3.11. Correlation Functions of Charged Particle in Magnetic Field and Harmonic Potential

As shown in §2.19, we can define a 2-D analog of (3.237)

$$\boldsymbol{G}^{p}_{\omega^2,B}(\tau,\ \tau') = \boldsymbol{D}^{-1}_{\omega^2,B}(\tau-\tau') \tag{3.318a}$$

where [see (2.691)]

$$\boldsymbol{D}_{\omega,B}(\tau) = \begin{pmatrix} -\partial_\tau^2 + \omega^2 - \omega_B^2 & -2\,i\,\omega_B\,\partial_\tau \\ 2\,i\,\omega_B\,\partial_\tau & -\partial_\tau^2 + \omega^2 - \omega_B^2 \end{pmatrix} \tag{3.318b}$$

with Fourier components [see (2.694)]

$$\tilde{\boldsymbol{D}}_{\omega,B}(\omega_m) = \begin{pmatrix} \omega_m^2 + \omega^2 - \omega_B^2 & -2\,\omega_B\,\omega_m \\ 2\,\omega_B\,\omega_m & \omega_m^2 + \omega^2 - \omega_B^2 \end{pmatrix} \tag{3.318c}$$

Using the imaginary time version of (3.310), (3.318a) becomes

$$\boldsymbol{G}^{(2)}_{\omega^2,B}(\tau,\ \tau') = \frac{\hbar}{M}\,\boldsymbol{D}^{-1}_{\omega^2,B}(\tau,\ \tau') \tag{3.318}$$

Introducing the Fourier components of $\tilde{\boldsymbol{G}}^{(2)}_{\omega^2,B}(\omega_m)$ as

$$\boldsymbol{G}^{(2)}_{\omega^2,B}(\tau,\ \tau') = \frac{1}{\beta\,\hbar}\sum_{m=-\infty}^{\infty} \tilde{\boldsymbol{G}}^{(2)}_{\omega^2,B}(\omega_m)\,e^{-i\,\omega_m(\tau-\tau')} \tag{3.319}$$

(3.318) implies

$$\begin{aligned}\tilde{\boldsymbol{G}}^{(2)}_{\omega^2,B}(\omega_m) &= \frac{\hbar}{M}\,\tilde{\boldsymbol{D}}^{-1}_{\omega,B}(\omega_m) \\ &= \frac{\hbar}{M\,\mathbb{D}(\omega_m)}\begin{pmatrix} \omega_m^2 + \omega^2 - \omega_B^2 & 2\,\omega_B\,\omega_m \\ -2\,\omega_B\,\omega_m & \omega_m^2 + \omega^2 - \omega_B^2 \end{pmatrix}\end{aligned} \tag{3.320}$$

where [see (2.696-7)]

$$\begin{aligned}\mathbb{D}(\omega_m) &= \det\tilde{\boldsymbol{D}}_{\omega,B}(\omega_m) \\ &= \left(\omega_m^2 + \omega_+^2\right)\left(\omega_m^2 + \omega_-^2\right)\end{aligned} \tag{3.320a}$$

with

$$\omega_\pm = \omega \pm \omega_B \tag{3.320b}$$

Using

$$\begin{aligned}\omega_+^2 + \omega_-^2 &= (\omega+\omega_B)^2 + (\omega-\omega_B)^2 = 2\left(\omega^2 + \omega_B^2\right) \\ \omega_+^2 - \omega_-^2 &= (\omega+\omega_B)^2 - (\omega-\omega_B)^2 = 4\,\omega\,\omega_B\end{aligned}$$

we have

$$\omega^2 = \frac{1}{2}\left(\omega_+^2 + \omega_-^2\right) - \omega_B^2 \qquad 4\,\omega_B = \frac{1}{\omega}\left(\omega_+^2 - \omega_-^2\right)$$

so that

$$\begin{aligned}\omega_m^2 + \omega^2 - \omega_B^2 &= \omega_m^2 + \frac{1}{2}\left(\omega_+^2 + \omega_-^2\right) - 2\,\omega_B^2 \\ &= \frac{1}{2}\left[\left(\omega_m^2 + \omega_+^2\right) + \left(\omega_m^2 + \omega_-^2\right) - 4\,\omega_B^2\right] \\ &= \frac{1}{2}\left[\left(\omega_m^2 + \omega_+^2\right) + \left(\omega_m^2 + \omega_-^2\right) - \frac{\omega_B}{\omega}\left(\omega_+^2 - \omega_-^2\right)\right] \\ &= \frac{1}{2}\left\{\left(\omega_m^2 + \omega_+^2\right) + \left(\omega_m^2 + \omega_-^2\right) - \frac{\omega_B}{\omega}\left[\left(\omega_m^2 + \omega_+^2\right) - \left(\omega_m^2 + \omega_-^2\right)\right]\right\}\end{aligned}$$

The diagonal elements of (3.320) thus become

$$\frac{\hbar\left(\omega_m^2 + \omega^2 - \omega_B^2\right)}{M\left(\omega_m^2 + \omega_+^2\right)\left(\omega_m^2 + \omega_-^2\right)}$$

$$= \frac{\hbar}{2M}\left[\frac{1}{\omega_m^2+\omega_+^2}+\frac{1}{\omega_m^2+\omega_-^2}+\frac{\omega_B}{\omega}\left(\frac{1}{\omega_m^2+\omega_+^2}-\frac{1}{\omega_m^2+\omega_-^2}\right)\right] \tag{3.321}$$

$$= \frac{\hbar}{2M\omega}\left(\frac{\omega_+}{\omega_m^2+\omega_+^2}+\frac{\omega_-}{\omega_m^2+\omega_-^2}\right)$$

Recall

$$G^p_{\omega^2,e}(\tau-\tau') = \frac{1}{\beta\hbar}\sum_{m=-\infty}^{\infty}\frac{1}{\omega_m^2+\omega^2}e^{-i\omega_m(\tau-\tau')} \tag{3.245}$$

implies

$$G^p_{\omega^2,e}(\tau) = \frac{1}{2\omega}\frac{\cosh\omega\left(\frac{1}{2}\beta\hbar-\tau\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} \qquad \tau\in[0,\ \beta\hbar) \tag{3.248}$$

$$= \frac{1}{2\omega}\frac{\cosh\omega\left(\frac{1}{2}\beta\hbar-|\tau|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} \qquad \tau\in[-\beta\hbar,\ \beta\hbar)$$

The diagonal elements of $\boldsymbol{G}^{(2)}_{\omega^2,B}(\tau,\ \tau')$ are therefore

$$G^{(2)}_{\omega^2,B,xx}(\tau,\ \tau') = G^{(2)}_{\omega^2,B,yy}(\tau,\ \tau')$$

$$= \frac{\hbar}{2M\omega}\left[\omega_+ G^p_{\omega_+^2,e}(\tau-\tau')+\omega_- G^p_{\omega_-^2,e}(\tau-\tau')\right]$$

$$= \frac{\hbar}{4M\omega}\left[\frac{\cosh\omega_+\left(\frac{1}{2}\beta\hbar-|\tau-\tau'|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega_+\right)}+\frac{\cosh\omega_-\left(\frac{1}{2}\beta\hbar-|\tau-\tau'|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega_-\right)}\right] \tag{3.322}$$

Similarly, using

$$\omega_B = \frac{1}{4\omega}\left[\left(\omega_m^2+\omega_+^2\right)-\left(\omega_m^2+\omega_-^2\right)\right]$$

the off-diagonal elements of (3.320) are

$$\mp\frac{2\hbar\omega_B\omega_m}{M\left(\omega_m^2+\omega_+^2\right)\left(\omega_m^2+\omega_-^2\right)} = \mp\frac{\hbar\omega_m}{2M\omega}\left(\frac{1}{\omega_m^2+\omega_+^2}-\frac{1}{\omega_m^2+\omega_-^2}\right)\Bigg] \tag{3.323}$$

From (3.245), we have

$$i\partial_\tau G^p_{\omega^2,e}(\tau-\tau') = \frac{1}{\beta\hbar}\sum_{m=-\infty}^{\infty}\frac{\omega_m}{\omega_m^2+\omega^2}e^{-i\omega_m(\tau-\tau')} \tag{3.323a}$$

Therefore, the off-diagonal elements of $\boldsymbol{G}^{(2)}_{\omega^2,B}(\tau,\ \tau')$ are

$$G^{(2)}_{\omega^2,B,xy}(\tau,\ \tau') = -G^{(2)}_{\omega^2,B,yx}(\tau,\ \tau')$$

$$= \frac{\hbar}{2M\omega}\left[i\partial_\tau G^p_{\omega_+^2,e}(\tau-\tau')-i\partial_\tau G^p_{\omega_-^2,e}(\tau-\tau')\right]$$

$$= \frac{\hbar}{2M\omega}i\partial_\tau\left[\frac{\cosh\omega_+\left(\frac{1}{2}\beta\hbar-|\tau-\tau'|\right)}{2\omega_+\sinh\left(\frac{1}{2}\beta\hbar\omega_+\right)}-\frac{\cosh\omega_-\left(\frac{1}{2}\beta\hbar-|\tau-\tau'|\right)}{2\omega_-\sinh\left(\frac{1}{2}\beta\hbar\omega_-\right)}\right] \tag{3.324}$$

$$= \frac{\hbar\epsilon(\tau-\tau')}{2M\omega i}\left[\frac{\sinh\omega_+\left(\frac{1}{2}\beta\hbar-|\tau-\tau'|\right)}{2\omega_+\sinh\left(\frac{1}{2}\beta\hbar\omega_+\right)}-\frac{\sinh\omega_-\left(\frac{1}{2}\beta\hbar-|\tau-\tau'|\right)}{2\omega_-\sinh\left(\frac{1}{2}\beta\hbar\omega_-\right)}\right] \tag{3.325}$$

where we’ve used

$$\partial_\tau\cosh\omega(a-|\tau|) = \begin{cases}-\omega\sinh\omega(a-\tau) & \tau>0\\ \omega\sinh\omega(a+\tau) & \tau<0\end{cases}$$

$$= -\epsilon(\tau)\sinh\omega(a-|\tau|)$$

with

$$\epsilon(\tau) = \begin{cases} 1 & \tau > 0 \\ -1 & \tau < 0 \end{cases}$$

For a free particle in a magnetic field, $\omega = 0$ so that (3.320b) becomes $\omega_\pm = \omega_B$.

Using

$$\omega = \frac{1}{2}(\omega_+ - \omega_-)$$

(3.322) gives

$$G^{(2)}_{0,B,xx}(\tau,\ \tau') = G^{(2)}_{0,B,yy}(\tau,\ \tau')$$

$$= \lim_{\omega_+,\,\omega_- \to \omega_B} \frac{\hbar}{M(\omega_+ - \omega_-)}\left[\omega_+ G^p_{\omega_+^2,e}(\tau - \tau') + \omega_- G^p_{\omega_-^2,e}(\tau - \tau')\right]$$

$$= \frac{\hbar}{M}\partial_{\omega_B}\left[\omega_B \partial_{\omega_B} G^p_{\omega_B^2,e}(\tau - \tau')\right]$$

$$= \frac{\hbar}{2M}\partial_{\omega_B}\frac{\cosh\omega_B\left(\frac{1}{2}\beta\hbar - |\tau - \tau'|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega_B\right)} \qquad (3.27a)$$

$$= \frac{\hbar}{2M\sinh^2\left(\frac{1}{2}\beta\hbar\omega_B\right)}\left[-\frac{1}{2}\beta\hbar\cosh\omega_B\left(\frac{1}{2}\beta\hbar - |\tau - \tau'|\right)\cosh\left(\frac{1}{2}\beta\hbar\omega_B\right)\right.$$
$$\left. + \left(\frac{1}{2}\beta\hbar - |\tau - \tau'|\right)\sinh\omega_B\left(\frac{1}{2}\beta\hbar - |\tau - \tau'|\right)\sinh\left(\frac{1}{2}\beta\hbar\omega_B\right)\right]$$

$$= \frac{\hbar}{4M\sinh^2\left(\frac{1}{2}\beta\hbar\omega_B\right)}\left\{-\frac{1}{2}\beta\hbar\left[\cosh\omega_B\left(\beta\hbar - |\tau - \tau'|\right) + \cosh\left(\omega_B|\tau - \tau'|\right)\right]\right.$$
$$\left. + \left(\frac{1}{2}\beta\hbar - |\tau - \tau'|\right)\left[\cosh\omega_B\left(\beta\hbar - |\tau - \tau'|\right) - \cosh\left(\omega_B|\tau - \tau'|\right)\right]\right\}$$

$$= \frac{\hbar}{4M\sinh^2\left(\frac{1}{2}\beta\hbar\omega_B\right)}\left\{-\left(\beta\hbar - |\tau - \tau'|\right)\cosh\left(\omega_B|\tau - \tau'|\right)\right. \qquad (3.327b)$$
$$\left. - |\tau - \tau'|\left[\cosh\omega_B\left(\beta\hbar - |\tau - \tau'|\right)\right]\right\}$$

Similarly, the off-diagonal elements of $\mathbf{G}^{(2)}_{0,B}(\tau,\ \tau')$ are

$$G^{(2)}_{0,B,xy}(\tau,\ \tau') = -G^{(2)}_{0,B,yx}(\tau,\ \tau')$$

$$= \lim_{\omega_+,\,\omega_- \to \omega_B} \frac{\hbar}{M(\omega_+ - \omega_-)}\left[i\partial_\tau G^p_{\omega_+^2,e}(\tau - \tau') - i\partial_\tau G^p_{\omega_-^2,e}(\tau - \tau')\right]$$

$$= \frac{\hbar}{M}i\partial_{\omega_B}\partial_\tau G^p_{\omega_B^2,e}(\tau - \tau') \qquad (3.328a)$$