omega\,(t-t_a)}\right) + \frac{i}{\omega} \left(e^{i\,\omega\,(t-t_b)} - 1\right) \right]$$
$$= -\frac{i}{\omega} j^2 \int_{t_a}^{t_b} d\,t \left[2 - e^{-i\,\omega\,(t-t_a)} - e^{i\,\omega\,(t-t_b)} \right]$$
$$= -\frac{i}{\omega} j^2 \left[2\,(t_b - t_a) - \frac{i}{\omega} \left(e^{-i\,\omega\,(t_b-t_a)} - 1\right) + \frac{i}{\omega} \left(1 - e^{i\,\omega\,(t_a-t_b)}\right) \right]$$
$$= -2\,\frac{i}{\omega} j^2 \left[t_b - t_a - \frac{i}{\omega} \left(e^{-i\,\omega\,(t_b-t_a)} - 1\right) \right]$$

so that

$$\tilde{\mathcal{A}}_{j,\,\mathrm{fl}} = j^2\, \frac{e^{-i\,\omega\,(t_b-t_a)} - 1}{2\,M\,\omega^3} \tan\!\left[\frac{\omega}{2}\,(t_b - t_a)\right] + \frac{j^2}{2\,M\,\omega^2} \left[t_b - t_a - \frac{i}{\omega} \left(e^{-i\,\omega\,(t_b-t_a)} - 1\right) \right]$$
$$= \frac{j^2}{2\,M\,\omega^3} \left\{ \omega\,(t_b - t_a) + \left(e^{-i\,\omega\,(t_b-t_a)} - 1\right) \left(-i + \tan\!\left[\frac{\omega}{2}\,(t_b - t_a)\right] \right) \right\} \qquad (3.180e)$$

Using

$$\left(e^{-i\,\theta} - 1\right)\left(-i + \tan\frac{\theta}{2}\right) = \left(e^{-i\,\theta} - 1\right)\left(-i + \frac{e^{i\,\theta/2} - e^{-i\,\theta/2}}{i\left(e^{i\,\theta/2} + e^{-i\,\theta/2}\right)}\right)$$
$$= \left(e^{-i\,\theta} - 1\right) \frac{2\,e^{i\,\theta/2}}{i\left(e^{i\,\theta/2} + e^{-i\,\theta/2}\right)}$$
$$= \frac{2\left(e^{-i\,\theta/2} - e^{i\,\theta/2}\right)}{i\left(e^{i\,\theta/2} + e^{-i\,\theta/2}\right)}$$
$$= -2\tan\frac{\theta}{2}$$

(3.180e) becomes

$$\tilde{\mathcal{A}}_{j,\,\mathrm{fl}} = \frac{j^2}{2\,M\,\omega^3} \left\{ \omega\,(t_b - t_a) - 2\tan\!\left[\frac{\omega}{2}\,(t_b - t_a)\right] \right\} \qquad (3.180f)$$

Thus,

$$\mathcal{A}_j = \mathcal{A}_{j,\mathrm{cl}} + \tilde{\mathcal{A}}_{j,\,\mathrm{fl}}$$
$$= \frac{j}{\omega} \tan\!\left[\frac{\omega}{2}\,(t_b - t_a)\right](x_b + x_a) + \frac{j^2}{2\,M\,\omega^3} \left\{ \omega\,(t_b - t_a) - 2\tan\!\left[\frac{\omega}{2}\,(t_b - t_a)\right] \right\} \qquad (3.181)$$

This result can also be obtained more directly from the action

$$\mathcal{A} = \int_{t_a}^{t_b} d\,t \left(\frac{1}{2} M\,\dot{x}^2 - \frac{1}{2} M\,\omega^2\,x^2 + j\,x \right) \qquad (3.182)$$
$$= \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} M\,\dot{x}^2 - \frac{1}{2} M\,\omega^2 \left(x - \frac{j}{M\,\omega^2} \right)^2 + \frac{j^2}{2\,M\,\omega^2} \right]$$
$$= \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} M\,\dot{x}^2 - \frac{1}{2} M\,\omega^2 \left(x - \frac{j}{M\,\omega^2} \right)^2 \right] + \frac{j^2}{2\,M\,\omega^2}\,(t_b - t_a) \qquad (3.183)$$

which corresponds to a source free harmonic oscillator centered at $x_0 = \frac{j}{M\,\omega^2}$ with an energy shift $\frac{j^2}{2\,M\,\omega^2}$. The evolution amplitude is therefore

$$(x_b\, t_b \mid x_a\, t_a)_\omega^{j=\text{const}} = \sqrt{\frac{M\,\omega}{2\,\pi\, i\,\hbar \sin\omega(t_b - t_a)}}\ \exp\!\left(\frac{i}{\hbar}\, \frac{j^2}{2\,M\,\omega^2}\,(t_b - t_a) \right. \tag{3.184}$$

$$+ \frac{i}{\hbar}\, \frac{M\,\omega}{2 \sin\omega(t_b - t_a)} \left\{ \left[\left(x_b - \frac{j}{M\,\omega^2} \right)^2 + \left(x_a - \frac{j}{M\,\omega^2} \right)^2 \right] \cos\omega\,(t_b - t_a) \right.$$

$$\left.\left. - 2 \left(x_b - \frac{j}{M\,\omega^2} \right) \left(x_a - \frac{j}{M\,\omega^2} \right) \right\} \right)$$

In the free particle $\omega \to 0$ limit, we can expand $\omega \cot \omega\,(t_b - t_a)$ & $\dfrac{\omega}{2 \sin\omega(t_b - t_a)}$ to the 4th power and get [see “3.06._Code.nb”]

$$(x_b\, t_b \mid x_a\, t_a)_0^{j=\text{const}} = \sqrt{\frac{M}{2\,\pi\, i\,\hbar\,(t_b - t_a)}}\ \exp\!\left(\frac{i}{\hbar} \left\{ \frac{1}{2}\, M\, \frac{(x_b - x_a)^2}{t_b - t_a} \right.\right. \tag{3.185}$$

$$\left.\left. + \frac{1}{2}\, j\,(x_b + x_a)\,(t_b - t_a) - \frac{j^2}{24\, M^2}\,(t_b - t_a)^4 \right\} \right)$$

As a cross check, we consider the free particle classical action

$$\mathcal{A}_{j,\text{cl}} = \int_{t_a}^{t_b} d\,t \left(\frac{1}{2}\, M\, \dot{x}_{j,\text{cl}}^2 + j\, x_{j,\text{cl}} \right) \tag{3.186}$$

which, by (3.168), should equal to the exponent in (3.185).

The Euler-Lagrange eq. for (3.186) is simply

$$M\, \ddot{x}_{j,\text{cl}} = j \tag{3.187}$$

$$\rightarrow \qquad \dot{x}_{j,\text{cl}} = \frac{j}{M}\,(t - t_a) + C_1$$

$$x_{j,\text{cl}} = \frac{j}{2\,M}\,(t - t_a)^2 + C_1 (t - t_a) + C_2 \tag{3.187a}$$

Inserting the B.C.

$$x_{j,\text{cl}}(t_b) = x_b \qquad\qquad x_{j,\text{cl}}(t_a) = x_a$$

we get

$$\frac{j}{2\,M}\,(t_b - t_a)^2 + C_1\,(t_b - t_a) + C_2 = x_b$$

$$C_2 = x_a$$

which gives

$$C_1 = \frac{1}{t_b - t_a} \left[x_b - x_a - \frac{j}{2\,M}\,(t_b - t_a)^2 \right]$$

so that

$$x_{j,\text{cl}} = \frac{j}{2\,M}\,(t - t_a)^2 + \left[x_b - x_a - \frac{j}{2\,M}\,(t_b - t_a)^2 \right] \frac{t - t_a}{t_b - t_a} + x_a \tag{3.188}$$

$$\rightarrow \qquad \dot{x}_{j,\text{cl}} = \frac{j}{M}\,(t - t_a) + \frac{1}{t_b - t_a} \left[x_b - x_a - \frac{j}{2\,M}\,(t_b - t_a)^2 \right] \tag{3.188a}$$

(3.186) can be evaluated as

$$\mathcal{A}_{j,\text{cl}} = \frac{1}{2}\, M\, x_{j,\text{cl}}\, \dot{x}_{j,\text{cl}} \Big|_{t_a}^{t_b} + \int_{t_a}^{t_b} d\,t \left(-\frac{1}{2}\, M\, \ddot{x}_{j,\text{cl}} + j \right) x_{j,\text{cl}}$$

$$= \frac{1}{2} M x_{j,\text{cl}} \dot{x}_{j,\text{cl}} \Big|_{t_a}^{t_b} + \frac{1}{2} j \int_{t_a}^{t_b} d t \, x_{j,\text{cl}} \tag{3.188b}$$

With a little help from *Mathematica* [see “3.06._Code.nb”], we get

$$\mathcal{A}_{j,\text{cl}} = \frac{1}{2} M \frac{(x_b - x_a)^2}{t_b - t_a} + \frac{1}{2} (t_b - t_a)(x_b + x_a) j - \frac{1}{24 M} (t_b - t_a)^3 j^2 \tag{3.189}$$

which is simply the exponent in (3.185) as predicted.

$x_{j,\text{cl}}$

Finally, $(x_b t_b \mid x_a t_a)_\omega^j$ in (3.168) can also be calculated using

$$x(t) = x_{j,\text{cl}}(t) + \delta x(t) \tag{3.190}$$

where

$$\ddot{x}_{j,\text{cl}}(t) + \omega^2 x_{j,\text{cl}}(t) = \frac{j(t)}{M} \tag{3.191}$$

instead of

$$x(t) = x_{\text{cl}}(t) + \delta x(t)$$
$$\ddot{x}_{\text{cl}}(t) + \omega^2 x_{\text{cl}}(t) = 0 \tag{3.191a}$$

as was the case in obtaining (3.169).

The general solution to (3.191) is

$$x_{j,\text{cl}}(t) = c_1 \sin\omega(t - t_1) + c_2 \sin\omega(t - t_2) + \frac{1}{M} \int_{t_a}^{t_b} d t' \, G_{\omega^2}(t, t') \, j(t') \tag{3.192a}$$

where c_1, c_2, t_1, t_2 are arbitrary constants. Inserting the B.C.

$$x(t_b) = x_b \qquad\qquad x(t_a) = x_a$$

we get

$$x_{j,\text{cl}}(t) = \frac{x_b \sin\omega(t_b - t) + x_a \sin\omega(t - t_a)}{\sin\omega(t_b - t_a)} + \frac{1}{M} \int_{t_a}^{t_b} d t' \, G_{\omega^2}(t, t') \, j(t') \tag{3.192}$$

The Green function thus appears at the classical level instead of the quantum fluctuation $F_\omega^j(t_b, t_a)$ [see (3.170)].

The classical action in (3.168-9) now reads

$$\mathcal{A}_{\text{cl}}^j = \int_{t_a}^{t_b} d t \left[\frac{1}{2} M \left(\dot{x}_{j,\text{cl}}^2 - \omega^2 x_{j,\text{cl}}^2\right) + j \, x_{j,\text{cl}} \right]$$

$$= \frac{1}{2} M x_{j,\text{cl}} \dot{x}_{j,\text{cl}} \Big|_{t_a}^{t_b} + \int_{t_a}^{t_b} d t \left[\frac{1}{2} M \left(-\ddot{x}_{j,\text{cl}} x_{j,\text{cl}} - \omega^2 x_{j,\text{cl}}^2\right) + j \, x_{j,\text{cl}} \right] \qquad \text{[Integration by part.]}$$

$$= \frac{1}{2} M x_{j,\text{cl}} \dot{x}_{j,\text{cl}} \Big|_{t_a}^{t_b} + \frac{1}{2} \int_{t_a}^{t_b} d t \, j \, x_{j,\text{cl}} \qquad \text{[(3.191) used.]}$$

$$= \frac{1}{2} M (x_b \dot{x}_b - x_a \dot{x}_a) \Big|_{x = x_{j,\text{cl}}} + \frac{1}{2} \int_{t_a}^{t_b} d t \, j(t) \, x_{j,\text{cl}}(t) \tag{3.193}$$

Inserting $x_{j,\text{cl}}$ from (3.192) & $G_{\omega^2}(t, t')$ from (3.36) leads once more to (3.168).