3.7. Time Evolution Amplitude at Fixed Path Average

From the evolution amplitude for a constant source [see (3.168)]

$$(x_b\, t_b \mid x_a\, t_a)^j_\omega = \int \mathcal{D}x \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt \left[\frac{1}{2}M\left(\dot{x}^2 - \omega^2 x^2\right) + j\,x\right]\right\}$$

$$= e^{\frac{i}{\hbar}\mathcal{A}^j_{\rm cl}}\, F^j_\omega(t_b, t_a)$$

we construct its Fourier transform with respect to j as

$$(x_b\, t_b \mid x_a\, t_a)^{x_0}_\omega = \frac{t_b - t_a}{2\pi\hbar}\int_{-\infty}^{\infty} dj\, e^{-i\,j\,(t_b - t_a)\,x_0/\hbar}\,(x_b\, t_b \mid x_a\, t_a)^j_\omega \qquad (3.195a)$$

$$= \int \mathcal{D}x\, \frac{t_b - t_a}{2\pi\hbar}\int_{-\infty}^{\infty} dj \exp\left[-\frac{i}{\hbar}\, j\,(t_b - t_a)\left(x_0 - \frac{1}{t_b - t_a}\int_{t_a}^{t_b} dt\, x\right)\right]$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt\left[\frac{1}{2}M\left(\dot{x}^2 - \omega^2 x^2\right)\right]\right\}$$

$$= \int \mathcal{D}x\, \delta(x_0 - \overline{x}) \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt\left[\frac{1}{2}M\left(\dot{x}^2 - \omega^2 x^2\right)\right]\right\} \qquad (3.195)$$

where

$$\overline{x} = \frac{1}{t_b - t_a}\int_{t_a}^{t_b} dt\, x(t) \qquad (3.195b)$$

is the average position of the particle. $(x_b\, t_b \mid x_a\, t_a)^{x_0}_\omega$ is therefore a path integral over only paths with average position equal to x_0.

$(x_b\, t_b \mid x_a\, t_a)^j_\omega$ can be written as

$$(x_b\, t_b \mid x_a\, t_a)^j_\omega = (x_b\, t_b \mid x_a\, t_a)_\omega\, e^{i\mathcal{A}_j/\hbar}$$

where $(x_b\, t_b \mid x_a\, t_a)_\omega$ is the evolution amplitude for $j = 0$ and [see (3.181)]

$$\mathcal{A}_j = \frac{j}{\omega}\tan\left[\frac{\omega}{2}(t_b - t_a)\right](x_b + x_a) + \frac{j^2}{2M\omega^3}\left\{\omega\,(t_b - t_a) - 2\tan\left[\frac{\omega}{2}(t_b - t_a)\right]\Big)\right\} \qquad (3.196a)$$

Hence, (3.195a) can be written as

$$(x_b\, t_b \mid x_a\, t_a)^{x_0}_\omega = (x_b\, t_b \mid x_a\, t_a)_\omega\, \frac{t_b - t_a}{2\pi\hbar}\int_{-\infty}^{\infty} dj\, \exp\left\{\frac{i}{\hbar}\left[\mathcal{A}_j - j\,(t_b - t_a)\,x_0\right]\right\} \qquad (3.196b)$$

Using (3.196a), we have

$$\mathcal{A}_j - j\,(t_b - t_a)\,x_0 = \frac{j^2}{2M\omega^3}\left\{\omega\,(t_b - t_a) - 2\tan\left[\frac{\omega}{2}(t_b - t_a)\right]\Big)\right.$$

$$+ j\left\{\frac{1}{\omega}\tan\left[\frac{\omega}{2}(t_b - t_a)\right](x_b + x_a) - (t_b - t_a)\,x_0\right\}$$

$$= \frac{1}{2M\omega^3}\left\{\omega\,(t_b - t_a) - 2\tan\left[\frac{\omega}{2}(t_b - t_a)\right]\right\}$$

$$\times\left\{j^2 + 2M\omega^3\,\frac{\frac{1}{\omega}\tan\left[\frac{\omega}{2}(t_b - t_a)\right](x_b + x_a) - (t_b - t_a)\,x_0}{\omega\,(t_b - t_a) - 2\tan\left[\frac{\omega}{2}(t_b - t_a)\right]}\, j\right\}$$

$$= \frac{1}{2M\omega^3}\left\{\omega\,(t_b - t_a) - 2\tan\left[\frac{\omega}{2}(t_b - t_a)\right]\right\}\left[(j - j_0)^2 - j_0^2\right] \qquad (3.196c)$$

where

$$j_0 = M\,\omega^2 \,\frac{\tan\left[\frac{\omega}{2}(t_b-t_a)\right](x_b+x_a) - \omega\,(t_b-t_a)\,x_0}{\omega\,(t_b-t_a) - 2\tan\left[\frac{\omega}{2}(t_b-t_a)\right]} \tag{3.196d}$$

Setting

$$\mathcal{A}^{x_0} = -\frac{1}{2\,M\,\omega^3}\left\{\omega\,(t_b-t_a) - 2\tan\left[\frac{\omega}{2}(t_b-t_a)\right]\right\} j_0^2 \tag{3.196e}$$

(3.196c) becomes

$$\mathcal{A}_j - j\,(t_b-t_a)\,x_0 = \frac{1}{2\,M\,\omega^3}\left\{\omega\,(t_b-t_a) - 2\tan\left[\frac{\omega}{2}(t_b-t_a)\right]\right\}(j-j_0)^2 + \mathcal{A}^{x_0} \tag{3.196}$$

Using (3.196d), we can write (3.196e) as

$$\mathcal{A}^{x_0} = -\frac{M\,\omega\left\{\tan\left[\frac{\omega}{2}(t_b-t_a)\right](x_b+x_a) - \omega\,(t_b-t_a)\,x_0\right\}^2}{2\left\{\omega\,(t_b-t_a) - 2\tan\left[\frac{\omega}{2}(t_b-t_a)\right]\right\}} \tag{3.196f}$$

With the quadratic form (3.196), the integral in (3.196b) is a simple Gaussian which gives

$$\frac{t_b-t_a}{2\,\pi\,\hbar}\int_{-\infty}^{\infty} d\,j\;\exp\left\{\frac{i}{\hbar}\left[\mathcal{A}_j - j\,(t_b-t_a)\,x_0\right]\right\} = \frac{t_b-t_a}{2\,\pi\,\hbar}\sqrt{\frac{2\,M\,\omega^3\,\pi\,i\,\hbar}{\omega\,(t_b-t_a) - 2\tan\left[\frac{\omega}{2}(t_b-t_a)\right]}}$$

$$= \sqrt{\frac{i\,M\,\omega^3\,(t_b-t_a)^2/\,2\,\pi\,\hbar}{\omega\,(t_b-t_a) - 2\tan\left[\frac{\omega}{2}(t_b-t_a)\right]}}$$

so that

$$(x_b\,t_b \mid x_a\,t_a)_\omega^{x_0} = (x_b\,t_b \mid x_a\,t_a)_\omega \sqrt{\frac{i\,M\,\omega^3\,(t_b-t_a)^2/\,2\,\pi\,\hbar}{\omega\,(t_b-t_a) - 2\tan\left[\frac{\omega}{2}(t_b-t_a)\right]}}\;\exp\left(\frac{i}{\hbar}\mathcal{A}^{x_0}\right) \tag{3.197}$$

The quantum mechanical version of the partition function at x_0 is given by

$$Z_\omega^{x_0} = \int_{-\infty}^{\infty} d\,x\,(x\,t_b \mid x\,t_a)_\omega^{x_0} \tag{3.197a}$$

Using [see (2.173)]

$$(x\,t_b \mid x\,t_a)_\omega = \sqrt{\frac{M}{2\,\pi\,i\,\hbar}}\sqrt{\frac{\omega}{\sin\omega\,(t_b-t_a)}}\;\exp\left\{\frac{i}{\hbar}\frac{M\,\omega\,x^2}{\sin\omega(t_b-t_a)}\left[\cos\omega(t_b-t_a) - 1\right]\right\}$$

$$= \sqrt{\frac{M}{2\,\pi\,i\,\hbar}}\sqrt{\frac{\omega}{\sin\omega\,(t_b-t_a)}}\;\exp\left\{-\frac{i}{\hbar}M\,\omega\,x^2\tan\left[\frac{\omega}{2}(t_b-t_a)\right]\right\} \tag{3.197b}$$

and [see (3.196f)]

$$\mathcal{A}^{x_0}\Big|_{x_b=x_a=x} = -\frac{M\,\omega\left\{2\tan\left[\frac{\omega}{2}(t_b-t_a)\right]x - \omega\,(t_b-t_a)\,x_0\right\}^2}{2\left\{\omega\,(t_b-t_a) - 2\tan\left[\frac{\omega}{2}(t_b-t_a)\right]\right\}}$$

(3.197a) becomes

$$Z_\omega^{x_0} = \frac{M}{2\,\pi\,\hbar}\sqrt{\frac{\omega^4\,(t_b-t_a)^2}{\left\{\omega\,(t_b-t_a) - 2\tan\left[\frac{\omega}{2}(t_b-t_a)\right]\right\}\sin\omega\,(t_b-t_a)}}$$

$$\times\int_{-\infty}^{\infty} d\,x\,\exp\left(-\frac{i}{\hbar}M\,\omega\left\{x^2\tan\left[\frac{\omega}{2}(t_b-t_a)\right] + \frac{\left\{2\tan\left[\frac{\omega}{2}(t_b-t_a)\right]x - \omega\,(t_b-t_a)\,x_0\right\}^2}{2\left\{\omega\,(t_b-t_a) - 2\tan\left[\frac{\omega}{2}(t_b-t_a)\right]\right\}}\right\}\right)$$

The Gaussian integral is evaluated in “3.07._Code.nb”, giving

$$Z_\omega^{x_0} = \frac{M}{2\pi\hbar}\sqrt{\frac{\omega^4 (t_b - t_a)^2 \hbar \pi}{i M \omega^2 (t_b - t_a) \tan\left[\frac{\omega}{2} (t_b - t_a)\right] \sin\omega (t_b - t_a)}}\, \exp\left[-\frac{i}{2\hbar} M \omega^2 (t_b - t_a) x_0^2\right]$$

$$= \sqrt{\frac{M \omega^2 (t_b - t_a)}{8 \pi i \hbar \sin^2\left[\frac{\omega}{2} (t_b - t_a)\right]}}\, \exp\left[-\frac{i}{2\hbar} M \omega^2 (t_b - t_a) x_0^2\right]$$

$$= \sqrt{\frac{M}{2\pi i \hbar (t_b - t_a)}}\, \frac{\omega (t_b - t_a)/2}{\sin\left[\frac{\omega}{2} (t_b - t_a)\right]} \exp\left[-\frac{i}{2\hbar} M \omega^2 (t_b - t_a) x_0^2\right] \tag{3.198}$$

As a check, we integrate over x_0 to get

$$Z_\omega = \int_{-\infty}^{\infty} d x_0\, Z_\omega^{x_0} = \sqrt{\frac{M}{2\pi i \hbar (t_b - t_a)}}\, \frac{\omega (t_b - t_a)/2}{\sin\left[\frac{\omega}{2} (t_b - t_a)\right]} \sqrt{\frac{2\hbar\pi}{i M \omega^2 (t_b - t_a)}}$$

$$= \frac{1}{2 i \sin\left[\frac{\omega}{2} (t_b - t_a)\right]} \tag{3.198a}$$

in agreement with (2.410).

The open version is given by

$$Z_\omega^{\text{open}, x_0} = \int_{-\infty}^{\infty} d x_b \int_{-\infty}^{\infty} d x_a\, (x_b t_b \mid x_a t_a)_\omega^{x_0} \tag{3.199a}$$

(3.197b) is replaced by

$$(x_b t_b \mid x_a t_a)_\omega = \sqrt{\frac{M}{2\pi i \hbar}} \sqrt{\frac{\omega}{\sin\omega (t_b - t_a)}}\, \exp\left(\frac{i}{\hbar} \mathcal{A}\right)$$

where

$$\mathcal{A} = \frac{M\omega}{2 \sin\omega(t_b - t_a)}\left[\left(x_b^2 + x_a^2\right) \cos\omega(t_b - t_a) - 2 x_b x_a\right]\Big\}$$

$$= \frac{M\omega}{2 \sin\omega(t_b - t_a)}\left\{\left(x_b^2 + x_a^2\right)\left[\cos\omega(t_b - t_a) - 1\right] + (x_b - x_a)^2\right\}$$

Using

$$\frac{\cos\theta - 1}{\sin\theta} = \tan\frac{\theta}{2} \qquad \sin\theta = \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$$

we have

$$\mathcal{A} = \frac{M\omega}{2}\left\{-\left(x_b^2 + x_a^2\right) \tan\left[\frac{\omega}{2} (t_b - t_a)\right] + \frac{(x_b - x_a)^2}{2\tan\left[\frac{\omega}{2} (t_b - t_a)\right]}\left(1 + \tan^2\left[\frac{\omega}{2} (t_b - t_a)\right]\right)\right\}$$

$$= \frac{M\omega}{2}\left\{-\left(x_b^2 + x_a^2\right) \tan\left[\frac{\omega}{2} (t_b - t_a)\right] + \frac{(x_b - x_a)^2}{2\tan\left[\frac{\omega}{2} (t_b - t_a)\right]}\left(1 + \tan^2\left[\frac{\omega}{2} (t_b - t_a)\right]\right)\right\}$$

Together with $\mathcal{A}^{x_0}$ given by (3.196f), (3.199a) becomes

$$Z_\omega^{\text{open},\,x_0} = \frac{M}{2\pi\hbar}\sqrt{\frac{\omega^4\,(t_b-t_a)^2}{\left\{\omega\,(t_b-t_a)-2\tan\left[\frac{\omega}{2}\,(t_b-t_a)\right]\right\}\sin\omega\,(t_b-t_a)}}$$

$$\times\int_{-\infty}^{\infty} d\,x\exp\Bigg(-\frac{i}{\hbar}\,\frac{M\omega}{2}\Bigg\{\left(x_b^2+x_a^2\right)\tan\left[\frac{\omega}{2}\,(t_b-t_a)\right]$$

$$-\frac{(x_b-x_a)^2}{2\tan\left[\frac{\omega}{2}\,(t_b-t_a)\right]}\left(1+\tan^2\left[\frac{\omega}{2}\,(t_b-t_a)\right]\right)$$

$$+$$

$$\left\{\tan\left[\frac{\omega}{2}\,(t_b-t_a)\right](x_b+x_a)-\omega\,(t_b-t_a)\,x_0\right\}^2\Big/\left(\omega\,(t_b-t_a)-2\tan\left[\frac{\omega}{2}\,(t_b-t_a)\right]\right)\Bigg\}\Bigg)$$

The Gaussian integral is evaluated in “3.07._Code.nb”, giving

$$Z_\omega^{\text{open},\,x_0} = \frac{1}{\omega}\sqrt{\frac{\omega^4\,(t_b-t_a)^2}{\omega\,(t_b-t_a)\sin\omega\,(t_b-t_a)}}\;\exp\left[-\frac{i}{2\hbar}\,M\,\omega^2(t_b-t_a)\,x_0^2\right]$$

$$=\sqrt{\frac{\omega\,(t_b-t_a)}{\sin\omega\,(t_b-t_a)}}\;\exp\left[-\frac{i}{2\hbar}\,M\,\omega^2(t_b-t_a)\,x_0^2\right] \qquad (3.199)$$

Integrating over x_0 gives

$$Z_\omega^{\text{open}} = \int_{-\infty}^{\infty} d\,x_0\,Z_\omega^{\text{open},\,x_0}$$

$$=\sqrt{\frac{\omega\,(t_b-t_a)}{\sin\omega\,(t_b-t_a)}}\;\sqrt{\frac{\pi\hbar}{i\,M\,\omega^2(t_b-t_a)}}$$

$$=\sqrt{\frac{2\pi\hbar}{i\,M\,\omega\sin\omega\,(t_b-t_a)}}$$

in agreement with (2.411).