3.8. External Source in Quantum-Statistical Path Integral

This section studies the quantum statistical evolution amplitude

$$(x_b,\ \beta\hbar \mid x_a\, 0\,)^j_\omega = \int \mathcal{D}\, x \exp\left\{ -\frac{1}{\hbar} \int_0^{\beta\hbar} d\,\tau \left[\frac{1}{2} M \left(\dot{x}^2 + \omega^2 x^2 \right) - j(\tau)\, x \right] \right\} \tag{3.200}$$

which will be evaluated in two different ways.

3.8.1. Continuation of Real-Time Result

The easiest way to obtain the result is by applying the analytic continuation

$$t = -i\,\tau \qquad\qquad t_b - t_a = -i\,(\tau_b - \tau_a) = -i\,\beta\,\hbar \tag{3.200a}$$

to the real-time results.

Using the subscript "e" to denote the Euclidean (imaginary-time) version of a quantity, we get from (3.168) & (3.170)

$$\begin{aligned}(x_b,\ \beta\hbar \mid x_a\, 0\,)^j_\omega &= e^{-\frac{1}{\hbar}\mathcal{A}^j_{\text{cl},\,e}}\, F^j_\omega(\tau_b,\ \tau_a) \\ &= e^{-\frac{1}{\hbar}\mathcal{A}^{\text{ext}}_e[j]}\, F_\omega(\tau_b,\ \tau_a) \\ &= \sqrt{\frac{M}{2\,\pi\,\hbar}}\ \sqrt{\frac{\omega}{\sinh\beta\,\hbar\,\omega}}\ \exp\left\{ -\frac{1}{\hbar}\mathcal{A}^{\text{ext}}_e[j] \right\}\end{aligned} \tag{3.201}$$

where

$$\mathcal{A}_e(\tau) = -i\,\mathcal{A}(t)\Big|_{t=-i\,\tau} \tag{3.201a}$$

$$F_\omega(\tau_b,\ \tau_a) = \sqrt{\frac{M}{2\,\pi\,\hbar}}\ \sqrt{\frac{\omega}{\sinh\beta\,\hbar\,\omega}} \qquad \text{[(3.170) used.]} \tag{3.201b}$$

$$\begin{aligned}\mathcal{A}^{\text{ext}}_e[j] &= \mathcal{A}^j_{\text{cl},e} + \tilde{\mathcal{A}}_{j,\,\text{fl},e} \\ &= \mathcal{A}_{\omega,\text{cl},e} + \mathcal{A}_{j,\text{cl},e} + \tilde{\mathcal{A}}_{j,\,\text{fl},\,e} \qquad \text{[(3.169) used.]} \\ &\equiv \mathcal{A}_e + \mathcal{A}^j_{1,e} + \mathcal{A}^j_{2,e} \equiv \mathcal{A}_e + \mathcal{A}^j_e\end{aligned} \tag{3.202}$$

with

$$\mathcal{A}_{\omega,\text{cl},e} \equiv \mathcal{A}_e = \frac{M\,\omega}{2\sinh\beta\,\hbar\,\omega}\left[\left(x_b^2 + x_a^2 \right)\cosh\beta\,\hbar\,\omega\ - 2\,x_b\,x_a \right] \tag{3.203}$$

$$\begin{aligned}\mathcal{A}_{j,\text{cl},e} \equiv \mathcal{A}^j_{1,e} &= \frac{1}{\sinh\beta\,\hbar\,\omega} \int_{\tau_a}^{\tau_b} d\,\tau \left[x_b \sinh\omega(\tau - \tau_a) + x_a \sinh\omega\,(\tau_b - \tau) \right] j(\tau) \\ &= -\frac{1}{\sinh\beta\,\hbar\,\omega} \int_0^{\beta\hbar} d\,\tau \left[x_b \sinh\omega\,\tau + x_a \sinh\omega\,(\beta\,\hbar - \tau) \right] j(\tau)\end{aligned} \tag{3.204}$$

$$\begin{aligned}\tilde{\mathcal{A}}_{j,\,\text{fl},\,e} \equiv \mathcal{A}^j_{2,e} &= -\frac{1}{M} \int_{\tau_a}^{\tau_b} d\,\tau \int_{\tau_a}^{\tau} d\,\tau'\, j(\tau)\, G_{\omega^2,\,e}(\tau,\ \tau')\, j(\tau') \qquad \text{[(3.167) used.]} \\ &= -\frac{1}{M} \int_0^{\beta\hbar} d\,\tau \int_0^{\tau} d\,\tau'\, j(\tau)\, G_{\omega^2,\,e}(\tau,\ \tau')\, j(\tau')\end{aligned} \tag{3.205}$$

$$G_{\omega^2,\,e}(\tau,\ \tau') = \frac{1}{\omega\sinh\beta\,\hbar\,\omega} \sinh\omega(\tau_b - \tau)\sinh\omega(\tau' - \tau_a) \qquad \tau > \tau'$$

Shifting the origin to τ_a, we have the equivalent form

$$G_{\omega^2,\,e}(\tau,\ \tau') = \frac{1}{\omega\sinh\beta\,\hbar\,\omega} \sinh\omega(\beta\,\hbar - \tau)\sinh\omega\,\tau' \qquad \tau > \tau' \tag{3.206a}$$

Alternatively, analytic continuation of the differential eq. [see (3.26)]

$$\left(\partial_t^2+\omega^2\right) G_{\omega^2}(t,\, t') = -\delta(t-t')$$

gives

$$\left(-\partial_\tau^2+\omega^2\right) G_{\omega^2}(-i\,\tau,\, -i\,\tau') = -\delta\left[-i\,(\tau-\ \tau')\right] = \frac{1}{i}\,\delta(\tau-\tau')$$

Defining

$$G_{\omega^2,\,e}(\tau,\ \tau') = iG_{\omega^2}(-i\,\tau,\, -i\,\tau') \qquad (3.208)$$

we get

$$\left(-\partial_\tau^2+\omega^2\right) G_{\omega^2,\,e}(\tau,\ \tau') = \delta(\tau-\tau') \qquad (3.207)$$

For the Dirichlet B.C., the solution is the Euclidean version of (3.36), namely

$$G_{\omega^2,\,e}(\tau,\ \tau') = \frac{1}{\omega \sinh \beta\,\hbar\,\omega} \sinh\omega\,(\tau_b-\tau_>) \sinh\omega\,(\tau_<-\tau_a)$$

Shifting the origin to τ_a, we have

$$\begin{aligned} G_{\omega^2,\,e}(\tau,\ \tau') &= \frac{1}{\omega \sinh \beta\,\hbar\,\omega} \sinh\omega\,(\beta\,\hbar-\tau_>) \sinh\omega\,\tau_< \\ &= \frac{1}{2\,\omega \sinh \beta\,\hbar\,\omega}\left[\cosh\omega\,(\beta\,\hbar-\tau_>+\tau_<) - \cosh\omega\,(\beta\,\hbar-\tau_>-\tau_<)\right] \\ &= \frac{1}{2\,\omega \sinh \beta\,\hbar\,\omega}\left[\cosh\omega\,(\beta\,\hbar-|\ \tau-\tau'\ |) - \cosh\omega\,(\beta\,\hbar-\tau-\tau')\right] \qquad (2.206) \end{aligned}$$

The Euclidean versions of (3.173-4) are

$$\begin{aligned} A_e(\omega) \equiv i\,A(\omega)\Big|_{t_b-t_a=-i\,\beta\,\hbar} &= \frac{1}{M\,\omega}\int_{\tau_a}^{\tau_b} d\,\tau\, e^{-\omega\,(\,\tau-\tau_a\,)}\, j(\tau) \\ &= \frac{1}{M\,\omega}\int_0^{\beta\,\hbar} d\,\tau\ e^{-\omega\,\tau}\ j(\tau) \qquad (3.211) \end{aligned}$$

$$\begin{aligned} B_e(\omega) \equiv i\,B(\omega)\Big|_{t_b-t_a=-i\,\beta\,\hbar} &= \frac{1}{M\,\omega}\int_{\tau_a}^{\tau_b} d\,\tau\, e^{-\omega\,(\,\tau_b-\tau\,)}\, j(\tau) \\ &= \frac{1}{M\,\omega}\int_0^{\beta\,\hbar} d\,\tau\, e^{-\omega\,(\,\beta\,\hbar-\tau\,)}\, j(\tau) \\ &= -e^{-\beta\,\hbar\,\omega}\, A_e\,(-\omega) \qquad (3.212) \end{aligned}$$

Either by using (3.21-2) on (3.204) or, more easily, by analytic continuing (3.175), we get

$$\begin{aligned} \mathcal{A}_{j,\text{cl},e} &= \mathcal{A}_{1,e}^{j} \\ &= \frac{M\,\omega}{2 \sinh \beta\,\hbar\,\omega}\left\{x_b\left[e^{\beta\,\hbar\,\omega}\, B_e\,(\omega) - A_e\,(\omega)\right] + x_a\left[e^{\beta\,\hbar\,\omega}\, A_e\,(\omega) - B_e\,(\omega)\right]\right\} \qquad (3.209) \end{aligned}$$

Similarly, (3.205) or (3.176) gives

$$\begin{aligned} \tilde{\mathcal{A}}_{j,\,\text{fl},\,e} &= \mathcal{A}_{2,e}^{j} \\ &= \frac{M\,\omega}{8 \sinh \beta\,\hbar\,\omega}\left\{e^{\beta\,\hbar\,\omega}\left[A_e\,(\omega)^2 + B_e\,(\omega)^2\right] - 2\,B_e\,(\omega)\,A_e\,(\omega)\right\} \qquad (3.210) \\ &\quad - \frac{1}{4\,M\,\omega}\int_0^{\beta\,\hbar} d\,\tau \int_0^{\beta\,\hbar} d\,\tau'\, j(\tau)\, e^{-\,\omega\,|\,\tau-\tau'\,|}\, j(\tau') \end{aligned}$$

The quantum statistical partition function is just the trace of the quantum statistical evolution amplitude:

$$Z_\omega^j = \int_{-\infty}^{\infty} d\,x\,(\,x,\ \beta\,\hbar\ |\ x\,0\,)_\omega^j$$

$$= F_\omega(\beta\hbar) \int_{-\infty}^{\infty} dx \exp\left\{-\frac{1}{\hbar}\mathcal{A}_e^{\text{ext}}[j]\right\}_{x_b=x_a=x} \tag{3.213a}$$

For a source-free oscillator,

$$\mathcal{A}_e^{\text{ext}}[j] = \mathcal{A}_{\omega,\text{cl},e} = \mathcal{A}_e$$

so that, with the help of (3.201b) & (3.203), we have

$$Z_\omega = F_\omega(\beta\hbar) \int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{\hbar}\mathcal{A}_e\right)_{x_b=x_a=x}$$

$$= \sqrt{\frac{M}{2\pi\hbar}}\sqrt{\frac{\omega}{\sinh\beta\hbar\omega}} \int_{-\infty}^{\infty} dx \exp\left\{-\frac{1}{\hbar}\frac{M\omega x^2}{\sinh\beta\hbar\omega}\left(\cosh\beta\hbar\omega - 1\right)\right\}$$

$$= \sqrt{\frac{M}{2\pi\hbar}}\sqrt{\frac{\omega}{\sinh\beta\hbar\omega}}\sqrt{\frac{\pi\hbar\sinh\beta\hbar\omega}{M\omega(\cosh\beta\hbar\omega - 1)}}$$

$$= \frac{1}{\sqrt{2(\cosh\beta\hbar\omega - 1)}}$$

$$= \frac{1}{2\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} \tag{3.214}$$

In the presence of a source, there is another x-dependent term given by [see (3.209)]

$$\mathcal{A}_{j,\text{cl},e} = \mathcal{A}_{1,e}^{j} = \frac{M\omega x}{2\sinh\beta\hbar\omega}\left[e^{\beta\hbar\omega}B_e(\omega) - A_e(\omega) + e^{\beta\hbar\omega}A_e(\omega) - B_e(\omega)\right]$$

$$= \frac{M\omega x}{2\sinh\beta\hbar\omega}\left(e^{\beta\hbar\omega} - 1\right)\left[B_e(\omega) + A_e(\omega)\right]$$

The x-integral thus becomes

$$\int_{-\infty}^{\infty} dx \exp\left\{-\frac{1}{\hbar}\frac{M\omega}{\sinh\beta\hbar\omega}\left[\left(\cosh\beta\hbar\omega - 1\right)x^2 + \frac{1}{2}\left(e^{\beta\hbar\omega} - 1\right)(B_e + A_e)\,x\right]\right\}$$

$$= \int_{-\infty}^{\infty} dx \exp\left\{-\frac{1}{\hbar}\frac{M\omega\left(\cosh\beta\hbar\omega - 1\right)}{\sinh\beta\hbar\omega}\left[x + \frac{1}{4}\left(\frac{e^{\beta\hbar\omega} - 1}{\cosh\beta\hbar\omega - 1}\right)(B_e + A_e)\right]^2\right\} \tag{3.214a}$$

$$\times \exp\left(-\frac{1}{\hbar}\mathcal{A}_{r,e}^{j}\right)$$

where

$$\mathcal{A}_{r,e}^{j} = -\frac{M\omega\left(\cosh\beta\hbar\omega - 1\right)}{\sinh\beta\hbar\omega}\left[\frac{1}{4}\left(\frac{e^{\beta\hbar\omega} - 1}{\cosh\beta\hbar\omega - 1}\right)(B_e + A_e)\right]^2 \tag{3.214b}$$

Using

$$\frac{\left(e^{\beta\hbar\omega} - 1\right)^2}{\cosh\beta\hbar\omega - 1} = \frac{\left(e^{\beta\hbar\omega} - 1\right)^2}{2\sinh^2(\beta\hbar\omega/2)} = \frac{2\left(e^{\beta\hbar\omega} - 1\right)^2}{\left(e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}\right)^2} = 2e^{\beta\hbar\omega}$$

we have

$$\mathcal{A}_{r,e}^{j} = -\frac{M\omega e^{\beta\hbar\omega}}{8\sinh\beta\hbar\omega}(B_e + A_e)^2 \tag{3.216}$$

Since the x-integral in (3.214a) gives the same result as the source-free case (3.214), (3.213a) becomes

$$Z_\omega^j = Z_\omega \exp\left[-\frac{1}{\hbar}\left(\mathcal{A}^j_{r,e} + \tilde{\mathcal{A}}_{j,\text{fl},e}\right)\right]$$

$$= Z_\omega \exp\left[-\frac{1}{\hbar}\left(\mathcal{A}^j_{r,e} + \mathcal{A}^j_{2,e}\right)\right] \quad (3.216a)$$

$$= Z_\omega \exp\left(-\frac{1}{\hbar}\mathcal{A}^j_e\right) \quad (3.220)$$

where [see (3.210) & (3.216)]

$$\mathcal{A}^j_e \equiv \mathcal{A}^j_{r,e} + \tilde{\mathcal{A}}_{j,\text{fl},e} = \mathcal{A}^j_{r,e} + \mathcal{A}^j_{2,e} \quad (3.215)$$

$$= -\frac{M\omega e^{\beta\hbar\omega}}{8\sinh\beta\hbar\omega}(B_e + A_e)^2 + \frac{M\omega}{8\sinh\beta\hbar\omega}\left(e^{\beta\hbar\omega}\left(A_e^2 + B_e^2\right) - 2A_e B_e\right)$$
$$- \frac{1}{4M\omega}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, j(\tau)\, e^{-\omega|\tau-\tau'|}\, j(\tau')\,\Big]$$

Using

$$-\frac{M\omega e^{\beta\hbar\omega}}{8\sinh\beta\hbar\omega}(B_e + A_e)^2 + \frac{M\omega}{8\sinh\beta\hbar\omega}\left(e^{\beta\hbar\omega}\left(A_e^2 + B_e^2\right) - 2A_e B_e\right)$$
$$= -\frac{M\omega}{4\sinh\beta\hbar\omega}\left(e^{\beta\hbar\omega} + 1\right)A_e B_e$$
$$= -\frac{M\omega e^{\beta\hbar\omega/2}\cosh(\beta\hbar\omega/2)}{2\sinh\beta\hbar\omega}A_e B_e$$
$$= -\frac{M\omega e^{\beta\hbar\omega/2}}{4\sinh(\beta\hbar\omega/2)}A_e B_e$$

(3.215) becomes

$$\mathcal{A}^j_e = -\frac{M\omega e^{\beta\hbar\omega/2}}{4\sinh(\beta\hbar\omega/2)}A_e B_e - \frac{1}{4M\omega}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, j(\tau)\, e^{-\omega|\tau-\tau'|}\, j(\tau') \quad (3.217)$$

Using (3.211-2), we have

$$\frac{M\omega e^{\beta\hbar\omega/2}}{4\sinh(\beta\hbar\omega/2)}A_e B_e$$
$$= \frac{e^{\beta\hbar\omega/2}}{4M\omega\sinh(\beta\hbar\omega/2)}\int_0^{\beta\hbar} d\tau\, e^{-\omega\tau}\, j(\tau)\int_0^{\beta\hbar} d\tau'\, e^{-\omega(\beta\hbar-\tau)}\, j(\tau')$$
$$= \frac{e^{-\beta\hbar\omega/2}}{4M\omega\sinh(\beta\hbar\omega/2)}\int_0^{\beta\hbar} d\tau\int_0^{\beta\hbar} d\tau'\, e^{-\omega(\tau-\tau')}\, j(\tau)\, j(\tau')$$
$$= \frac{1}{2M\omega\left(e^{\beta\hbar\omega} - 1\right)}\int_0^{\beta\hbar} d\tau\int_0^{\beta\hbar} d\tau'\, e^{-\omega(\tau-\tau')}\, j(\tau)\, j(\tau')$$
$$= \frac{1}{4M\omega\left(e^{\beta\hbar\omega} - 1\right)}\int_0^{\beta\hbar} d\tau\int_0^{\beta\hbar} d\tau'\left[e^{-\omega(\tau-\tau')} + e^{\omega(\tau-\tau')}\right] j(\tau)\, j(\tau')$$

where we've used $\tau \leftrightarrow \tau'$ to obtain the 2nd term in the square bracket.

(3.217) thus becomes

$$\mathcal{A}^j_e = -\frac{1}{4M\omega}\int_0^{\beta\hbar} d\tau\int_0^{\beta\hbar} d\tau'\, j(\tau)\, j(\tau')\left\{\frac{1}{e^{\beta\hbar\omega} - 1}\left[e^{-\omega(\tau-\tau')} + e^{\omega(\tau-\tau')}\right] + e^{-\omega|\tau-\tau'|}\right\}$$
$$= -\frac{1}{4M\omega}\int_0^{\beta\hbar} d\tau\int_0^{\beta\hbar} d\tau'\,\frac{j(\tau)\, j(\tau')}{e^{\beta\hbar\omega} - 1}\begin{cases} e^{\beta\hbar\omega - \omega(\tau-\tau')} + e^{\omega(\tau-\tau')} & \tau > \tau' \\ e^{-\omega(\tau-\tau')} + e^{\beta\hbar\omega + \omega(\tau-\tau')} & \tau < \tau' \end{cases}$$

$$= -\frac{1}{4M\omega}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, j(\tau)\, j(\tau') \left(\frac{e^{\omega(\beta\hbar - |\tau-\tau'|)} + e^{\omega|\tau-\tau'|}}{e^{\beta\hbar\omega} - 1} \right)$$

$$= -\frac{1}{4M\omega}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, j(\tau)\, j(\tau')\, \frac{\cosh\left[\omega\left(\frac{1}{2}\beta\hbar - |\tau - \tau'|\right)\right]}{\sinh(\beta\hbar\omega/2)} \qquad (3.218)$$

$$= -\frac{1}{2M}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, j(\tau)\, j(\tau')\, G^p_{\omega^2,e}(\tau - \tau') \qquad (3.218a)$$

where

$$G^p_{\omega^2,e}(\tau) = \frac{\cosh\left[\omega\left(\frac{1}{2}\beta\hbar - \tau\right)\right]}{2\omega\sinh(\beta\hbar\omega/2)} \qquad \tau \in [0, \beta\hbar) \qquad (3.219)$$

$$= i\, G^p_{\omega^2}(-i\tau)\big|_{t_b - t_a = -i\beta\hbar}$$

is just the Euclidean version of the periodic Green function in (3.99).

For completeness, we shall also calculate the partition function for open paths, $Z^{\text{open}}_\omega[j]$.

To begin, the x-integral now involves [see (3.203-4) & (3.209)]

$$\mathcal{I} = \int_{-\infty}^{\infty} dx_b \int_{-\infty}^{\infty} dx_a \exp\left[-\frac{1}{\hbar}\left(\mathcal{A}_{\omega,\text{cl},e} + \mathcal{A}_{j,\text{cl},e}\right)\right]$$

$$= \int_{-\infty}^{\infty} dx_b \int_{-\infty}^{\infty} dx_a \exp\left\{-\frac{M\omega}{2\hbar\sinh\beta\hbar\omega}\left[\left(x_b^2 + x_a^2\right)\cosh\beta\hbar\omega - 2x_b x_a \right.\right. \qquad (3.221a)$$

$$\left.\left. + x_b\left(e^{\beta\hbar\omega} B_e - A_e\right) + x_a\left(e^{\beta\hbar\omega} A_e - B_e\right)\right]\right\}$$

Using *Mathematica* [see “3.08._Code.nb”] to do the integral, we get

$$\mathcal{I} = \frac{2\pi\hbar}{M\omega}\exp\left[\frac{M\omega}{4\hbar}\left(1 + \coth\beta\hbar\omega\right)\left(A_e^2 + B_e^2\right)\right] \qquad (3.221b)$$

Thus, (3.213a) becomes

$$Z^{\text{open}}_\omega[j] = \sqrt{\frac{2\pi\hbar}{M\omega\sinh\beta\hbar\omega}}\, \exp\left(-\frac{1}{\hbar}\mathcal{A}^{\text{open}}\right) \qquad (3.221)$$

where

$$\mathcal{A}^{\text{open}} = -\frac{M\omega}{4}\left(1 + \coth\beta\hbar\omega\right)\left(A_e^2 + B_e^2\right) + \mathcal{A}^j_{2,e}$$

$$= -\frac{M\omega}{4}\,\frac{e^{\beta\hbar\omega}}{\sinh\beta\hbar\omega}\left(A_e^2 + B_e^2\right) + \mathcal{A}^j_{2,e} \qquad (3.222a)$$

Using (3.211-2) &

$$\mathcal{A}^{\text{open}} = -\frac{M\omega}{8\sinh\beta\hbar\omega}\left[e^{\beta\hbar\omega}\left(A_e^2 + B_e^2\right) + 2A_e B_e\right]$$

$$-\frac{1}{4M\omega}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, j(\tau)\, e^{-\omega|\tau-\tau'|}\, j(\tau')$$

$$= -\frac{1}{8M\omega\sinh\beta\hbar\omega}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, j(\tau)\, j(\tau')$$

$$\times\left\{e^{\beta\hbar\omega}\left[e^{-\omega\tau} e^{-\omega\tau'} + e^{-\omega(\beta\hbar-\tau)} e^{-\omega(\beta\hbar-\tau')}\right] + 2e^{-\omega\tau} e^{-\omega(\beta\hbar-\tau')}\right.$$

$$\left. + \left(e^{\beta\hbar\omega} - e^{-\beta\hbar\omega}\right) e^{-\omega|\tau-\tau'|}\right\}$$

$$= -\frac{1}{8M\omega\sinh\beta\hbar\omega}\int_0^{\beta\hbar}d\tau\int_0^{\beta\hbar}d\tau'\ j(\tau)\,j(\tau')\Big[e^{-\omega(\tau+\tau'-\beta\hbar)}+e^{\omega(\tau+\tau'-\beta\hbar)} \quad (3.222b)$$
$$+2e^{-\omega(\tau-\tau'+\beta\hbar)}+e^{-\omega(|\tau-\tau'|-\beta\hbar)}-e^{-\omega(|\tau-\tau'|+\beta\hbar)}\Big]$$

For any symmetric function $f(\tau',\tau)=f(\tau,\tau')$,

$$\int_0^{\beta\hbar}d\tau\int_0^{\beta\hbar}d\tau'\ f(\tau,\tau')$$
$$=\int_0^{\beta\hbar}d\tau\left[\int_0^{\tau}d\tau'+\int_\tau^{\beta\hbar}d\tau'\right]f(\tau,\tau')$$
$$=\left[\int_0^{\beta\hbar}d\tau\int_0^{\tau}d\tau'+\int_0^{\beta\hbar}d\tau'\int_0^{\tau'}d\tau\right]f(\tau,\tau') \qquad \text{[c.f. (3.167-a)]}$$
$$=2\int_0^{\beta\hbar}d\tau\int_0^{\tau}d\tau'\,f(\tau,\tau') \qquad (3.222c)$$

While the 1st two terms inside the square bracket in (3.222b) can be re-written using (3.222c), the last 3 terms require special treatment as follows

$$\int_0^{\beta\hbar}d\tau\int_0^{\beta\hbar}d\tau'\ j(\tau)\,j(\tau')\left[2e^{-\omega(\tau-\tau'+\beta\hbar)}+e^{-\omega(|\tau-\tau'|-\beta\hbar)}-e^{-\omega(|\tau-\tau'|+\beta\hbar)}\right]$$
$$=\int_0^{\beta\hbar}d\tau\left\{\int_0^{\tau}d\tau'\left[e^{-\omega(\tau-\tau'+\beta\hbar)}+e^{-\omega(\tau-\tau'-\beta\hbar)}\right]\right.$$
$$\left.+\int_\tau^{\beta\hbar}d\tau'\left[2e^{-\omega(\tau-\tau'+\beta\hbar)}+e^{\omega(\tau-\tau'+\beta\hbar)}-e^{\omega(\tau-\tau'-\beta\hbar)}\right]\right\}j(\tau)\,j(\tau')$$
$$=\int_0^{\beta\hbar}d\tau\int_0^{\tau}d\tau'\left[e^{-\omega(\tau-\tau'+\beta\hbar)}+e^{-\omega(\tau-\tau'-\beta\hbar)}\right]j(\tau)\,j(\tau')$$
$$+\int_0^{\beta\hbar}d\tau'\int_\tau^{\beta\hbar}d\tau\left[2e^{-\omega(\tau-\tau'+\beta\hbar)}+e^{\omega(\tau-\tau'+\beta\hbar)}-e^{\omega(\tau-\tau'-\beta\hbar)}\right]j(\tau)\,j(\tau')$$
$$=\int_0^{\beta\hbar}d\tau\int_0^{\tau}d\tau'\left[e^{-\omega(\tau-\tau'+\beta\hbar)}+e^{-\omega(\tau-\tau'-\beta\hbar)}\right.$$
$$\left.+2e^{\omega(\tau-\tau'-\beta\hbar)}+e^{-\omega(\tau-\tau'-\beta\hbar)}-e^{-\omega(\tau-\tau'+\beta\hbar)}\right]j(\tau)\,j(\tau')$$
$$=2\int_0^{\beta\hbar}d\tau\int_0^{\tau}d\tau'\left[e^{\omega(\tau-\tau'-\beta\hbar)}+e^{-\omega(\tau-\tau'-\beta\hbar)}\right]j(\tau)\,j(\tau')$$
$$=4\int_0^{\beta\hbar}d\tau\int_0^{\tau}d\tau'\ j(\tau)\,j(\tau')\cosh[\omega(\tau-\tau'-\beta\hbar)]$$

(3.222b) can therefore be written as

$$\mathcal{A}^{\text{open}}=-\frac{1}{2M\omega\sinh\beta\hbar\omega}\int_0^{\beta\hbar}d\tau\int_0^{\tau}d\tau'\ j(\tau)\,j(\tau')$$
$$\times\left\{\cosh[\omega(\tau+\tau'-\beta\hbar)]+\cosh[\omega(\tau-\tau'-\beta\hbar)]\right\}$$
$$=-\frac{1}{M\omega\sinh\beta\hbar\omega}\int_0^{\beta\hbar}d\tau\int_0^{\beta\hbar}d\tau'\ j(\tau)\,j(\tau')\cosh\omega(\tau-\beta\hbar)\cosh\omega\tau'$$
$$=-\frac{1}{M}\int_0^{\beta\hbar}d\tau\int_0^{\beta\hbar}d\tau'\ j(\tau)\,j(\tau')\,G^{\text{open}}_{\omega^2,e}(\tau,\tau') \qquad (3.230)$$

where

$$G^{\text{open}}_{\omega^2,e}(\tau,\tau')=\frac{\cosh\omega(\beta\hbar-\tau)\cosh\omega\tau'}{\omega\sinh\beta\hbar\omega} \qquad \text{for } \tau>\tau'$$
$$=\frac{\cosh\omega(\beta\hbar-\tau_>)\cosh\omega\tau_<}{\omega\sinh\beta\hbar\omega} \qquad (3.231)$$

For small ω, we have [see "3.08._Code.nb"],

$$G^{\text{open}}_{\omega^2,e}(\tau,\,\tau') = \frac{1}{\beta\hbar\omega^2} + \frac{1}{3}\beta\hbar - \tau_> + \frac{1}{2\beta\hbar}\left(\tau_>^2 + \tau_<^2\right) + O\left[\omega^2\right] \qquad (3.232a)$$

Using

$$\tau_> = \begin{cases} \tau & \tau > \tau' \\ \tau' & \tau < \tau' \end{cases} = \frac{1}{2}\left(\left|\tau - \tau'\right| + \tau + \tau'\right)$$

(3.232a) becomes

$$G^{\text{open}}_{\omega^2,e}(\tau,\,\tau') = \frac{1}{\beta\hbar\omega^2} + \frac{1}{3}\beta\hbar - \frac{1}{2}\left|\tau - \tau'\right| - \frac{1}{2}(\tau + \tau') + \frac{1}{2\beta\hbar}\left(\tau^2 + \tau'^2\right) + O\left[\omega^2\right] \qquad (3.232)$$

which is the imaginary-time version of (3.157).

3.8.2. Calculation at Imaginary Time

We now calculate Z^j_ω directly from

$$Z^j_\omega = \oint_{x_b = x_a} \mathcal{D}\,x(\tau)\exp\left(-\frac{1}{\hbar}\mathcal{A}_e[j]\right) \qquad (3.233)$$

with the Euclidean action

$$\mathcal{A}_e[j] = \int_0^{\beta\hbar} d\tau\left[\frac{1}{2}M\left(\dot{x}^2 + \omega^2 x^2\right) - j(\tau)\,x\right] \qquad (3.234)$$

Using the periodic B.C., the surface terms from a partial integration cancel out & we have

$$\mathcal{A}_e[j] = \int_0^{\beta\hbar} d\tau\left[\frac{1}{2}M\,x\left(-\partial_\tau^2 + \omega^2\right)x - j(\tau)\,x\right] \qquad (3.235)$$

Define

$$D_{\omega^2,e}(\tau,\,\tau') \equiv \left(-\partial_\tau^2 + \omega^2\right)\delta(\tau - \tau') \qquad \tau - \tau' \in [0,\,\beta\hbar) \qquad (3.236)$$

$$\rightarrow \quad G^p_{\omega^2,e}(\tau,\,\tau') = \left(-\partial_\tau^2 + \omega^2\right)^{-1}\delta(\tau - \tau') = G^p_{\omega^2,e}(\tau - \tau') = D^{-1}_{\omega^2,e}(\tau,\,\tau') \qquad (3.237)$$

Setting

$$x(\tau) = x'(\tau) + \frac{1}{M}\int_0^{\beta\hbar} d\tau'\,G^p_{\omega^2,e}(\tau,\,\tau')\,j(\tau') \qquad (3.238)$$

we have

$$\begin{aligned} &x(\tau)\left(-\partial_\tau^2 + \omega^2\right)x(\tau) \\ &= x'(\tau)\left(-\partial_\tau^2 + \omega^2\right)x'(\tau) \\ &\quad + \frac{1}{M}\int_0^{\beta\hbar} d\tau'\,G^p_{\omega^2,e}(\tau,\,\tau')\,j(\tau')\left(-\partial_\tau^2 + \omega^2\right)x'(\tau) \\ &\quad + \frac{1}{M}\int_0^{\beta\hbar} d\tau'\,x'(\tau)\left(-\partial_\tau^2 + \omega^2\right)G^p_{\omega^2,e}(\tau,\,\tau')\,j(\tau') \\ &\quad + \frac{1}{M^2}\int_0^{\beta\hbar} d\tau'\,G^p_{\omega^2,e}(\tau,\,\tau')\,j(\tau')\left(-\partial_\tau^2 + \omega^2\right)\int_0^{\beta\hbar} d\tau''\,G^p_{\omega^2,e}(\tau,\,\tau'')\,j(\tau'') \\ &= x'(\tau)\left(-\partial_\tau^2 + \omega^2\right)x'(\tau) + \frac{2}{M}j(\tau)\,x'(\tau) + \frac{1}{M^2}\int_0^{\beta\hbar} d\tau'\,j(\tau)\,G^p_{\omega^2,e}(\tau,\,\tau')\,j(\tau') \end{aligned}$$

and

$$j(\tau)\,x(\tau) = j(\tau)\,x'(\tau) + \frac{1}{M}\int_0^{\beta\hbar} d\tau'\,j(\tau)\,G^p_{\omega^2,e}(\tau,\,\tau')\,j(\tau')$$

(3.235) thus becomes

$$\mathcal{A}_e[j] = \int_0^{\beta\hbar} d\tau \frac{1}{2} M x' \left(-\partial_\tau^2 + \omega^2\right) x' - \frac{1}{2M} \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' \, j(\tau) \, G^p_{\omega^2,e}(\tau, \tau') \, j(\tau') \qquad (3.239)$$

For $j = 0$, we have [c.f. (2.408a)]

$$\begin{aligned} Z_\omega &= \oint_{x'_b = x'_a} \mathcal{D} x' \exp\left(-\frac{1}{\hbar} \int_0^{\beta\hbar} d\tau \frac{1}{2} M x' \left(-\partial_\tau^2 + \omega^2\right) x'\right) \\ &= \oint_{x'_b = x'_a} \mathcal{D} x' \exp\left(-\frac{1}{\hbar} \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' \frac{1}{2} M x'(\tau) \, D_{\omega^2, e}(\tau, \tau') \, x'(\tau')\right) \\ &= \frac{1}{\sqrt{\det D_{\omega^2, e}}} \end{aligned} \qquad (3.240)$$

For the periodic B.C.,

$$\begin{aligned} \det D_{\omega^2, e} &= \prod_{m=-\infty}^{\infty} \left(\omega_m^2 + \omega^2\right) \qquad \omega_m = \frac{2 m \pi}{\beta\hbar} \\ &= \exp\left[\sum_{m=-\infty}^{\infty} \ln\left(\omega_m^2 + \omega^2\right)\right] \qquad (3.241) \\ &= \exp\left[2 \sum_{m=1}^{\infty} \ln\left(\frac{\omega_m^2}{\omega^2} + 1\right)\right] \qquad \text{[(2.527) used.]} \\ &= \exp\left\{2 \ln\left[2 \sinh\left(\frac{1}{2} \beta\hbar\omega\right)\right]\right\} \qquad \text{[See §2.15.3]} \\ &= 4 \sinh^2\left(\frac{1}{2} \beta\hbar\omega\right) \end{aligned} \qquad (3.241a)$$

(3.240) thus becomes

$$Z_\omega = \frac{1}{2 \sinh\left(\frac{1}{2} \beta\hbar\omega\right)} \qquad (3.242)$$

As with (3.220),

$$Z^j_\omega = Z_\omega \exp\left\{-\frac{1}{\hbar} \mathcal{A}_e[j]\right\} \qquad (3.243)$$

where

$$\mathcal{A}^j_e[j] = -\frac{1}{2M} \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' \, j(\tau) \, G^p_{\omega^2,e}(\tau, \tau') \, j(\tau') \qquad (3.244)$$

is the j-dependent part of $\mathcal{A}_e[j]$ in (3.239).

Using the set of orthonormal, complete, and periodic eigenfunctions

$$\phi_m(\tau) = \frac{1}{\sqrt{\beta\hbar}} e^{-i \omega_m \tau} \qquad (3.244a)$$

as basis, (3.237) becomes

$$\begin{aligned} G^p_{\omega^2,e}(\tau - \tau') &= \left(-\partial_\tau^2 + \omega^2\right)^{-1} \delta(\tau - \tau') \\ &= \frac{1}{\beta\hbar} \left(-\partial_\tau^2 + \omega^2\right)^{-1} \sum_{m=-\infty}^{\infty} e^{-i \omega_m (\tau - \tau')} \\ &= \frac{1}{\beta\hbar} \sum_{m=-\infty}^{\infty} \frac{1}{\omega_m^2 + \omega^2} e^{-i \omega_m (\tau - \tau')} \end{aligned} \qquad (3.245)$$

For low temperatures, $\beta >> 1$ so that $\omega_m << 1$, we have

$$\frac{1}{\beta\hbar} \sum_m f(\omega_m) = \frac{\Delta\omega_m}{2\pi} \sum_m f(\omega_m) \approx \int \frac{d\omega_m}{2\pi} f(\omega_m) \qquad \omega_m = \frac{2 m \pi}{\beta\hbar}$$

so that

$$G^{p}_{\omega^2,e}(\tau-\tau') = \int_{-\infty}^{\infty} \frac{d\,\omega_m}{2\pi} \frac{1}{\omega_m^2+\omega^2} e^{-i\,\omega_m(\tau-\tau')} \qquad \beta\to\infty$$

$$= \int_{-\infty}^{\infty} \frac{d\,\omega_m}{2\pi} \frac{1}{2\,i\,\omega}\left(\frac{1}{\omega_m - i\,\omega} - \frac{1}{\omega_m + i\,\omega}\right) e^{-i\,\omega_m(\tau-\tau')}$$

$$= \frac{1}{2\,\omega}\begin{cases} e^{-\omega(\tau-\tau')} & \tau>\tau' \quad C.W. \\ e^{\omega(\tau-\tau')} & \tau<\tau' \quad C.C.W. \end{cases} \text{ contour closed in } \begin{matrix}\text{lower}\\ \text{upper}\end{matrix} \text{ plane}$$

$$= \frac{1}{2\,\omega} e^{-\omega\,|\tau-\tau'|} \tag{3.246}$$

For finite tempertures, we use the Poisson summation formula to write

$$G^{p}_{\omega^2,e}(\tau-\tau') = \sum_{n=-\infty}^{\infty} \int \frac{d\,\mu}{2\pi} e^{i\,\beta\,\hbar\,\mu\,n} \frac{1}{\mu^2+\omega^2} e^{-i\,\mu(\tau-\tau')}$$

$$= \frac{1}{2\,\omega} \sum_{n=-\infty}^{\infty} e^{-\omega\,|\tau-\tau'+n\,\beta\,\hbar|} \tag{3.246a}$$

Since $G^{p}_{\omega^2,e}$ is periodic, we need consider only the primary interval $\tau\in[0,\ \beta\,\hbar)$ for which

$$G^{p}_{\omega^2,e}(\tau) = \frac{1}{2\,\omega}\left(\sum_{n=0}^{\infty} e^{-\omega(\tau+n\,\beta\,\hbar)} + \sum_{n=-\infty}^{-1} e^{\omega(\tau+n\,\beta\,\hbar)}\right)$$

$$= \frac{1}{2\,\omega}\left(\sum_{n=0}^{\infty} e^{-\omega(\tau+n\,\beta\,\hbar)} + \sum_{n=1}^{\infty} e^{\omega(\tau-n\,\beta\,\hbar)}\right)$$

$$= \frac{1}{2\,\omega}\left[e^{-\omega\,\tau} + (e^{-\omega\,\tau} + e^{\omega\,\tau}) \sum_{n=1}^{\infty} e^{-n\,\beta\,\hbar\,\omega}\right]$$

$$= \frac{1}{2\,\omega}\left[e^{-\omega\,\tau} + (e^{-\omega\,\tau} + e^{\omega\,\tau}) \frac{e^{-\beta\,\hbar\,\omega}}{1-e^{-\beta\,\hbar\,\omega}}\right]$$

$$= \frac{1}{2\,\omega}\left(\frac{e^{-\omega\,\tau} + e^{\omega(\tau-\beta\,\hbar)}}{1-e^{-\beta\,\hbar\,\omega}}\right)$$

$$= \frac{1}{2\,\omega} \frac{\cosh\omega\left(\frac{1}{2}\beta\,\hbar-\tau\right)}{\sinh\left(\frac{1}{2}\beta\,\hbar\,\omega\right)} \qquad \tau\in[0,\ \beta\,\hbar) \tag{3.248}$$

in agreement with (3.219). Since (3.248) is not explicitly periodic, values outside $[0,\ \beta\,\hbar)$ must be calculated using the periodicity.

For small ω, we have [see “3.08._Code.nb”],

$$G^{p}_{\omega^2,e}(\tau) = \frac{1}{\beta\,\hbar\,\omega^2} - \frac{1}{2}\tau + \frac{1}{2}\frac{\tau^2}{\beta\,\hbar} + \frac{1}{12}\beta\,\hbar + O\left[\omega^2\right] \tag{3.249}$$

Noting that the $m=0$ term in (3.245) is $\frac{1}{\beta\,\hbar\,\omega^2}$, we define

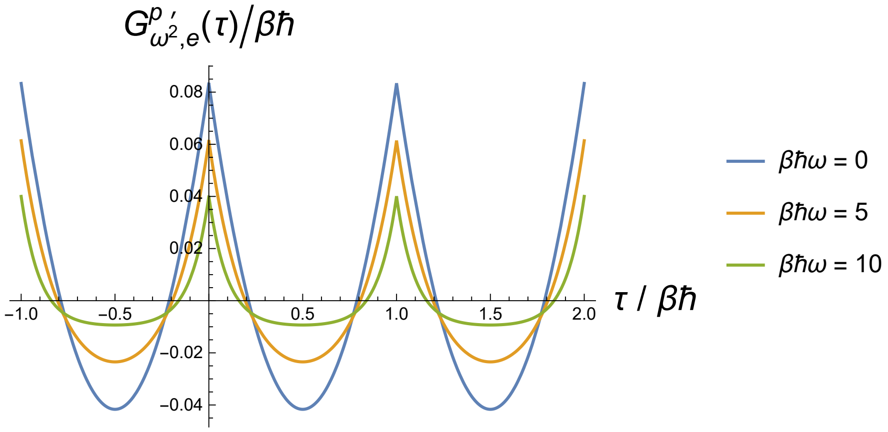
$$G^{p\,\prime}_{\omega^2,e}(\tau) \equiv G^{p}_{\omega^2,e}(\tau) - \frac{1}{\beta\,\hbar\,\omega^2} \tag{3.250}$$

$$= \frac{1}{\beta\,\hbar} \sum_{m\neq 0} \frac{1}{\omega_m^2+\omega^2} e^{-i\,\omega_m\,\tau} \tag{3.251a}$$

which has a finite $\omega\to 0$ limit

$$G^{p\,\prime}_{0,e}(\tau) = -\frac{1}{2}\tau + \frac{1}{2}\frac{\tau^2}{\beta\,\hbar} + \frac{1}{12}\beta\,\hbar \qquad \tau\in[0,\ \beta\,\hbar) \tag{3.251}$$

The following graph plots $G^{p\,\prime}_{\omega^2,e}(\tau)/\beta\hbar$ for 3 values of $\beta\hbar\omega$ [see “3.08._Code.nb”].



(3.251) can also be derived directly from (3.251a) [c.f. §2.15.6]. Setting $\omega = 0$, we have

$$G^{p\,\prime}_{0,e}(\tau) = \frac{1}{\beta\hbar}\sum_{m\neq 0}\frac{1}{\omega_m^2}e^{-i\,\omega_m\,\tau}$$

$$= \frac{1}{\beta\hbar}\sum_{m\neq 0}\frac{(-)^m}{\omega_m^2}e^{-i\,\omega_m\left(\tau-\frac{1}{2}\beta\hbar\right)} \qquad \frac{1}{2}\beta\hbar\,\omega_m = m\,\pi \qquad (3.252)$$

$$= \frac{1}{\beta\hbar}\left(\frac{\beta\hbar}{2\pi}\right)^2\sum_{m\neq 0}\frac{(-)^m}{m^2}\sum_{n=0}^{\infty}\frac{m^n}{n!}\left[-i\frac{2\pi}{\beta\hbar}\left(\tau-\frac{1}{2}\beta\hbar\right)\right]^n \qquad (3.252a)$$

Using

$$\sum_{m\neq 0}\frac{(-)^m}{m^{2-n}} = \sum_{m=1}^{\infty}\left[\frac{(-)^m}{m^{2-n}}+\frac{(-)^{-m+2-n}}{m^{2-n}}\right]$$

$$= \begin{cases} 2\sum_{m=1}^{\infty}\frac{(-)^m}{m^{2-n}} & n = \text{even} \\ 0 & n = \text{odd}\end{cases}$$

(3.252a) becomes

$$G^{p\,\prime}_{0,e}(\tau) = \frac{2}{\beta\hbar}\left(\frac{\beta\hbar}{2\pi}\right)^2\sum_{n=0,2,4,\ldots}\left(\sum_{m=1}^{\infty}\frac{(-)^m}{m^{2-n}}\right)\frac{1}{n!}\left[-i\frac{2\pi}{\beta\hbar}\left(\tau-\frac{1}{2}\beta\hbar\right)\right]^n \qquad (3.253)$$

$$= -\frac{2}{\beta\hbar}\left(\frac{\beta\hbar}{2\pi}\right)^2\sum_{n=0,2,4,\ldots}\eta(2-n)\frac{1}{n!}\left[-i\frac{2\pi}{\beta\hbar}\left(\tau-\frac{1}{2}\beta\hbar\right)\right]^n \qquad (3.253a)$$

where [see Abramowitz & Stegun, Formula. 23.2.19]

$$\eta(z) \equiv \sum_{m=1}^{\infty}\frac{(-)^{m-1}}{m^{2-n}} \qquad (3.254)$$

is the Riemann eta function related to the Riemann zeta function by

$$\eta(z) = \left(1-2^{1-z}\right)\zeta(z) \qquad (3.255)$$

Since [see Abramowitz & Stegun, Formulae 23.2.(14, 11 & 24)]

$$\zeta(-2\,n) = 0 \qquad \forall\; n = 1, 2, 3, \ldots \qquad (3.255a)$$

$$\zeta(0) = -\frac{1}{2} \qquad \zeta(2) = \frac{\pi^2}{6}$$

we have

$$\eta(2-n)=\begin{cases}\eta(2)=\frac{1}{2}\zeta(2)=\frac{\pi^2}{12} & n=0\\ \eta(0)=-\zeta(0)=\frac{1}{2} & n=2\\ 0 & n=4,6,8,\ \ldots.\end{cases} \tag{3.255b}$$

so that (3.253a) reduces to

$$\begin{aligned}G_{0,e}^{p\,\prime}(\tau)&=-\frac{2}{\beta\hbar}\left(\frac{\beta\hbar}{2\pi}\right)^2\left\{\eta(2)+\eta(0)\frac{1}{2!}\left[-i\frac{2\pi}{\beta\hbar}\left(\tau-\frac{1}{2}\beta\hbar\right)\right]^2\right\}\\&=-\frac{2}{\beta\hbar}\left(\frac{\beta\hbar}{2\pi}\right)^2\left\{\frac{\pi^2}{12}+\frac{1}{2}\frac{1}{2!}\left[-i\frac{2\pi}{\beta\hbar}\left(\tau-\frac{1}{2}\beta\hbar\right)\right]^2\right\}\\&=-\frac{2}{\beta\hbar}\left(\frac{\beta\hbar}{2\pi}\right)^2\left[\frac{\pi^2}{12}-\frac{1}{4}\left(\frac{2\pi}{\beta\hbar}\right)^2\left(\tau-\frac{1}{2}\beta\hbar\right)^2\right]\\&=-\frac{\beta\hbar}{24}+\frac{1}{2\beta\hbar}\left(\tau-\frac{1}{2}\beta\hbar\right)^2\\&=\frac{1}{2\beta\hbar}\tau^2-\frac{1}{2}\tau+\frac{1}{12}\beta\hbar\end{aligned} \tag{3.255}$$

in agreement with (3.251).

The Bernoulli polynomials are defined as [see Gradshteyn & Ryzhik, Formula 9.620]

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_k\,x^{n-k} \tag{3.259}$$

where B_n are the Bernoulli numbers given by [G&R, 9.612]

$$B_n=\sum_{k=0}^{n}\binom{n}{k}B_k \tag{3.259a}$$

They can also be obtained from the generating function [G&R, 9.621]

$$\frac{e^{x\,t}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^{n-1}}{n!} \tag{3.260}$$

and have the expansion [G&R, 9.622]

$$B_{2n}(x)=\frac{(-)^{n-1}\,2\,(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos 2k\pi x}{k^{2n}} \qquad 0\le x\le 1,\quad n=1,2,3,\ \ldots \tag{3.261}$$

Some special values are [G&R, 9.627]

$$B_1(x)=x-\frac{1}{2},\qquad B_2(x)=x^2-x+\frac{1}{6} \tag{3.262}$$

(3.255) can therefore be written as

$$\begin{aligned}G_{0,e}^{p\,\prime}(\tau)&=\frac{1}{2}\beta\hbar\left[\left(\frac{\tau}{\beta\hbar}\right)^2-\frac{\tau}{\beta\hbar}+\frac{1}{6}\right]\\&=\frac{1}{2}\beta\hbar B_2\left(\frac{\tau}{\beta\hbar}\right)\end{aligned} \tag{3.258}$$

This relation is not accidental but arises from the relation [A&S 23.2.16]

$$\zeta(2n)=\frac{(2\pi)^{2n}}{2\,(2n)!}\left|B_{2n}\right| \qquad n=1,2,3,\ \ldots \tag{3.262a}$$

Anti-periodic B.C.

The anti-periodic Green function is related to the periodic one by the substitution

$$\omega_m = \frac{2\pi}{\beta\hbar} m \qquad \leftrightarrow \qquad \omega_m^f = \frac{2\pi}{\beta\hbar}\left(m + \frac{1}{2}\right) \qquad (3.263a)$$

Using the Poisson summation formula for fermions [see (3.105)]

$$\sum_{m=-\infty}^{\infty} f\left(m+\frac{1}{2}\right) = \int_{-\infty}^{\infty} d\mu \sum_{n=-\infty}^{\infty} (-)^n e^{2\pi i \mu n} f(\mu) \qquad (3.263b)$$

(3.246a) becomes

$$G_{\omega^2,e}^a(\tau-\tau') = \frac{1}{2\omega} \sum_{n=-\infty}^{\infty} (-)^n e^{-\omega|\tau-\tau'+n\beta\hbar|} \qquad (3.263c)$$

Since $G_{\omega^2,e}^a$ is anti-periodic, we need consider only the primary interval $\tau \in [0, \beta\hbar)$ for which

$$\begin{aligned} G_{\omega^2,e}^a(\tau) &= \frac{1}{2\omega}\left(\sum_{n=0}^{\infty}(-)^n e^{-\omega(\tau+n\beta\hbar)} + \sum_{n=-\infty}^{-1}(-)^n e^{\omega(\tau+n\beta\hbar)}\right) \\ &= \frac{1}{2\omega}\left(\sum_{n=0}^{\infty}(-)^n e^{-\omega(\tau+n\beta\hbar)} + \sum_{n=1}^{\infty}(-)^n e^{\omega(\tau-n\beta\hbar)}\right) \\ &= \frac{1}{2\omega}\left[e^{-\omega\tau} + (e^{-\omega\tau} + e^{\omega\tau})\sum_{n=1}^{\infty}(-)^n e^{-n\beta\hbar\omega}\right] \\ &= \frac{1}{2\omega}\left[e^{-\omega\tau} + (e^{-\omega\tau} + e^{\omega\tau})\frac{-e^{-\beta\hbar\omega}}{1+e^{-\beta\hbar\omega}}\right] \\ &= \frac{1}{2\omega}\left(\frac{e^{-\omega\tau} - e^{\omega(\tau-\beta\hbar)}}{1+e^{-\beta\hbar\omega}}\right) \\ &= \frac{1}{2\omega}\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar-\tau\right)}{\cosh\left(\frac{1}{2}\beta\hbar\omega\right)} \qquad \tau \in [0, \beta\hbar) \qquad (3.263) \end{aligned}$$

Caution: (3.263) differs from Kleinert's version by a negative sign.

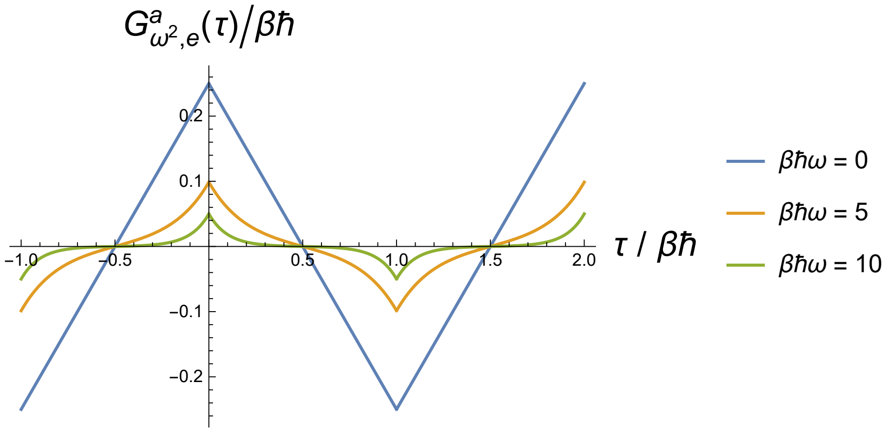
Using (3.208), the analytic continuation of (3.113) gives

$$\begin{aligned} G_{\omega^2,e}(\tau) &= i\, G_{\omega^2}(-i\tau) \\ &= -i\frac{\sin\omega\left[-i\left(\tau-\frac{1}{2}\beta\hbar\right)\right]}{2\omega\cos\left(-i\frac{\omega}{2}\beta\hbar\right)} \\ &= -\frac{\sinh\omega\left(\tau-\frac{1}{2}\beta\hbar\right)}{2\omega\cosh\left(\frac{\omega}{2}\beta\hbar\right)} \end{aligned}$$

which agrees with (3.263).

Since (3.263) is not explicitly anti-periodic, values outside $[0, \beta\hbar)$ must be calculated using the anti-periodicity.

The following graph plots $G_{\omega^2,e}^a(\tau)/\beta\hbar$ for 3 values of $\beta\hbar\omega$ [see "3.08._Code.nb"].



Caution: This graph differs from Kleinert's Fig.3.4 by a negative sign.

For $\omega << 1$, (3.263) becomes [see "3.08._Code.nb"],

$$G^a_{\omega^2,e}(\tau) = -\frac{1}{2}\tau + \frac{1}{4}\beta\hbar + O[\omega^2] \qquad \tau \in [0,\ \beta\hbar) \qquad (3.264)$$

Unlike the periodic version (3.249), there is no singularity at $\omega = 0$.

Similar to the periodic case, we now try to obtain (3.264) directly from the fermionic version of (3.245). Setting $\omega = 0$, we have

$$G^a_{0,e}(\tau) = \frac{1}{\beta\hbar}\sum_{m=-\infty}^{\infty}\frac{1}{\omega_m^{f2}} e^{-i\omega_m^f \tau} \qquad (3.265)$$

Using

$$\omega_{-m}^f = \frac{2\pi}{\beta\hbar}\left(-m+\frac{1}{2}\right) = -\frac{2\pi}{\beta\hbar}\left[(m-1)+\frac{1}{2}\right] = -\omega_{m-1}^f$$

(3.265) becomes

$$\begin{aligned} G^a_{0,e}(\tau) &= \frac{1}{\beta\hbar}\left[\sum_{m=0}^{\infty}\frac{1}{\omega_m^{f2}} e^{-i\omega_m^f\tau} + \sum_{m=1}^{\infty}\frac{1}{\omega_{-m}^{f2}} e^{-i\omega_{-m}^f\tau}\right] \\ &= \frac{1}{\beta\hbar}\left[\sum_{m=0}^{\infty}\frac{1}{\omega_m^{f2}} e^{-i\omega_m^f\tau} + \sum_{m=1}^{\infty}\frac{1}{\omega_{m-1}^{f2}} e^{i\omega_{m-1}^f\tau}\right] \\ &= \frac{1}{\beta\hbar}\left[\sum_{m=0}^{\infty}\frac{1}{\omega_m^{f2}} e^{-i\omega_m^f\tau} + \sum_{m=0}^{\infty}\frac{1}{\omega_m^{f2}} e^{i\omega_m^f\tau}\right] \\ &= \frac{2}{\beta\hbar}\sum_{m=0}^{\infty}\frac{1}{\omega_m^{f2}}\cos(\omega_m^f\tau) \end{aligned} \qquad (3.266a)$$

Caution: the sum $2\sum_{m=0}^{\infty}$ in (3.266a) cannot be converted to the form $\sum_{m=-\infty}^{\infty}$ as given in Kleinert's (3.266) for obvious reasons.

Using

$$\tfrac{1}{2}\beta\hbar\omega_m^f = \pi\left(m+\frac{1}{2}\right)$$

$$\rightarrow \qquad \sin\left[\omega_m^f\left(\tau-\frac{1}{2}\beta\hbar\right)\right] = \sin(\omega_m^f\tau)\cos\pi\left(m+\frac{1}{2}\right) + \cos(\omega_m^f\tau)\sin\pi\left(m+\frac{1}{2}\right)$$
$$= (-)^m\cos(\omega_m^f\tau)$$

(3.266a) becomes

$$G_{0,e}^{a}(\tau) = \frac{2}{\beta\hbar}\sum_{m=0}^{\infty}\frac{(-)^m}{\omega_m^{f2}}\sin\left[\omega_m^f\left(\tau-\frac{1}{2}\beta\hbar\right)\right] \qquad (3.266)$$

$$= \frac{2}{\beta\hbar}\sum_{m=0}^{\infty}\frac{(-)^m}{\omega_m^{f2}}\sum_{n=1,3,5,\dots}\frac{(-)^{(n-1)/2}}{n!}\left[\omega_m^f\left(\tau-\frac{1}{2}\beta\hbar\right)\right]^n$$

$$= \frac{2}{\beta\hbar}\left(\frac{\beta\hbar}{2\pi}\right)^2\sum_{n=1,3,5,\dots}\left(\sum_{m=0}^{\infty}\frac{(-)^m}{\left(m+\frac{1}{2}\right)^{2-n}}\right)\frac{(-)^{(n-1)/2}}{n!}\left[\frac{2\pi}{\beta\hbar}\left(\tau-\frac{1}{2}\beta\hbar\right)\right]^n \qquad (3.267)$$

Consider [see Abramowitz & Stegun, Formulae 23.2.21]

$$\beta(n) = \sum_{m=0}^{\infty}\frac{(-)^m}{(2m+1)^n} \qquad n = 1, 2, 3, \dots$$

$$= \frac{1}{2^n}\sum_{m=0}^{\infty}\frac{(-)^m}{\left(m+\frac{1}{2}\right)^n} \qquad (3.268)$$

which is related to a generalization of the Reimann zeta function called the Hurwitz zeta function

$$\zeta(s, q) \equiv \sum_{m=0}^{\infty}\frac{1}{(m+q)^s} \qquad \text{Re}\, s > 1 \;\;\&\;\; \text{Re}\, q > 0 \qquad (3.269)$$

Indeed,

$$\sum_{m=0}^{\infty}\frac{(-)^m}{(m+q)^s} = \sum_{k=0}^{\infty}\left[\frac{1}{(2k+q)^s} - \frac{1}{(2k+1+q)^s}\right]$$

$$= \frac{1}{2^s}\sum_{k=0}^{\infty}\left[\frac{1}{\left(k+\frac{1}{2}q\right)^s} - \frac{1}{\left[k+\frac{1}{2}(1+q)\right]^s}\right]$$

$$= \frac{1}{2^s}\left[\zeta\left(s, \frac{q}{2}\right) - \zeta\left(s, \frac{1}{2}(1+q)\right)\right] \qquad (3.270)$$

(3.268) thus becomes

$$\beta(n) = \frac{1}{2^{2n}}\left[\zeta\left(n, \frac{1}{4}\right) - \zeta\left(n, \frac{3}{4}\right)\right] \qquad (3.271)$$

Near $s = 1$, we have [see G&R, 9.533.2]

$$\zeta(s, q) = \frac{1}{s-1} - \psi(q) + O[s-1] \qquad (3.272)$$

where the psi function is defined as [G&R 8.360]

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x)$$

Hence, for $n = 1$, (3.271) gives

$$\beta(1) = \frac{1}{2^2}\left[\zeta\left(1, \frac{1}{4}\right) - \zeta\left(1, \frac{3}{4}\right)\right]$$

$$= \frac{1}{2^2}\left[-\psi\left(\frac{1}{4}\right) + \psi\left(\frac{3}{4}\right)\right] \qquad (3.273a)$$

Using [G&R 8.366.4-5]

$$\psi\left(\frac{1}{4}\right) = -\gamma - \frac{\pi}{2} - 3\ln 2 \qquad (3.274)$$

$$\psi\left(\frac{3}{4}\right) = -\gamma + \frac{\pi}{2} - 3\ln 2$$

where

$\gamma = 0.577215 \dots =$ Euler's constant

(3.273a) becomes

$$\beta(1) = \frac{\pi}{4} \tag{3.273}$$

in agreement with [A&S 23.2.30].

Caution: G&R use C to denote Euler's constant and set $\gamma = e^C = 1.781072$ .... [see G&R, page xxviii].

In the form of (3.268), $\beta(-n)$ seems to be divergent. However, the form (3.271) gives finite results. Unfortunately, the claim that

$$\beta(-n) = 0 \qquad \text{for } n = 1, 3, 5, \ldots$$

does not hold numerically [see "3.08._Code.nb"]. Thus, (3.267) cannot be evaluated as described in Kleinert's text.

Periodic Source $j(\tau)$

Using [see (3.246a)],

$$G^p_{\omega^2,e}(\tau, \tau') = \frac{1}{2\,\omega} \sum_{n=-\infty}^{\infty} e^{-\omega\,|\tau-\tau'+n\,\beta\,\hbar|} \tag{3.277}$$

the source term (3.244) can be written as

$$\begin{aligned}
\mathcal{A}^j_e[j] &= -\frac{1}{2\,M} \int_0^{\beta\hbar} d\,\tau \int_0^{\beta\hbar} d\,\tau'\, j(\tau)\, G^p_{\omega^2,e}(\tau, \tau')\, j(\tau') \\
&= -\frac{1}{4\,M\,\omega} \sum_{n=-\infty}^{\infty} \int_0^{\beta\hbar} d\,\tau \int_0^{\beta\hbar} d\,\tau'\, j(\tau)\, j(\tau')\, e^{-\omega\,|\tau-\tau'+n\,\beta\,\hbar|} \\
&= -\frac{1}{4\,M\,\omega} \sum_{n=-\infty}^{\infty} \int_0^{\beta\hbar} d\,\tau \int_{-n\,\beta\hbar}^{(1-n)\,\beta\hbar} d\,\tau''\, j(\tau)\, j(\tau''+n\,\beta\,\hbar)\, e^{-\omega\,|\tau-\tau''|} \\
&= -\frac{1}{4\,M\,\omega} \sum_{n=-\infty}^{\infty} \int_0^{\beta\hbar} d\,\tau \int_{n\,\beta\hbar}^{(1+n)\,\beta\hbar} d\,\tau''\, j(\tau)\, j(\tau''-n\,\beta\,\hbar)\, e^{-\omega\,|\tau-\tau''|}
\end{aligned} \tag{3.276a}$$

For a periodic source,

$$j(\tau'-n\,\beta\,\hbar) = j(\tau) \qquad \forall\; n$$

(3.276a) becomes

$$\begin{aligned}
\mathcal{A}^j_e[j] &= -\frac{1}{4\,M\,\omega} \int_0^{\beta\hbar} d\,\tau \sum_{n=-\infty}^{\infty} \int_{n\,\beta\hbar}^{(n+1)\,\beta\hbar} d\,\tau'\, j(\tau)\, j(\tau')\, e^{-\omega\,|\tau-\tau'|} \\
&= -\frac{1}{4\,M\,\omega} \int_0^{\beta\hbar} d\,\tau \int_{-\infty}^{\infty} d\,\tau'\, j(\tau)\, j(\tau')\, e^{-\omega\,|\tau-\tau'|}
\end{aligned} \tag{3.276}$$

Periodic or Anti-periodic Potential $\Omega(\tau)$

The following is simply the Euclidean version of the Wronski construction discussed in §3.5.

The eq. for the Green function is [c.f. (3.158a)]

$$\left[\,\partial_\tau^2 - \Omega^2(\tau)\,\right] G^{p,a}_{\Omega^2,e}(\tau, \tau') = \delta^{p,a}(\tau-\tau') \tag{3.278}$$

where [c.f. (3.159)]

$$\delta^{p,a}(\tau-\tau') = \sum_{n=-\infty}^{\infty} (\pm)^n\, \delta(\tau-\tau'-n\,\beta\,\hbar) \tag{3.279}$$

Note that except for the defining eq, there is no explicit appearance of t^n in any eq. in §3.5. Furthermore, derivatives of t in them appear homogeneously. Therefore, the Euclidean versions of the results can be obtained by simply replacing t with τ without worrying about factors of $-i$. From (3.166), we get

$$G^{p,a}_{\Omega^2,e}(\tau,\tau') = G_{\Omega^2,e}(\tau,\tau') + \frac{\left[\Delta(\tau,\tau_a)\pm\Delta(\tau_b,\tau)\right]\left[\Delta(\tau_b,\tau')\pm\Delta(\tau',\tau_a)\right]}{\Delta(\tau_b,\tau_a)\,\overline{\Delta}^{p,a}(\tau_a,\tau_b)} \tag{3.280}$$

where [see (3.59)]

$$G_{\Omega^2,e}(\tau,\tau') = -\frac{1}{\Delta(\tau_b,\tau_a)}\left(\Theta(\tau-\tau')\,\Delta(\tau_b,\tau)\,\Delta(\tau',\tau_a) + \Theta(\tau'-\tau)\,\Delta(\tau,\tau_a)\,\Delta(\tau_b,\tau')\right) \tag{3.281}$$

and [see (3.164c-d) & (3.165)]

$$W = \begin{vmatrix} \xi & \eta \\ \dot\xi & \dot\eta \end{vmatrix} = \xi\,\dot\eta - \dot\xi\,\eta \tag{3.282a}$$

$$\Delta(\tau,\tau') = \frac{1}{W}\left[\,\xi(\tau)\,\eta(\tau') - \xi(\tau')\,\eta(\tau)\,\right] \tag{3.282}$$

$$\overline{\Delta}^{p,a}(\tau_a,\tau_b) = 2 \pm \left[\partial_{\tau_b}\Delta(\tau_b,\tau_a) + \partial_{\tau_a}\Delta(\tau_a,\tau_b)\right] \tag{3.283}$$

ω

For a time-independent potential ω, the defining eq. is

$$(-\partial_\tau - \omega)\,G^{p,a}_{\omega,e}(\tau,\tau') = \delta^{p,a}(\tau-\tau') \tag{3.284a}$$

Similar to (3.245-6), we have

$$\begin{aligned}
G^{p}_{\omega,e}(\tau-\tau') &= (-\partial_\tau-\omega)^{-1}\,\delta(\tau-\tau') \\
&= \frac{1}{\beta\hbar}(-\partial_\tau-\omega)^{-1}\sum_{m=-\infty}^{\infty} e^{-i\omega_m(\tau-\tau')} \\
&= \frac{1}{\beta\hbar}\sum_{m=-\infty}^{\infty}\frac{1}{i\,\omega_m-\omega}\,e^{-i\omega_m(\tau-\tau')} \\
&= \sum_{n=-\infty}^{\infty}\int\frac{d\mu}{2\pi}\,e^{i\beta\hbar\mu n}\,\frac{1}{i\mu-\omega}\,e^{-i\mu(\tau-\tau')}
\end{aligned}$$

$$= \sum_{n=-\infty}^{\infty}\begin{cases} e^{-\omega(\tau-\tau'-n\beta\hbar)} & \tau-\tau'-n\beta\hbar>0 \quad \text{C.W.} \\ 0 & \tau-\tau'-n\beta\hbar<0 \quad \text{C.C.W.}\end{cases} \quad \text{contour closed in } \begin{matrix}\text{lower}\\ \text{upper}\end{matrix} \text{ plane}$$

$$= \sum_{n=-\infty}^{\infty}\Theta(\tau-\tau'-n\beta\hbar)\,e^{-\omega|\tau-\tau'-n\beta\hbar|}$$

Since $G^{p}_{\omega,e}$ is periodic, we need consider only the primary interval $\tau\in[0,\ \beta\hbar)$ for which [c.f. (3.248)]

$$G^{p}_{\omega,e}(\tau) = \sum_{n=0}^{\infty} e^{-\omega(\tau+n\beta\hbar)}$$

$$= e^{-\omega\tau}\,\frac{1}{1-e^{-\beta\hbar\omega}} = e^{-\omega\tau}\left(1+n^{b}_{\omega}\right) \tag{3.284}$$

$$= e^{-\omega\tau}\,\frac{e^{\beta\hbar\omega/2}}{e^{\beta\hbar\omega/2}-e^{-\beta\hbar\omega/2}} = e^{-\omega(\tau-\beta\hbar/2)}\,\frac{1}{2\sinh(\beta\hbar\omega/2)} \tag{3.284a}$$

Similarly, following (3.263), we have

$$\begin{aligned}
G^{a}_{\omega,e}(\tau) &= \frac{1}{\beta\hbar}\sum_{m=-\infty}^{\infty}\frac{1}{i\,\omega^f_m-\omega}\,e^{-i\omega^f_m\tau} \\
&= \sum_{n=-\infty}^{\infty}(-)^n\,\Theta(\tau-n\beta\hbar)\,e^{-\omega|\tau-n\beta\hbar|}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} (-)^n e^{-\omega(\tau + n\beta\hbar)} \qquad \tau \in [0, \beta\hbar)$$

$$= e^{-\omega\tau} \frac{1}{1 + e^{-\beta\hbar\omega}} = e^{-\omega\tau}(1 - n_\omega^f) \qquad (3.285)$$

$$= e^{-\omega\tau} \frac{e^{\beta\hbar\omega/2}}{e^{\beta\hbar\omega/2} + e^{-\beta\hbar\omega/2}} = e^{-\omega(\tau - \beta\hbar/2)} \frac{1}{2\cosh(\beta\hbar\omega/2)} \qquad (3.285a)$$

$\Omega(\tau)$

For a time-dependent potential $\Omega(\tau)$, the defining eq. is

$$[-\partial_\tau - \Omega(\tau)]\, G_{\omega,e}^{p,a}(\tau, \tau') = \delta^{p,a}(\tau - \tau') \qquad (3.287)$$

The analytic continuation of (3.120) is

$$G_\Omega(\tau, \tau') = \bar{\Theta}(\tau - \tau') \exp\left[-\int_{\tau'}^{\tau} d\tau'' \,\Omega(\tau'')\right] \qquad (3.288)$$

The periodic & anti-periodic versions are obtained by superposition:

$$G_{\omega,e}^{p,a}(\tau, \tau') = \sum_{n=-\infty}^{\infty} (\pm)^n G_\Omega(\tau - \tau' + n\beta\hbar) \qquad \tau - \tau' \in [0, \beta\hbar)$$

$$= \sum_{n=0}^{\infty} (\pm)^n \exp\left[-\int_{\tau'}^{\tau + n\beta\hbar} d\tau'' \,\Omega(\tau'')\right] \qquad (3.289)$$

which reduces to (3.284-5) for $\Omega(\tau) = \omega$.