3.9. Lattice Green Function

The time-sliced version of the defining eq. (3.207) is

$$\left(-\nabla\overline{\nabla}+\omega^2\right) G_{\omega^2,e}(\tau_n,\ \tau_{n'}) = \delta_{n\,n'} \qquad (3.290a)$$

$G_{\omega^2,e}(\tau,\ \tau')$ can be expressed in terms of its Fourier transform as

$$G_{\omega^2,e}(\tau,\ \tau') = \int_{-\infty}^{\infty} \frac{d\,\omega'}{2\,\pi}\, e^{-i\,\omega'(\tau-\tau')}\, G_{\omega^2,e}(\omega') \qquad (3.290b)$$

which is valid also for the time-sliced case.

Using the techniques of §2.2.3, we put (3.290b) into (3.290a) to get

$$\left[\,\Omega(\omega')\,\overline{\Omega}(\omega') + \omega^2\,\right] G_{\omega^2,e}(\omega') = 1 \qquad (3.290c)$$

where [see 2.106-8)]

$$\nabla\, f(\omega') = i\,\Omega(\omega')\, f(\omega')$$

$$\Omega(\omega') = \frac{e^{i\,\omega'\,\epsilon} - 1}{\epsilon} \qquad \overline{\Omega}(\omega') = \Omega^*(\omega')$$

$$\Omega(\omega')\,\overline{\Omega}(\omega') = \frac{2}{\epsilon^2}\left[1 - \cos(\omega'\,\epsilon)\right] \qquad (3.290d)$$

(3.290b) thus becomes

$$G_{\omega^2,e}(\tau,\ \tau') = \int \frac{d\,\omega'}{2\,\pi}\, e^{-i\,\omega'(\tau-\tau')}\, \frac{\epsilon^2}{2\,[1\ -\ \cos(\omega'\,\epsilon)\,]\ +\ \epsilon^2\,\omega^2} \qquad (3.292)$$

Note that (3.292) can also be interpreted as the spectral representation of the operator $\left(-\nabla\overline{\nabla}+\omega^2\right)^{-1}$ with respect to the orthonormal basis functions

$$\phi_{\omega'}(\tau) = \frac{1}{\sqrt{2\,\pi}}\, e^{i\,\omega'\,\tau} \qquad (3.292a)$$

The poles in (3.292) are given by

$$2\,[1\ -\ \cos(\omega'\,\epsilon)\,]\ +\ \epsilon^2\,\omega^2 = 0$$

$$\rightarrow \qquad 4\sin^2\frac{\omega'\,\epsilon}{2} + \epsilon^2\,\omega^2 = 0$$

$$\sin\frac{\omega'\,\epsilon}{2} = \pm i\,\frac{\epsilon\,\omega}{2} \qquad (3.292b)$$

Introducing the auxiliary frequency $\tilde{\omega}_e$ by [see (2.481-a)]

$$\sinh\frac{\epsilon\,\tilde{\omega}_e}{2} = \frac{\epsilon\,\omega}{2} \qquad (3.291)$$

(3.292b) becomes

$$\sin\frac{\omega'\,\epsilon}{2} = \pm i\sinh\frac{\epsilon\,\tilde{\omega}_e}{2}$$

$$\rightarrow \qquad \omega' = \pm i\,\tilde{\omega}_e \qquad (3.292c)$$

Using

$$\mathrm{Res}\left[\frac{1}{f(z)};\, z = z_0\right] = \frac{1}{f'(z_0)} \qquad \text{where} \quad f(z_0) = 0$$

we have

$$\mathrm{Res}\left[\frac{e^{-i\,\omega'(\tau-\tau')}}{2\,[1\ -\ \cos(\omega'\,\epsilon)\,]\ +\ \epsilon^2\,\omega^2};\ \omega' = \pm i\,\tilde{\omega}_e\right] = \left.\frac{e^{-i\,\omega'(\tau-\tau')}}{2\,\epsilon\sin(\omega'\,\epsilon)}\right|_{\omega' = \pm i\,\tilde{\omega}_e}$$

(3.292) thus becomes

$$G_{\omega^2,e}(\tau,\tau') = \begin{cases} -i \left.\dfrac{\epsilon^2 e^{-i\omega'(\tau-\tau')}}{2\epsilon\sin(\omega'\epsilon)}\right|_{\omega'=-i\tilde{\omega}_e} & \tau>\tau' \\ i \left.\dfrac{\epsilon^2 e^{-i\omega'(\tau-\tau')}}{2\epsilon\sin(\omega'\epsilon)}\right|_{\omega'=i\tilde{\omega}_e} & \tau<\tau' \end{cases}$$

$$= \frac{\epsilon}{2\sinh\epsilon\tilde{\omega}_e} e^{-\tilde{\omega}_e|\tau-\tau'|}$$

$$= \frac{\sinh\frac{\epsilon\tilde{\omega}_e}{2}}{\omega\sinh\epsilon\tilde{\omega}_e} e^{-\tilde{\omega}_e|\tau-\tau'|}$$

$$= \frac{1}{2\omega\cosh\frac{\epsilon\tilde{\omega}_e}{2}} e^{-\tilde{\omega}_e|\tau-\tau'|} \qquad (3.290)$$

The periodic Green function is [c.f. (3.245) & (3.246a)]

$$G^p_{\omega^2,e}(\tau,\tau') = \frac{1}{\beta\hbar}\sum_{m=-\infty}^{\infty}\frac{\epsilon^2}{2[1-\cos(\omega_m\epsilon)]+\epsilon^2\omega^2} e^{-i\omega_m(\tau-\tau')}$$

$$= \frac{1}{2\omega\cosh\frac{\epsilon\tilde{\omega}_e}{2}}\sum_{n=-\infty}^{\infty}\exp[\tilde{\omega}_e\mid\tau-\tau'+n\beta\hbar\mid] \qquad (3.294a)$$

The sum was already evaluated in (3.248). Hence

$$G^p_{\omega^2,e}(\tau) = \frac{1}{2\omega\cosh\frac{\epsilon\tilde{\omega}_e}{2}}\frac{\cosh\omega\left(\frac{1}{2}\beta\hbar-\tau\right)}{\sinh\left(\frac{1}{2}\beta\hbar\tilde{\omega}_e\right)} \qquad \tau\in[0,\ \beta\hbar) \qquad (3.294)$$

Kleinert's Version

Below is a derivation following Kleinert's description in the paragraph between (3.293) & (3.294).

Using

$$\int_0^\infty d\,s\,e^{-a\,s} = \frac{1}{a}$$

we can write (3.292) as

$$G_{\omega^2,e}(\tau,\tau') = \int_0^\infty d\,s\int_{-\infty}^{\infty}\frac{d\,\omega'}{2\pi}e^{-i\omega'(\tau-\tau')} \qquad (3.293)$$

$$\times\exp\left[-s\,\frac{2[1-\cos(\omega'\epsilon)]+\epsilon^2\omega^2}{\epsilon^2}\right]$$

The ω' integral is

$$\mathcal{I} = \int_{-\infty}^{\infty}\frac{d\,\omega'}{2\pi}\exp\left[-i\omega'(\tau-\tau')+2\frac{s}{\epsilon^2}\cos(\omega'\epsilon)\right]$$

$$= \int_0^\infty\frac{d\,\omega'}{2\pi}\left[e^{-i\omega'(\tau-\tau')}+e^{i\omega'(\tau-\tau')}\right]\exp\left[2\frac{s}{\epsilon^2}\cos(\omega'\epsilon)\right]$$

$$= \int_0^\infty\frac{d\,\omega'}{\pi}\cos[\omega'(\tau-\tau')]\exp\left[2\frac{s}{\epsilon^2}\cos(\omega'\epsilon)\right] \qquad (3.293a)$$

In order to use [see A&S, 9.6.24]

$$K_\nu(z) = \int_0^\infty d\,t\cosh(\nu\,t)\,e^{-z\cosh t} \qquad (3.293b)$$

we set

$$i\,x = \omega'\epsilon$$

so that (3.293a) becomes

$$\mathcal{I} = \frac{i}{\epsilon}\int_0^\infty \frac{dx}{\pi}\cosh\left[x\left(\frac{\tau-\tau'}{\epsilon}\right)\right]\exp\left(2\,\frac{s}{\epsilon^2}\cosh x\right)$$

$$= \frac{i}{\epsilon\,\pi}K_{(\tau-\tau')/\epsilon}\left(-2\,\frac{s}{\epsilon^2}\right) \qquad (3.293c)$$

(3.293) thus becomes

$$G_{\omega^2,e}(\tau,\,\tau') = \frac{i}{\epsilon\,\pi}\int_0^\infty ds\,K_{(\tau-\tau')/\epsilon}\left(-2\,\frac{s}{\epsilon^2}\right)\exp\left[-s\,\frac{2+\epsilon^2\,\omega^2}{\epsilon^2}\right] \qquad (3.293d)$$

In order to use [see G&R, 6.611.3]

$$\int_0^\infty dx\,e^{-\alpha x}K_\nu(\beta x)$$

$$= \frac{\pi}{2\sqrt{\alpha^2-\beta^2}\,\sin\pi\nu}\left[\left(\frac{\alpha+\sqrt{\alpha^2-\beta^2}}{\beta}\right)^\nu - \left(\frac{\beta}{\alpha+\sqrt{\alpha^2-\beta^2}}\right)^\nu\right]$$

$$= \frac{\pi}{2\sqrt{\alpha^2-\beta^2}\,\sin\pi\nu}\left[\left(\frac{\alpha+\sqrt{\alpha^2-\beta^2}}{\beta}\right)^\nu - \left(\frac{\alpha-\sqrt{\alpha^2-\beta^2}}{\beta}\right)^\nu\right] \qquad (3.293e)$$

we set

$$\alpha = \frac{2+\epsilon^2\,\omega^2}{\epsilon^2} \qquad \beta = -\frac{2}{\epsilon^2} \qquad \nu = \frac{\tau-\tau'}{\epsilon} \qquad (3.293f)$$

so that

$$\sqrt{\alpha^2-\beta^2} = \frac{1}{\epsilon^2}\sqrt{\left(2+\epsilon^2\,\omega^2\right)^2-4} = \frac{2}{\epsilon^2}\sinh\left(\epsilon\,\tilde{\omega}_e\right) \qquad (3.293g)$$

where we introduced the auxiliary frequency $\tilde{\omega}_e$ by [see (2.481-a)]

$$\sinh\frac{\epsilon\,\tilde{\omega}_e}{2} = \frac{\epsilon\,\omega}{2} \qquad \cosh(\epsilon\,\tilde{\omega}_e) = \frac{1}{2}\left(2+\epsilon^2\,\omega^2\right) \qquad (3.291)$$

$$\to \qquad \alpha = \frac{2}{\epsilon^2}\cosh(\epsilon\,\tilde{\omega}_e) \qquad (3.293h)$$

(3.293d) thus becomes

$$G_{\omega^2,e}(\tau,\,\tau') = \frac{i}{\epsilon\,\pi}\,\frac{\epsilon^2\,\pi}{4\sinh\left(\epsilon\,\tilde{\omega}_e\right)\sin\left(\pi\,\frac{\tau-\tau'}{\epsilon}\right)}$$

$$\times\left\{\left[-\cosh(\epsilon\,\tilde{\omega}_e)-\sinh(\epsilon\,\tilde{\omega}_e)\right]^{(\tau-\tau')/\epsilon} - \left[-\cosh(\epsilon\,\tilde{\omega}_e)+\sinh(\epsilon\,\tilde{\omega}_e)\right]^{(\tau-\tau')/\epsilon}\right\}$$

$$= \frac{i\,\epsilon}{4\sinh\left(\epsilon\,\tilde{\omega}_e\right)\sin\left(\pi\,\frac{\tau-\tau'}{\epsilon}\right)}\,(-1)^{(\tau-\tau')/\epsilon}\left[e^{\tilde{\omega}_e(\tau-\tau')} - e^{-\tilde{\omega}_e(\tau-\tau')}\right] \qquad (3.293i)$$

Setting

$$\tau-\tau' = \left(n+\frac{1}{2}\right)\epsilon = \frac{2n+1}{2N+1}\,\beta\,\hbar \qquad n = 0,\,1,\,\ldots,\,N$$

$$= \frac{1}{2}\,\epsilon,\;\frac{3}{2}\,\epsilon,\;\ldots,\;\left(N+\frac{1}{2}\right)\epsilon = \beta\,\hbar$$

$$\to \qquad \sin\left(\pi\,\frac{\tau-\tau'}{\epsilon}\right) = \sin\left[\pi\left(n+\frac{1}{2}\right)\right] = (-)^n = \frac{1}{i}\,(-1)^{(\tau-\tau')/\epsilon}$$

$$\therefore \qquad G_{\omega^2,e}(\tau,\,\tau') = -\frac{\epsilon}{4\sinh\left(\epsilon\,\tilde{\omega}_e\right)}\left[e^{\tilde{\omega}_e(\tau-\tau')} - e^{-\tilde{\omega}_e(\tau-\tau')}\right]$$

$$= -\frac{\epsilon}{2\sinh(\epsilon\,\tilde{\omega}_e)} \sinh[\tilde{\omega}_e\,(\tau - \tau')]$$

$$= -\frac{\sinh\frac{\epsilon\,\tilde{\omega}_e}{2}}{\omega\sinh(\epsilon\,\tilde{\omega}_e)} \sinh[\tilde{\omega}_e\,(\tau - \tau')] \qquad \text{[(3.291) used.]}$$

$$= -\frac{1}{2\,\omega\cosh\left(\frac{1}{2}\,\epsilon\,\tilde{\omega}_e\right)} \sinh[\tilde{\omega}_e\,(\tau - \tau')] \qquad (3.293j)$$

For the causal Green function, the case $\tau < \tau'$ is obtained from (3.293j) by the switch $\tau \leftrightarrow \tau'$. Hence,

$$G_{\omega^2,e}(\tau,\ \tau') = -\frac{1}{2\,\omega\cosh\left(\frac{1}{2}\,\epsilon\,\tilde{\omega}_e\right)} \sinh[\tilde{\omega}_e\ |\ \tau - \tau'\ |] \qquad (3.290)$$

Unfortunately, (3.290) is not acceptible since it diverges as $|\ \tau - \tau'\ | \to \infty$.