3.12. Correlation Functions in Canonical Path Integral

Correlation functions of both position & momentum variables are defined as

$$G_{\omega^2}^{(m,n)}(\tau_1, \ldots, \tau_m; \tau_1, \ldots, \tau_n) = \langle x(\tau_1), \ldots x(\tau_m)\, p(\tau_1), \ldots, p(\tau_n) \rangle$$

$$= \frac{1}{Z} \oint \mathcal{D}\, x(\tau) \int \frac{\mathcal{D}\, p(\tau)}{2\pi\hbar} x(\tau_1), \ldots x(\tau_m)\, p(\tau_1), \ldots, p(\tau_n) \exp\left(-\frac{1}{\hbar} \mathcal{A}_e\right) \qquad (3.329)$$

They can be calculated from a generalization of the generating functional (3.298a) to

$$Z[j, k] = \oint \mathcal{D}\, x \int \frac{\mathcal{D}\, p}{2\pi\hbar} \exp\left(-\frac{1}{\hbar} \mathcal{A}_e[j, k]\right) \qquad (3.330)$$

where

$$\mathcal{A}_e[j, k] = \mathcal{A}_e + \mathcal{A}_e^j + \mathcal{A}_e^k \qquad (3.330a)$$

$$\mathcal{A}_e^j = -\int_0^{\beta\hbar} d\tau\, j(\tau)\, x(\tau) \qquad (3.330b)$$

$$\mathcal{A}_e^k = -\int_0^{\beta\hbar} d\tau\, k(\tau)\, p(\tau) = -i M \int_0^{\beta\hbar} d\tau\, k(\tau)\, \dot{x}(\tau) \qquad (3.330c)$$

3.12.1. Harmonic Correlation Functions

For the harmonic oscillator, the generating function is denoted as $Z_\omega[j, k]$ with

$$\mathcal{A}_e[j, k] = \int_0^{\beta\hbar} d\tau \left[-i p \dot{x} + \frac{1}{2M} p^2 + \frac{1}{2} M \omega^2 x^2 - j(\tau)\, x - k(\tau)\, p \right] \qquad (3.331)$$

In terms of the phase space vectors

$$\mathbf{V}(\tau) = \begin{pmatrix} x(\tau) \\ p(\tau) \end{pmatrix} \qquad \mathbf{J}(\tau) = \begin{pmatrix} j(\tau) \\ k(\tau) \end{pmatrix} \qquad (3.332a)$$

(3.331) can be written in matrix form as

$$\mathcal{A}_e[\mathbf{J}] = \int_0^{\beta\hbar} d\tau \left[\frac{1}{2} \int_0^{\beta\hbar} d\tau'\, \mathbf{V}^T(\tau)\, \mathbf{D}_{\omega^2,e}(\tau, \tau')\, \mathbf{V}(\tau') - \mathbf{V}^T(\tau)\, \mathbf{J}(\tau) \right] \qquad (3.332)$$

where

$$\mathbf{D}_{\omega^2,e}(\tau, \tau') = \begin{pmatrix} M\omega^2 & i\partial_\tau \\ -i\partial_\tau & \frac{1}{M} \end{pmatrix} \delta(\tau - \tau') \qquad \tau - \tau' \in [0, \beta\hbar) \quad (3.333)$$

$$= \delta(\tau - \tau') \begin{pmatrix} M\omega^2 & i\partial_{\tau'} \\ -i\partial_{\tau'} & \frac{1}{M} \end{pmatrix} \qquad (3.333a)$$

is a Hermitian operator. Since $\mathbf{V}$ & $\mathbf{J}$ are real, so is $\mathcal{A}_e[\mathbf{J}]$, as it should be.

The equivalence of (3.333) & (3.333a) can be proved by showing, by partial integrations, that they give the same

$$\int d\tau \int d\tau'\, f(\tau)\, \mathbf{D}_{\omega^2,e}(\tau, \tau')\, g(\tau') \qquad \forall\, f, g$$

That (3.332) is equivalent to (3.331) is seen as follows:

$$\mathbf{V}^T(\tau)\, \mathbf{J}(\tau) = \begin{pmatrix} x(\tau) & p(\tau) \end{pmatrix} \begin{pmatrix} j(\tau) \\ k(\tau) \end{pmatrix} = j(\tau)\, x(\tau) + k(\tau)\, p(\tau) \qquad (3.333b)$$

and

$$\mathbf{V}^T(\tau)\, \mathbf{D}_{\omega^2,e}(\tau, \tau')\, \mathbf{V}(\tau')$$

$$= \begin{pmatrix} x(\tau) & p(\tau) \end{pmatrix} \begin{pmatrix} M\omega^2\, \delta(\tau-\tau')\, x(\tau') + i\,\partial_\tau \delta(\tau-\tau')\, p(\tau') \\ -i\,\partial_\tau \delta(\tau-\tau')\, x(\tau') + \frac{1}{M}\, \delta(\tau-\tau')\, p(\tau') \end{pmatrix}$$

$$= M\omega^2\, \delta(\tau-\tau')\, x(\tau)\, x(\tau') + i\, x(\tau)\, \partial_\tau \delta(\tau-\tau')\, p(\tau')$$
$$-i\, p(\tau)\, \partial_\tau \delta(\tau-\tau')\, x(\tau') + \frac{1}{M}\, \delta(\tau-\tau')\, p(\tau)\, p(\tau')$$

Using

$$\int_0^{\beta\hbar} d\tau\; f(\tau)\, \partial_\tau \delta(\tau-\tau') = -\dot{f}(\tau')$$

we have

$$\int_0^{\beta\hbar} d\tau' \int_0^{\beta\hbar} d\tau\; \boldsymbol{V}(\tau)^T\, \boldsymbol{D}_{\omega^2,e}(\tau,\, \tau')\, \boldsymbol{V}(\tau')$$
$$= \int_0^{\beta\hbar} d\tau' \left[M\omega^2\, x^2(\tau') - i\,\dot{x}(\tau')\, p(\tau') + i\,\dot{p}(\tau')\, x(\tau') + \frac{1}{M}\, p^2(\tau') \right]$$
$$= \int_0^{\beta\hbar} d\tau' \left[M\omega^2\, x^2(\tau') - 2\, i\,\dot{x}(\tau')\, p(\tau') + \frac{1}{M}\, p^2(\tau') \right] \qquad (3.333c)$$

Putting (3.333b-c) into (3.332) thus reproduces (3.331).

As in (3.318a), we have

$$\mathbb{G}^p_{\omega^2,e}(\tau,\, \tau') = \boldsymbol{D}^{-1}_{\omega^2,e}(\tau-\tau')$$
$$= \frac{1}{-\partial_\tau^2+\omega^2} \begin{pmatrix} \frac{1}{M} & -i\,\partial_\tau \\ i\,\partial_\tau & M\omega^2 \end{pmatrix} \delta(\tau-\tau') \qquad (3.334)$$
$$= \delta(\tau-\tau')\, \frac{1}{-\partial_{\tau'}^2+\omega^2} \begin{pmatrix} \frac{1}{M} & -i\,\partial_{\tau'} \\ i\,\partial_{\tau'} & M\omega^2 \end{pmatrix}$$

The meaning of the inverse is as follows

$$\int d\tau'\, \boldsymbol{D}^{-1}_{\omega^2,e}(\tau,\, \tau')\, \boldsymbol{D}_{\omega^2,e}(\tau'-\tau'')$$
$$= \int d\tau'\, \frac{1}{-\partial_\tau^2+\omega^2} \begin{pmatrix} \frac{1}{M} & -i\,\partial_\tau \\ i\,\partial_\tau & M\omega^2 \end{pmatrix} \delta(\tau-\tau') \begin{pmatrix} M\omega^2 & i\,\partial_{\tau'} \\ -i\,\partial_{\tau'} & \frac{1}{M} \end{pmatrix} \delta(\tau'-\tau'')$$
$$= \int d\tau'\, \frac{1}{-\partial_\tau^2+\omega^2} \begin{pmatrix} \frac{1}{M} & -i\,\partial_\tau \\ i\,\partial_\tau & M\omega^2 \end{pmatrix} \delta(\tau-\tau')\, \delta(\tau'-\tau'') \begin{pmatrix} M\omega^2 & i\,\partial_{\tau''} \\ -i\,\partial_{\tau''} & \frac{1}{M} \end{pmatrix}$$
$$= \frac{1}{-\partial_\tau^2+\omega^2} \begin{pmatrix} \frac{1}{M} & -i\,\partial_\tau \\ i\,\partial_\tau & M\omega^2 \end{pmatrix} \delta(\tau-\tau'') \begin{pmatrix} M\omega^2 & i\,\partial_{\tau''} \\ -i\,\partial_{\tau''} & \frac{1}{M} \end{pmatrix}$$
$$= \frac{1}{-\partial_\tau^2+\omega^2} \begin{pmatrix} \frac{1}{M} & -i\,\partial_\tau \\ i\,\partial_\tau & M\omega^2 \end{pmatrix} \begin{pmatrix} M\omega^2 & i\,\partial_{\tau''} \\ -i\,\partial_{\tau''} & \frac{1}{M} \end{pmatrix} \delta(\tau-\tau'')$$
$$= \frac{1}{-\partial_\tau^2+\omega^2} \begin{pmatrix} -\partial_\tau^2+\omega^2 & 0 \\ 0 & -\partial_\tau^2+\omega^2 \end{pmatrix} \delta(\tau-\tau'')$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(\tau-\tau'')$$

$\mathbb{G}^{p}_{\omega^2,e}(\tau, \tau')$

Note that (3.334) can be written as

$$\mathbb{G}^{p}_{\omega^2,e}(\tau,\tau') = \begin{pmatrix} \frac{1}{M} & -i\partial_\tau \\ i\partial_\tau & M\omega^2 \end{pmatrix} \frac{1}{-\partial_\tau^2+\omega^2}\,\delta(\tau-\tau')$$

$$= \begin{pmatrix} \frac{1}{M} & -i\partial_\tau \\ i\partial_\tau & M\omega^2 \end{pmatrix} G^{p}_{\omega^2,e}(\tau,\tau') \qquad (3.345)$$

where $G^{p}_{\omega^2,e}(\tau, \tau')$ is the scalar Green function given by (3.237) or (3.248).

Now,

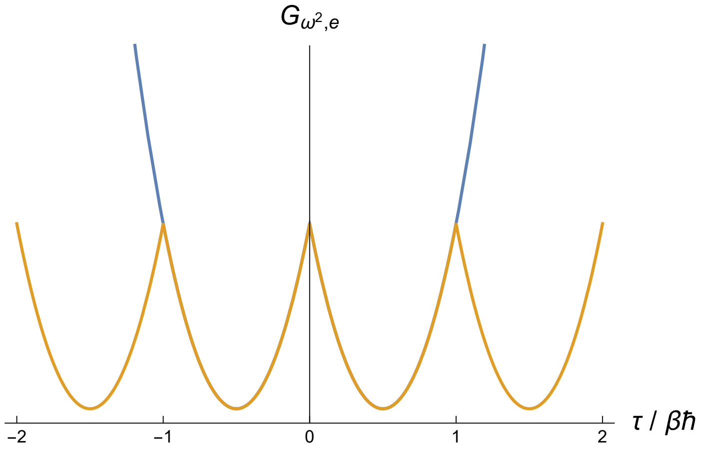
$$G^{p}_{\omega^2,e}(\tau) = \frac{1}{2\omega}\,\frac{\cosh\omega\left(\frac{1}{2}\beta\hbar - |\tau|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} \qquad \tau\in(-\beta\hbar,\ \beta\hbar)\ (3.248)$$

$$\rightarrow \qquad \partial_\tau G^{p}_{\omega^2,e}(\tau) = -\frac{\epsilon(\tau)}{2\omega}\,\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar - |\tau|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)}$$

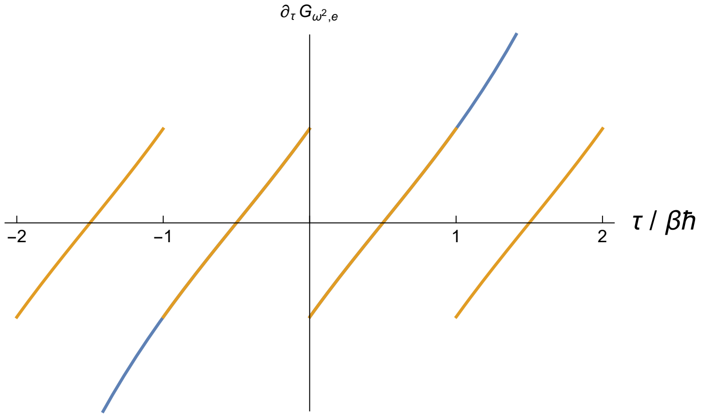
where

$$\epsilon(\tau) = \partial_\tau\,|\tau| = \begin{cases} 1 & \tau>0 \\ -1 & \tau<0 \end{cases}$$

Plot of $G_{\omega^2,e}(\tau)$ for $\omega = 1$ [See “3.12._Code.nb”]. Blue lines show extension of (3.248) outside the interval $(-\beta\hbar,\ \beta\hbar)$.



Plot of $\partial_\tau G_{\omega^2,e}(\tau)$ for $\omega = 1$ [See “3.12._Code.nb”]. Blue lines show extension of (3.248) outside the interval $(-\beta\hbar,\ \beta\hbar)$.



Thus, the period of both $G^p_{\omega^2,e}(\tau)$ & $\partial_\tau G^p_{\omega^2,e}(\tau)$ is $\beta\hbar$.

The presence of $\epsilon(\tau)$ means that there are discontinuities at $\tau = 0$ & $\beta\hbar$.

$$\partial_\tau G^p_{\omega^2,e}(0_+) = \partial_\tau G^p_{\omega^2,e}(0) = -\frac{1}{2\omega}\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} = -\frac{1}{2\omega}$$

$$\partial_\tau G^p_{\omega^2,e}(0_-) = \frac{1}{2\omega}\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} = \frac{1}{2\omega}$$

$$\partial_\tau G^p_{\omega^2,e}(\beta\hbar) = -\frac{1}{2\omega}\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar - \beta\hbar\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} = \frac{1}{2\omega}$$

$$\partial_\tau G^p_{\omega^2,e}(-\beta\hbar) = \frac{1}{2\omega}\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar - \beta\hbar\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} = -\frac{1}{2\omega} \qquad (3.345a)$$

Since

$$\tau,\ \tau' \in [0,\ \beta\hbar) \qquad \rightarrow \qquad \tau - \tau' \in (-\beta\hbar,\ \beta\hbar)$$

we have

$$\mathbb{G}^p_{\omega^2,e}(\tau,\ \tau') = \begin{pmatrix} G^p_{\omega^2,e,xx} & G^p_{\omega^2,e,xp} \\ G^p_{\omega^2,e,px} & G^p_{\omega^2,e,pp} \end{pmatrix}(\tau,\ \tau')$$

$$= \begin{pmatrix} \frac{1}{2M\omega}\frac{\cosh\omega\left(\frac{1}{2}\beta\hbar - \left|\tau-\tau'\right|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} & \frac{i\,\epsilon(\tau-\tau')}{2}\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar - \left|\tau-\tau'\right|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} \\ -\frac{i\,\epsilon(\tau-\tau')}{2}\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar - \left|\tau-\tau'\right|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} & \frac{1}{2}M\omega\frac{\cosh\omega\left(\frac{1}{2}\beta\hbar - \left|\tau-\tau'\right|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} \end{pmatrix}$$

$$\text{for} \quad \tau - \tau' \in (-\beta\hbar,\ \beta\hbar) \qquad (3.345b)$$

Thus,

$$\mathbb{G}^{p+}_{\omega^2,e}(\tau,\ \tau') = \mathbb{G}^p_{\omega^2,e}(\tau,\ \tau') = \mathbb{G}^{pT}_{\omega^2,e}(\tau',\ \tau) \qquad (3.345c)$$

Quadratic Completion

As in (3.238), we do the quadratic completion by setting

$$\boldsymbol{V}(\tau) = \boldsymbol{V}'(\tau) + \int_0^{\beta\hbar} d\tau_1\, \mathbb{G}^p_{\omega^2,e}(\tau,\ \tau_1)\,\boldsymbol{J}(\tau_1) \qquad (3.335)$$

$$\boldsymbol{V}^T(\tau) = \boldsymbol{V}'^T(\tau) + \int_0^{\beta\hbar} d\tau_1\, \boldsymbol{J}^T(\tau_1)\,\mathbb{G}^{pT}_{\omega^2,e}(\tau,\ \tau_1)$$

$$= \boldsymbol{V}'^T(\tau) + \int_0^{\beta\hbar} d\tau_1\, \boldsymbol{J}^T(\tau_1)\, \mathbb{G}^p_{\omega^2,e}(\tau_1, \tau) \quad \text{[(3.345c) used.]} \qquad (3.335a)$$

so that

$$\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, \boldsymbol{V}'^T(\tau)\, \boldsymbol{D}_{\omega^2,e}(\tau, \tau')\, \boldsymbol{V}'(\tau')$$
$$= \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' \left[\boldsymbol{V}^T(\tau) - \int_0^{\beta\hbar} d\tau_1\, \boldsymbol{J}^T(\tau_1)\, \mathbb{G}^p_{\omega^2,e}(\tau_1, \tau)\right] \boldsymbol{D}_{\omega^2,e}(\tau, \tau')$$
$$\times \left[\boldsymbol{V}(\tau') - \int_0^{\beta\hbar} d\tau_2\, \mathbb{G}^p_{\omega^2,e}(\tau', \tau_2)\, \boldsymbol{J}(\tau_2)\right]$$
$$= \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, \boldsymbol{V}^T(\tau)\, \boldsymbol{D}_{\omega^2,e}(\tau, \tau')\, \boldsymbol{V}(\tau')$$
$$- \int_0^{\beta\hbar} d\tau\, \boldsymbol{V}^T(\tau) \int_0^{\beta\hbar} d\tau_2\, \delta(\tau - \tau_2)\, \boldsymbol{J}(\tau_2)$$
$$- \int_0^{\beta\hbar} d\tau' \int_0^{\beta\hbar} d\tau_1\, \boldsymbol{J}^T(\tau_1)\, \delta(\tau_1 - \tau')\, \boldsymbol{V}(\tau')$$
$$+ \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau_1 \int_0^{\beta\hbar} d\tau_2\, \boldsymbol{J}^T(\tau_1)\, \mathbb{G}^p_{\omega^2,e}(\tau_1, \tau)\, \delta(\tau - \tau_2)\, \boldsymbol{J}(\tau_2)$$
$$= \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, \boldsymbol{V}^T(\tau)\, \boldsymbol{D}_{\omega^2,e}(\tau, \tau')\, \boldsymbol{V}(\tau') \qquad (3.335b)$$
$$- \int_0^{\beta\hbar} d\tau\, \boldsymbol{V}^T(\tau)\, \boldsymbol{J}(\tau) - \int_0^{\beta\hbar} d\tau\, \boldsymbol{J}^T(\tau)\, \boldsymbol{V}(\tau)$$
$$+ \int_0^{\beta\hbar} d\tau_1 \int_0^{\beta\hbar} d\tau_2\, \boldsymbol{J}^T(\tau_1)\, \mathbb{G}^p_{\omega^2,e}(\tau_1, \tau_2)\, \boldsymbol{J}(\tau_2)$$

Since

$$\boldsymbol{V}^T(\tau)\, \boldsymbol{J}(\tau) = \boldsymbol{J}^T(\tau)\, \boldsymbol{V}(\tau) \qquad (3.335c)$$

(3.332) becomes

$$\mathcal{A}_e[\boldsymbol{J}] = \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' \left[\frac{1}{2} \boldsymbol{V}'^T(\tau)\, \boldsymbol{D}_{\omega^2,e}(\tau, \tau')\, \boldsymbol{V}'(\tau') \right. \qquad (3.336)$$
$$\left. - \frac{1}{2} \boldsymbol{J}(\tau)^T\, \mathbb{G}^p_{\omega^2,e}(\tau, \tau')\, \boldsymbol{J}(\tau')\right]$$

which is simply the matrix version of (3.239) with M & $\frac{1}{M}$ absorbed into $\boldsymbol{D}_{\omega^2,e}$ & $\mathbb{G}^p_{\omega^2,e}$, respectively.

$J = 0$

For $\boldsymbol{J} = 0$, we have [see (3.240)]

$$Z_\omega = \frac{1}{\sqrt{\det \boldsymbol{D}_{\omega^2,e}}} \qquad (3.337)$$

To evaluate (3.337), we introduce the Fourier series

$$\boldsymbol{D}_{\omega^2,e}(\tau, \tau') = \frac{1}{\beta\hbar} \sum_{m=-\infty}^{\infty} \boldsymbol{D}_{\omega^2,e}(\omega_m)\, e^{-i\omega_m(\tau-\tau')} \qquad (3.338)$$

$$\rightarrow \quad \boldsymbol{D}_{\omega^2,e}(\omega_m) = \int_0^{\beta\hbar} d(\tau - \tau')\, D_{\omega^2,e}(\tau - \tau')\, e^{i\omega_m(\tau-\tau')}$$
$$= \int_0^{\beta\hbar} d(\tau - \tau')\, \delta(\tau - \tau') \begin{pmatrix} M\omega^2 & i\partial_{\tau'} \\ -i\partial_{\tau'} & \frac{1}{M} \end{pmatrix} e^{i\omega_m(\tau-\tau')}$$

$$= \begin{pmatrix} M\omega^2 & \omega_m \\ -\omega_m & \frac{1}{M} \end{pmatrix} \tag{3.339}$$

$$\therefore \qquad \det \boldsymbol{D}_{\omega^2, e}(\omega_m) = \omega^2 + \omega_m^2 \tag{3.340}$$

$$\mathbb{G}^p_{\omega^2, e}(\omega_m) = \boldsymbol{D}^{-1}_{\omega^2, e}(\omega_m) = \frac{1}{\omega^2 + \omega_m^2} \begin{pmatrix} \frac{1}{M} & -\omega_m \\ \omega_m & M\omega^2 \end{pmatrix} \tag{3.341}$$

From (3.241a), we have

$$\det \boldsymbol{D}_{\omega^2, e} = \prod_{m=-\infty}^{\infty} \left(\omega_m^2 + \omega^2\right) = 4\sinh^2\left(\frac{1}{2}\beta\hbar\omega\right) \tag{3.342a}$$

(3.337) then becomes

$$Z_\omega = \frac{1}{2\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} \tag{3.342}$$

$Z[\boldsymbol{J}]$

(3.330) can therefore be written as

$$Z[\boldsymbol{J}] = Z_\omega \exp\left(-\frac{1}{\hbar}\mathcal{A}_e^J[\boldsymbol{J}]\right) \tag{3.343}$$

where

$$\mathcal{A}_e^J[\boldsymbol{J}] = -\frac{1}{2}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' \, \boldsymbol{J}(\tau)^T \, \mathbb{G}^p_{\omega^2, e}(\tau, \tau') \, \boldsymbol{J}(\tau') \tag{3.344}$$

$$= \frac{1}{2}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' \, \boldsymbol{J}(\tau)^T \begin{pmatrix} \frac{1}{M} & -i\partial_\tau \\ i\partial_\tau & M\omega^2 \end{pmatrix} G^p_{\omega^2, e}(\tau, \tau') \, \boldsymbol{J}(\tau') \tag{3.344a}$$

The correlation functions are given by the functional derivatives of (3.343) so that

$$\mathbf{G}^{(2)}_{\omega^2, e}(\tau, \tau') = \begin{pmatrix} G^{(2)}_{\omega^2, e, xx}(\tau, \tau') & G^{(2)}_{\omega^2, e, xp}(\tau, \tau') \\ G^{(2)}_{\omega^2, e, px}(\tau, \tau') & G^{(2)}_{\omega^2, e, pp}(\tau, \tau') \end{pmatrix}$$

$$= \langle \boldsymbol{V}(\tau)\, \boldsymbol{V}^T(\tau') \rangle = \begin{pmatrix} \langle x(\tau)\, x(\tau') \rangle & \langle x(\tau)\, p(\tau') \rangle \\ \langle p(\tau)\, x(\tau') \rangle & \langle p(\tau)\, p(\tau') \rangle \end{pmatrix}$$

$$= \hbar \begin{pmatrix} \frac{\delta^2}{\delta j(\tau)\,\delta j(\tau')} & \frac{\delta^2}{\delta j(\tau)\,\delta k(\tau')} \\ \frac{\delta^2}{\delta k(\tau)\,\delta j(\tau')} & \frac{\delta^2}{\delta k(\tau)\,\delta k(\tau')} \end{pmatrix} \mathcal{A}_e^J[\boldsymbol{J}] \Bigg|_{J=0} \qquad \text{[See (3.301a).]}$$

$$= \hbar \begin{pmatrix} \frac{1}{M} G^p_{\omega^2, e}(\tau, \tau') & -i\partial_\tau G^p_{\omega^2, e}(\tau, \tau') \\ i\partial_\tau G^p_{\omega^2, e}(\tau, \tau') & M\omega^2 G^p_{\omega^2, e}(\tau, \tau') \end{pmatrix} \tag{3.346-9}$$

Consider now the source-free version of the action (3.331) as

$$\mathcal{A}_e[0, 0] = \int_0^{\beta\hbar} d\tau \left[-i p \dot{x} + \frac{1}{2M} p^2 + \frac{1}{2} M\omega^2 x^2\right]$$

$$= \int_0^{\beta\hbar} d\tau \left[\frac{1}{2M}(p - iM\dot{x})^2 + \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\omega^2 x^2\right] \tag{3.350}$$

The path integrations over p are therefore Gaussians centered at $p = iM\dot{x}$.

In other words, $p(\tau)$ fluctuates around the real-time classical momentum

$$p_{cl}(t) = M \frac{dx}{dt} = i M \dot{x} \tag{3.350a}$$

(3.346-9) give

$$\langle p(\tau)\, x(\tau') \rangle = i\hbar\, \partial_\tau G^p_{\omega^2,e}(\tau, \tau') \;= i M \partial_\tau \langle x(\tau)\, x(\tau') \rangle \;= \langle p_{cl}(\tau)\, x(\tau') \rangle$$

$$\langle x(\tau)\, p(\tau') \rangle = -i\hbar\, \partial_\tau G^p_{\omega^2,e}(\tau - \tau') \;= i\hbar\, \partial_{\tau'} G^p_{\omega^2,e}(\tau - \tau') = \langle x(\tau)\, p_{cl}(\tau') \rangle$$

so that the effects of fluctuations are cancelled out.

On the other hand,

$$\langle p(\tau)\, p(\tau') \rangle = M\hbar\, \omega^2\, G^p_{\omega^2,e}(\tau, \tau') \;= M^2 \omega^2 \langle x(\tau)\, x(\tau') \rangle \tag{3.351a}$$

but [see also (3.311d)]

$$\begin{aligned}\langle \dot{x}(\tau)\, \dot{x}(\tau') \rangle &= \partial_\tau \partial_{\tau'} \langle x(\tau)\, x(\tau') \rangle \\ &= \frac{\hbar}{M} \partial_\tau \partial_{\tau'} G^p_{\omega^2,e}(\tau - \tau') \\ &= -\frac{\hbar}{M} \partial_\tau^2 G^p_{\omega^2,e}(\tau - \tau')\end{aligned} \tag{3.351}$$

Thus,

$$\begin{aligned}&\langle p(\tau)\, p(\tau') \rangle - \langle p_{cl}(\tau)\, p_{cl}(\tau') \rangle \\ &= \langle p(\tau)\, p(\tau') \rangle + M^2 \langle \dot{x}(\tau)\, \dot{x}(\tau') \rangle \\ &= \hbar M \left(-\partial_\tau^2 + \omega^2\right) G^p_{\omega^2,e}(\tau - \tau') \\ &= \hbar M\, \delta(\tau - \tau') \;\; \text{[(3.114c) used.]}\end{aligned} \tag{3.352}$$

indicating the effects of fluctuations.

Alternative Approach

The action (3.331) corresponds to a Euclidean Hamiltonian

$$H = \frac{1}{2M} p^2 + \frac{1}{2} M \omega^2 x^2 - j(\tau)\, x - k(\tau)\, p \tag{A}$$

The Hamilton's eqs [see "EuclideanDynamics.pdf"] are

$$\dot{x} = -i \frac{\partial H}{\partial p} = -i \frac{p}{M} + i k \tag{B}$$

$$\dot{p} = i \frac{\partial H}{\partial x} = i M \omega^2 x - i j \tag{C}$$

$$\rightarrow \qquad p = i M \dot{x} + M k \tag{D}$$

$$-\ddot{x} + \omega^2 x = \frac{j}{M} - i \dot{k} \tag{E}$$

Note that in the eqs above x & p can either be classical variables or quantum mechanical Heisenberg operators.

Using

$$\left(-\partial_\tau^2 + \omega^2\right) G_{\omega^2,e}(\tau, \tau') = \delta(\tau - \tau') \tag{3.207}$$

we have

$$\begin{aligned}x(\tau) &= \int_0^{\beta\hbar} d\tau'\, G_{\omega^2,e}(\tau, \tau') \left[\frac{j(\tau')}{M} - i \dot{k}(\tau') \right] \\ &= \int_0^{\beta\hbar} d\tau' \left[\frac{j(\tau')}{M} - i k(\tau')\, \partial_\tau \right] G_{\omega^2,e}(\tau, \tau')\end{aligned} \tag{F}$$

where we've done a partial integration & made use of $\partial_\tau G(\tau,\ \tau') = -\partial_{\tau'} G(\tau,\ \tau')$.

$$\therefore \qquad \dot{x}(\tau) = \int_0^{\beta\hbar} d\tau' \left[\frac{j(\tau')}{M} \partial_\tau - i\,k(\tau')\,\partial_\tau^2 \right] G_{\omega^2,e}(\tau,\ \tau')$$

$$= i\,k(\tau) + \int_0^{\beta\hbar} d\tau' \left[\frac{j(\tau')}{M} \partial_\tau - i\,k(\tau')\,\omega^2 \right] G_{\omega^2,e}(\tau,\ \tau') \qquad \text{(G)}$$

(D) thus becomes

$$p(\tau) = \int_0^{\beta\hbar} d\tau' \left[i\,j(\tau')\,\partial_\tau + M\,k(\tau')\,\omega^2 \right] G_{\omega^2,e}(\tau,\ \tau') \qquad \text{(H)}$$

Combining (F) & (H) gives

$$\boldsymbol{V}(\tau) = \int_0^{\beta\hbar} d\tau' \begin{pmatrix} \frac{1}{M} & -i\,\partial_\tau \\ i\,\partial_\tau & M\,\omega^2 \end{pmatrix} G_{\omega^2,e}(\tau,\ \tau')\,\boldsymbol{J}(\tau') \qquad \text{(J)}$$

$$\equiv \int_0^{\beta\hbar} d\tau'\ \mathbb{G}^p_{\omega^2,e}(\tau,\ \tau')\,\boldsymbol{J}(\tau') \qquad \text{(K)}$$

in agreement with (3.345).

Uniform Magnetic Field

These results can be generalized immediately to the case of a particle in a uniform magnetic field. Applying the minimal substitution $\boldsymbol{p} \to \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}$ to (3.331) gives the canonical action as

$$\mathcal{A}_e[\boldsymbol{J}] = \int_0^{\beta\hbar} d\tau \left[-i\,\dot{\boldsymbol{x}}\cdot\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right) + \frac{1}{2M}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + \frac{1}{2} M\,\omega^2\,\boldsymbol{x}^2 - \boldsymbol{j}(\tau)\cdot\boldsymbol{x} - \boldsymbol{k}(\tau)\cdot\boldsymbol{P} \right]$$

$$= \int_0^{\beta\hbar} d\tau \left[-i\,\dot{\boldsymbol{x}}\cdot\boldsymbol{P} + \frac{1}{2M}\boldsymbol{P}^2 + \frac{1}{2} M\,\omega^2\,\boldsymbol{x}^2 - \boldsymbol{j}(\tau)\cdot\boldsymbol{x} - \boldsymbol{k}(\tau)\cdot\boldsymbol{P} \right] \qquad \text{(3.353a)}$$

$$= \int_0^{\beta\hbar} d\tau \left[\frac{1}{2M}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A} - i\,M\,\dot{\boldsymbol{x}}\right)^2 + \frac{1}{2} M\left(\dot{\boldsymbol{x}}^2 + \omega^2\,\boldsymbol{x}^2\right) - \boldsymbol{j}(\tau)\cdot\boldsymbol{x} - \boldsymbol{k}(\tau)\cdot\boldsymbol{P} \right] \qquad \text{(3.353)}$$

where

$$\boldsymbol{B} = B\,\hat{\boldsymbol{z}} \qquad \boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{x} = \frac{1}{2} B\,(-y,\ x,\ 0) \qquad \text{(3.353b)}$$

$$\boldsymbol{P} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{A} = \left(p_x + \frac{e\,B}{2\,c}\,y,\ p_y - \frac{e\,B}{2\,c}\,x,\ p_z\right)$$

$$= (p_x + M\,\omega_B\,y,\ p_y - M\,\omega_B\,x,\ p_z) \qquad \text{(3.353c)}$$

$$\omega_B = \frac{e\,B}{2\,M\,c} = \frac{1}{2}\,\omega_L \qquad \text{(3.353d)}$$

Using [cf.(3.332)]

$$\boldsymbol{V}(\tau) = \begin{pmatrix} \boldsymbol{x}(\tau) \\ \boldsymbol{P}(\tau) \end{pmatrix} = \begin{pmatrix} x(\tau) \\ y(\tau) \\ z(\tau) \\ P_x(\tau) \\ P_y(\tau) \\ P_z(\tau) \end{pmatrix} \qquad \boldsymbol{J}(\tau) = \begin{pmatrix} \boldsymbol{j}(\tau) \\ \boldsymbol{k}(\tau) \end{pmatrix} = \begin{pmatrix} j_x(\tau) \\ j_y(\tau) \\ j_z(\tau) \\ k_x(\tau) \\ k_y(\tau) \\ k_z(\tau) \end{pmatrix} \qquad \text{(3.353e)}$$

(3.353a) becomes

$$\mathcal{A}_e[\boldsymbol{J}] = \int_0^{\beta\hbar} d\tau \left[\frac{1}{2} \int_0^{\beta\hbar} d\tau'\ \boldsymbol{V}(\tau)^T\,\boldsymbol{D}_{\omega^2,B}(\tau,\ \tau')\,\boldsymbol{V}(\tau') - \boldsymbol{V}(\tau)^T\,\boldsymbol{J}(\tau) \right] \qquad \text{(3.353f)}$$

where [cf. (3.333)]

$$\boldsymbol{D}_{\omega^2,B}(\tau,\tau') = \begin{pmatrix} M\omega^2\,\mathbb{I} & i\,\mathbb{I}\,\partial_\tau \\ -i\,\mathbb{I}\,\partial_\tau & \frac{1}{M}\mathbb{I} \end{pmatrix} \delta(\tau-\tau') \qquad \tau-\tau' \in [0, \beta\hbar) \quad (3.353g)$$

where $\mathbb{I}$ is the 3×3 identity matrix. Note that $\boldsymbol{D}_{\omega^2,B}(\tau,\tau')$ is Hermitian and $\boldsymbol{V}$, $\boldsymbol{J}$ are real.

After the quadratic completion, we obtain [cf. (3.344)]

$$\mathcal{A}_e^J[\boldsymbol{J}] = \frac{1}{2}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, \boldsymbol{J}(\tau)^T \mathcal{G}^p_{\omega^2,B}(\tau,\tau')\, \boldsymbol{J}(\tau') \qquad (3.353h)$$

$$= \frac{1}{2}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, \boldsymbol{J}(\tau)^T \begin{pmatrix} \frac{1}{M}\mathbb{G}^p_{\omega^2,B}(\tau,\tau') & -i\partial_\tau \mathbb{G}^p_{\omega^2,B}(\tau,\tau') \\ i\partial_\tau \mathbb{G}^p_{\omega^2,B}(\tau,\tau') & M\omega^2\,\mathbb{G}^p_{\omega^2,B}(\tau,\tau') \end{pmatrix} \boldsymbol{J}(\tau')$$

where

$$\mathbb{G}^p_{\omega^2,B}(\tau,\tau') = \begin{pmatrix} G^p_{\omega^2,B,xx}(\tau,\tau') & G^p_{\omega^2,B,xy}(\tau,\tau') & 0 \\ G^p_{\omega^2,B,yx}(\tau,\tau') & G^p_{\omega^2,B,yy}(\tau,\tau') & 0 \\ 0 & 0 & G^p_{\omega^2,e}(\tau,\tau') \end{pmatrix} \qquad (3.353i)$$

with the x, y components given by (3.322) & (3.325).

The correlation functions are given by the functional derivatives of (3.353h) so that

$$\mathcal{G}^{(2)}_{\omega^2,B}(\tau,\tau') = \begin{pmatrix} \boldsymbol{G}^{(2)}_{\omega^2,B,\boldsymbol{xx}}(\tau,\tau') & \boldsymbol{G}^{(2)}_{\omega^2,B,\boldsymbol{xP}}(\tau,\tau') \\ \boldsymbol{G}^{(2)}_{\omega^2,B,\boldsymbol{Px}}(\tau,\tau') & \boldsymbol{G}^{(2)}_{\omega^2,B,\boldsymbol{PP}}(\tau,\tau') \end{pmatrix}$$

$$= \left\langle \boldsymbol{V}(\tau)\,\boldsymbol{V}^T(\tau')\right\rangle = \begin{pmatrix} \left\langle \boldsymbol{x}(\tau)\,\boldsymbol{x}^T(\tau')\right\rangle & \left\langle \boldsymbol{x}(\tau)\,\boldsymbol{P}^T(\tau')\right\rangle \\ \left\langle \boldsymbol{P}(\tau)\,\boldsymbol{x}^T(\tau')\right\rangle & \left\langle \boldsymbol{P}(\tau)\,\boldsymbol{P}^T(\tau')\right\rangle \end{pmatrix}$$

$$= \hbar \begin{pmatrix} \frac{\delta^2}{\delta\boldsymbol{j}(\tau)\,\delta\boldsymbol{j}^T(\tau')} & \frac{\delta^2}{\delta\boldsymbol{j}(\tau)\,\delta\boldsymbol{k}^T(\tau')} \\ \frac{\delta^2}{\delta\boldsymbol{k}(\tau)\,\delta\boldsymbol{j}^T(\tau')} & \frac{\delta^2}{\delta\boldsymbol{k}(\tau)\,\delta\boldsymbol{k}^T(\tau')} \end{pmatrix} \mathcal{A}_e^J[\boldsymbol{J}]$$

$$= \hbar \begin{pmatrix} \frac{1}{M}\mathbb{G}^p_{\omega^2,B}(\tau,\tau') & -i\partial_\tau \mathbb{G}^p_{\omega^2,B}(\tau,\tau') \\ i\partial_\tau \mathbb{G}^p_{\omega^2,B}(\tau,\tau') & M\omega^2\,\mathbb{G}^p_{\omega^2,B}(\tau,\tau') \end{pmatrix} \qquad (3.354a)$$

The results (3.354-9) in Kleinert's text can be obtained using

$$\left\langle \boldsymbol{x}(\tau)\,\boldsymbol{p}^T(\tau')\right\rangle = \left\langle \boldsymbol{x}(\tau)\,\boldsymbol{P}^T(\tau')\right\rangle + \frac{e}{c}\left\langle \boldsymbol{x}(\tau)\,\boldsymbol{A}^T(\tau')\right\rangle \qquad (3.354b)$$

with [see (3.353c)]

$$\frac{e}{c}\boldsymbol{A} = M\,\omega_B\,(y, -x, 0) \qquad (3.354c)$$

Finally, (3.357) & (3.369) also make use of (3.352).

3.12.2. Relations between Various Amplitudes

We can define other evolution amplitudes beside the periodic one considered so far.

For fixed spatial end-points, we have [cf. (2.14)]

$$(x_b, \beta\hbar \mid x_a\, 0)[j,k] = \int_{x(0)=x_a}^{x(\beta\hbar)=x_b} \mathcal{D}'x \int \frac{\mathcal{D}p}{2\pi\hbar} \exp\left\{-\frac{1}{\hbar}\mathcal{A}_e[j,k]\right\} \qquad (3.360)$$

$$= \left(\prod_{k=1}^{N} \int_{x_0=x_a}^{x_{N+1}=x_b} dx_k\right)\left(\prod_{n=1}^{N+1}\int \frac{dp_n}{2\pi\hbar}\right)\left\{-\frac{1}{\hbar}\mathcal{A}_e^N[j,k]\right\} \qquad (3.360a)$$

where [cf. (2.15)]

$$\mathcal{A}_e^N[j, k] = \sum_{n=1}^{N+1} \left[-i\, p_n\, (x_n - x_{n-1}) + \epsilon H(p_n, x_n) - j_n x_n - k_n p_n \right] \tag{3.360b}$$

For fixed momentum end-points, we have [cf. (2.38)]

$$(p_b, \beta\hbar \mid p_a\, 0)[j, k] = \int \mathcal{D} x \int_{p(0)=p_a}^{p(\beta\hbar)=p_b} \frac{\mathcal{D}' p}{2\pi\hbar} \exp\left\{ -\frac{1}{\hbar} \overline{\mathcal{A}}_e[j, k] \right\} \tag{3.361}$$

$$= \left(\prod_{k=1}^{N+1} \int d x_k \right) \left(\prod_{n=1}^{N} \int_{p_0=p_a}^{p_{N+1}=p_b} \frac{d p_n}{2\pi\hbar} \right) \left\{ -\frac{1}{\hbar} \overline{\mathcal{A}}_e^N[j, k] \right\} \tag{3.361a}$$

where [cf. (2.34a)]

$$\overline{\mathcal{A}}_e^N[j, k] = \sum_{n=0}^{N} \left[-i\, x_n\, (p_{n+1} - p_n) + \epsilon H(p_n, x_n) - j_n x_n - k_n p_n \right] \tag{3.361b}$$

Using the Euclidean version of (2.37), we have

$$(p_b, \beta\hbar \mid p_a\, 0)[j, k] \tag{3.362}$$

$$= \int_{-\infty}^{\infty} d x_b \int_{-\infty}^{\infty} d x_a\, e^{-i(p_b x_b - p_a x_a)/\hbar}\, (x_b, \beta\hbar \mid x_a\, 0)[j, k]$$

Consider the amplitude with vanishing endpoints

$$(0, \beta\hbar \mid 0\, 0)[j, k] = \int_{x(0)=0}^{x(\beta\hbar)=0} \mathcal{D}' x \int \frac{\mathcal{D} p}{2\pi\hbar} \exp\left\{ -\frac{1}{\hbar} \mathcal{A}_{e,0}[j, k] \right\} \tag{3.362a}$$

where

$$\mathcal{A}_{e,0}^N[j, k] = \mathcal{A}_e^N[j, k] \Big|_{x_{N+1}=x_0=0}$$

From (3.360b), we have

$$\mathcal{A}_e^N[j, k] = \mathcal{A}_{e,0}^N[j, k] - i\, p_{N+1} x_b + i\, p_1 x_a$$

$$= \mathcal{A}_{e,0}^N[j, k + i\, x_b\, \delta_{n, N+1} - i\, x_a\, \delta_{n,1}] \tag{3.365a}$$

where we've neglected the terms

$$\epsilon\, [H(p_{N+1}, x_b) - H(p_{N+1}, 0)] - j_{N+1} x_b \tag{3.365b}$$

since the 1st term vanishes as $\epsilon \to 0$ while the single point contribution of $j_{N+1} x_b$ vanishes since there is no integration in (3.360a) on x_{N+1}.

In the continuum limit, (3.365a) becomes

$$(x_b, \beta\hbar \mid x_a\, 0)[j, k] \tag{3.365}$$

$$= (0, \beta\hbar \mid 0\, 0)[j(\tau), k(\tau) + i\, x_b\, \delta(\tau - \beta\hbar) - i\, x_a\, \delta(\tau - 0_+)]$$

where $0_+ = \tau_1 - \tau_a$ is an infinitesimal positive number. Note that $\beta\hbar$ is outside the primary interval $[0, \beta\hbar)$ so that one should invoke the periodic B.C. to get $f(\beta\hbar) = f(0)$. These limiting considerations is necessary in order to keep track of the discontinuities of the off-diagonal elements of $\mathbb{G}^p_{\omega^2, e}(\tau, \tau')$ [see (3.345a)].

Thus, the amplitude for fixed endpoints is the same as that for vanishing endpoints with a modified source.

Consider the measure for path integrals on periodic functions

$$\int_{x(0)=x}^{x(\beta\hbar)=x} \mathcal{D}' x \int \frac{\mathcal{D} p}{2\pi\hbar} = \left(\prod_{k=1}^{N} \int_{x_0=x}^{x_{N+1}=x} d x_k \right) \left(\prod_{n=1}^{N+1} \int \frac{d p_n}{2\pi\hbar} \right)$$

$$= \left(\prod_{k=1}^{N+1} \int d x_k \right) \left(\prod_{n=1}^{N+1} \int \frac{d p_n}{2\pi\hbar} \right) \delta(x_{N+1} - x)$$

$$= \oint \mathcal{D} x \int \frac{\mathcal{D} p}{2\pi\hbar}\, \delta[x(\beta\hbar) - x] \tag{3.366}$$

$$= \int_{-\infty}^{\infty} \frac{d\,p_0}{2\,\pi\,\hbar}\, e^{i\,p_0\,[\,x(\beta\,\hbar) - x\,]/\hbar} \oint \mathcal{D}\,x \int \frac{\mathcal{D}\,p}{2\,\pi\,\hbar} \tag{3.367a}$$

Thus, (3.362a) can be written as

$$(\,0\,,\,\beta\,\hbar\,|\,0\,0\,)[j,\,k] = \int_{-\infty}^{\infty} \frac{d\,p_0}{2\,\pi\,\hbar}\, e^{i\,p_0\,x(\beta\,\hbar)/\hbar}\, Z[j,\,k] \qquad [\,x = 0\,]$$

$$= \int_{-\infty}^{\infty} \frac{d\,p_0}{2\,\pi\,\hbar}\, Z[\,j - i\,p_0\,\delta(\tau - \beta\,\hbar),\,k\,] \tag{3.367b}$$

Using (3.365), we have

$$(\,x_b\,,\,\beta\,\hbar\,|\,x_a\,0\,)[j,\,k]$$

$$= \int_{-\infty}^{\infty} \frac{d\,p_a}{2\,\pi\,\hbar}\, Z\Big[\,j - i\,p_a\,\delta(\tau - \beta\,\hbar),\ k + i\,x_b\,\delta(\tau - \beta\,\hbar) - i\,x_a\,\delta\,(\tau - 0_+)\Big] \tag{3.368}$$

Note that the effective source is complex.

3.12.3. Harmonic Generating Functionals

Let

$$\tilde{\boldsymbol{J}} = \begin{pmatrix} \tilde{j} \\ \tilde{k} \end{pmatrix} = \begin{pmatrix} j\,(\tau) - i\,p_a\,\delta(\tau - \beta\,\hbar) \\ k\,(\tau) + i\,x_b\,\delta(\tau - \beta\,\hbar) - i\,x_a\,\delta\,(\tau - 0_+) \end{pmatrix} \tag{3.369}$$

$$\equiv \boldsymbol{J} + i\boldsymbol{\varphi}$$

where

$$\boldsymbol{J} = \begin{pmatrix} j \\ k \end{pmatrix} \qquad \boldsymbol{\varphi} = \begin{pmatrix} -p_a\,\delta(\tau - \beta\,\hbar) \\ x_b\,\delta(\tau - \beta\,\hbar) - x_a\,\delta\,(\tau - 0_+) \end{pmatrix} \tag{3.369a}$$

are real.

The action for (3.368) is therefore

$$\mathcal{A}_e[\tilde{\boldsymbol{J}}] = \int_0^{\beta\hbar} d\,\tau \int_0^{\beta\hbar} d\,\tau'\,\left\{ \frac{1}{2}\,\boldsymbol{V}(\tau)^T\,\boldsymbol{D}_{\omega^2,e}(\tau,\,\tau')\,\boldsymbol{V}(\tau') - \boldsymbol{V}(\tau)^T\,\tilde{\boldsymbol{J}}(\tau)\right\} \tag{3.369b}$$

Applying the quadratic completion (3.336), we have

$$\mathcal{A}_e[\tilde{\boldsymbol{J}}] = \int_0^{\beta\hbar} d\,\tau \int_0^{\beta\hbar} d\,\tau'\,\left[\frac{1}{2}\,\boldsymbol{V}'^{\,T}(\tau)\,\boldsymbol{D}_{\omega^2,e}(\tau,\,\tau')\,\boldsymbol{V}'(\tau') - \frac{1}{2}\,\tilde{\boldsymbol{J}}(\tau)^T\,\mathbb{G}^p_{\omega^2,\,e}(\tau,\,\tau')\,\tilde{\boldsymbol{J}}(\tau')\right] \tag{3.369c}$$

so that

$$Z[\tilde{\boldsymbol{J}}] = Z_\omega \exp\!\left(-\frac{1}{\hbar}\,\mathcal{A}_e^J[\tilde{\boldsymbol{J}}]\right) \tag{3.370a}$$

where

$$\mathcal{A}_e^J[\tilde{\boldsymbol{J}}] = -\frac{1}{2}\int_0^{\beta\hbar} d\,\tau \int_0^{\beta\hbar} d\,\tau'\,\tilde{\boldsymbol{J}}(\tau)^T\,\mathbb{G}^p_{\omega^2,\,e}(\tau,\,\tau')\,\tilde{\boldsymbol{J}}(\tau') \tag{3.370b}$$

$$= -\frac{1}{2}\int_0^{\beta\hbar} d\,\tau \int_0^{\beta\hbar} d\,\tau'\,\Big\{\boldsymbol{J}(\tau)^T\,\mathbb{G}^p_{\omega^2,\,e}(\tau,\,\tau')\,\boldsymbol{J}(\tau') - \boldsymbol{\varphi}^T(\tau)\,\mathbb{G}^p_{\omega^2,\,e}(\tau,\,\tau')\,\boldsymbol{\varphi}(\tau') + i\boldsymbol{\varphi}^T(\tau)\,\mathbb{G}^p_{\omega^2,\,e}(\tau,\,\tau')\,\boldsymbol{J}(\tau') + i\,\boldsymbol{J}^T(\tau)\,\mathbb{G}^p_{\omega^2,\,e}(\tau,\,\tau')\,\boldsymbol{\varphi}(\tau')\Big\}$$

$$= -\frac{1}{2}\int_0^{\beta\hbar} d\,\tau \int_0^{\beta\hbar} d\,\tau'\,\Big\{\boldsymbol{J}(\tau)^T\,\mathbb{G}^p_{\omega^2,\,e}(\tau,\,\tau')\,\boldsymbol{J}(\tau') - \boldsymbol{\varphi}^T(\tau)\,\mathbb{G}^p_{\omega^2,\,e}(\tau,\,\tau')\,\boldsymbol{\varphi}(\tau') + 2\,i\boldsymbol{\varphi}^T(\tau)\,\mathbb{G}^p_{\omega^2,\,e}(\tau,\,\tau')\,\boldsymbol{J}(\tau')\Big\} \qquad [(3.345\,c)\text{ used.}]$$

$$\equiv \mathcal{A}_e^J[\boldsymbol{J}] + \mathcal{A}_e^J[\boldsymbol{\varphi}] + \mathcal{A}_e^J[\boldsymbol{J},\,\boldsymbol{\varphi}] \tag{3.370c}$$

with

$$\mathcal{A}_e^{\boldsymbol{J}}[\boldsymbol{J}] = -\frac{1}{2}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' \,\boldsymbol{J}(\tau)^T\, \mathbb{G}^p_{\omega^2,e}(\tau,\tau')\,\boldsymbol{J}(\tau') \tag{3.370d}$$

$$\mathcal{A}_e^{\boldsymbol{J}}[\boldsymbol{\varphi}] = \frac{1}{2}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' \,\boldsymbol{\varphi}^T(\tau)\, \mathbb{G}^p_{\omega^2,e}(\tau,\tau')\,\boldsymbol{\varphi}(\tau') \tag{3.370e}$$

$$\mathcal{A}_e^{\boldsymbol{J}}[\boldsymbol{J},\boldsymbol{\varphi}] = -i\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' \,\boldsymbol{\varphi}^T(\tau)\, \mathbb{G}^p_{\omega^2,e}(\tau,\tau')\,\boldsymbol{J}(\tau') \tag{3.370f}$$

To simplify the notations, we set [cf. (3.345)]

$$\mathbb{G}^p_{\omega^2,e}(\tau,\tau') = \begin{pmatrix} G^p_{xx}(\tau,\tau') & G^p_{xp}(\tau,\tau') \\ G^p_{px}(\tau,\tau') & G^p_{pp}(\tau,\tau') \end{pmatrix} \tag{3.370g}$$

Using

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} G^p_{xx} & G^p_{xp} \\ G^p_{px} & G^p_{pp} \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} = a\, G^p_{xx}\, a' + a\, G^p_{xp}\, b' + b\, G^p_{px}\, a' + b\, G^p_{pp}\, b'$$

(3.370e) becomes

$$\begin{aligned}
\mathcal{A}_e^{\boldsymbol{J}}[\boldsymbol{\varphi}] = \frac{1}{2}\Big\{ & p_a\, G^p_{xx}(\beta\hbar,\beta\hbar)\, p_a - p_a \left[G^p_{xp}(\beta\hbar,\beta\hbar)\, x_b - G^p_{xp}(\beta\hbar,0_+)\, x_a\right] \\
& - \left[x_b\, G^p_{px}(\beta\hbar,\beta\hbar) - x_a\, G^p_{px}(0_+,\beta\hbar)\right] p_a + x_b\, G^p_{pp}(\beta\hbar,\beta\hbar)\, x_b \\
& - x_b\, G^p_{pp}(\beta\hbar,0_+)\, x_a - x_a\, G^p_{pp}(0_+,\beta\hbar)\, x_b + x_a\, G^p_{pp}(0_+,0_+)\, x_a \Big\} \\
= \frac{1}{2}\Big\{ & p_a^2\, G^p_{xx}(\beta\hbar,\beta\hbar) - 2\, p_a \left[x_b\, G^p_{xp}(\beta\hbar,\beta\hbar) - x_a\, G^p_{xp}(\beta\hbar,0_+)\right] \\
& + x_b^2\, G^p_{pp}(\beta\hbar,\beta\hbar) - 2\, x_b\, x_a\, G^p_{pp}(\beta\hbar,0) + x_a^2\, G^p_{pp}(0,0) \Big\}
\end{aligned} \tag{3.370h}$$

Note that it is necessary to keep track of the limiting behavior only for the off-diagonal elements and we've used

$$G^p_{xp}(\tau,\tau') = -G^p_{xp}(\tau',\tau) = G^p_{px}(\tau',\tau) = -G^p_{px}(\tau,\tau')$$

$$G^p_{xx}(\tau,\tau') = G^p_{xx}(\tau',\tau) \qquad\qquad G^p_{pp}(\tau,\tau') = G^p_{pp}(\tau',\tau)$$

Invoking the periodicity, we have

$$\begin{aligned}
\mathcal{A}_e^{\boldsymbol{J}}[\boldsymbol{\varphi}] &= \frac{1}{2}\left\{ p_a^2\, G^p_{xx}(0,0) - 2\, p_a \left[x_b\, G^p_{xp}(0,0) - x_a\, G^p_{xp}(0,0_+)\right] + (x_b - x_a)^2\, G^p_{pp}(0,0) \right\} \\
&= \frac{1}{2}\left\{ p_a^2\, G^p_{xx}(0) - 2\, p_a \left[x_b\, G^p_{xp}(0) + x_a\, G^p_{xp}(0_+)\right] + (x_b - x_a)^2\, G^p_{pp}(0) \right\}
\end{aligned} \tag{3.370i}$$

where we've used

$$G^p_{ij}(\tau,\tau') = G^p_{ij}(\tau-\tau') \qquad\qquad \forall\ i, j = x, p$$

At this point, $G^p_{xp}(0,0) = G^p_{xp}(0)$ is ambiguous.

Similarly, (3.370f) becomes

$$\begin{aligned}
\mathcal{A}_e^{\boldsymbol{J}}[\boldsymbol{J},\boldsymbol{\varphi}] = -i\int_0^{\beta\hbar} d\tau \Big\{ & \left[-p_a\, G^p_{xx}(\beta\hbar,\tau) + x_b\, G^p_{xp}(\tau,\beta\hbar) - x_a\, G^p_{xp}\left(\tau,0_+\right)\right] j(\tau) \\
& + \left[-p_a\, G^p_{xp}(\beta\hbar,\tau) + x_b\, G^p_{pp}(\beta\hbar,\tau) - x_a\, G^p_{pp}(0_+,\tau)\right] k(\tau) \Big\} \\
= -i\int_0^{\beta\hbar} d\tau \Big\{ & \left[-p_a\, G^p_{xx}(0,\tau) - (x_b - x_a)\, G^p_{xp}(0,\tau)\right] j(\tau) \\
& + \left[-p_a\, G^p_{xp}(0,\tau) + (x_b - x_a)\, G^p_{pp}(0,\tau)\right] k(\tau) \Big\}
\end{aligned}$$

$$= -i \int_0^{\beta\hbar} d\tau \left\{ \left[-p_a\, G^p_{xx}(\tau) + (x_b - x_a)\, G^p_{xp}(\tau) \right] j(\tau) + \left[p_a\, G^p_{xp}(\tau) + (x_b - x_a)\, G^p_{pp}(\tau) \right] k(\tau) \right\} \tag{3.370j}$$

In terms of Kleinert's notations, (3.370a) becomes

$$Z[\tilde{\boldsymbol{J}}] = Z^{(0)}_\omega[0, 0]\, Z^{(1)}_\omega[\boldsymbol{J}]\, Z^p_\omega[\boldsymbol{J}] \tag{3.370}$$

with

$$Z^{(0)}_\omega[0, 0] = Z_\omega \exp\left(-\frac{1}{\hbar} \mathcal{A}^{\boldsymbol{J}}_e[\boldsymbol{\varphi}] \right) \tag{3.371}$$

$$Z^{(1)}_\omega[\boldsymbol{J}] = \exp\left(-\frac{1}{\hbar} \mathcal{A}^{\boldsymbol{J}}_e[\boldsymbol{J}, \boldsymbol{\varphi}] \right) \tag{3.372}$$

$$Z^p_\omega[\boldsymbol{J}] = \exp\left(-\frac{1}{\hbar} \mathcal{A}^{\boldsymbol{J}}_e[\boldsymbol{J}] \right) \tag{3.373}$$

Note that $Z^{(0)}_\omega[0, 0]$ is independent of $\boldsymbol{J}$.

In the present notations, (3.368) becomes

$$(\, x_b\, ,\, \beta\hbar \mid x_a\, 0\,)[j, k] = \int_{-\infty}^{\infty} \frac{d\, p_a}{2\pi\hbar} Z[\tilde{\boldsymbol{J}}\,] \tag{3.374a}$$

Since p_a comes from $\boldsymbol{\varphi}$, the p_a-integral in (3.374a) involves only $\mathcal{A}^{\boldsymbol{J}}_e[\boldsymbol{\varphi}]$ & $\mathcal{A}^{\boldsymbol{J}}_e[\boldsymbol{J}, \boldsymbol{\varphi}]$. From (3.370h-i), we have

$$\mathcal{I} = \int_{-\infty}^{\infty} \frac{d\, p_a}{2\pi\hbar} \exp\left(-\frac{1}{\hbar} \left\{ \frac{1}{2} p_a^2\, G^p_{xx}(0) - p_a \left[x_b\; G^p_{xp}(0) + x_a\, G^p_{xp}(0_+) \right] + i\, p_a \int_0^{\beta\hbar} d\tau \left[G^p_{xx}(\tau)\, j(\tau) - G^p_{xp}(\tau)\, k(\tau) \right] \right\} \right)$$

$$= \frac{1}{\sqrt{2\pi\hbar\, G^p_{xx}(0)}}\, e^{Q/\hbar} \tag{3.374b}$$

where

$$Q = \frac{1}{2\, G^p_{xx}(0)} \left\{ x_b\, G^p_{xp}(0) + x_a\, G^p_{xp}(0_+) - i \int_0^{\beta\hbar} d\tau \left[G^p_{xx}(\tau)\, j(\tau) - G^p_{xp}(\tau)\, k(\tau) \right] \right\}^2 \tag{3.374c}$$

The ambiguity concerning $G^p_{xp}(0)$ can be removed by matching $(\, x_b\, ,\, \beta\hbar \mid x_a\, 0\,)[0, 0]$ with known result.

$(\, x_b\, ,\, \beta\hbar \mid x_a\, 0\,)[0, 0]$

Collecting terms from (3.370j) & (3.374c), we get the factor independent of $\boldsymbol{J}$ as

$$(\, x_b\, ,\, \beta\hbar \mid x_a\, 0\,)[0, 0] = \frac{Z_\omega}{\sqrt{2\pi\hbar\, G^p_{xx}(0)}} \exp\left(-\frac{1}{\hbar} \mathcal{A}_0 \right) \tag{3.375}$$

where

$$\mathcal{A}_0 = \frac{1}{2} \left\{ -\frac{1}{G^p_{xx}(0)} \left[x_b\, G^p_{xp}(0) + x_a\, G^p_{xp}(0_+) \right]^2 + (x_b - x_a)^2\, G^p_{pp}(0) \right\} \tag{3.375a}$$

Using (3.345b), we have

$$G^p_{xx}(0) = \frac{1}{2M\omega} \coth\left(\frac{1}{2} \beta\hbar\omega \right) \qquad G^p_{pp}(0) = \frac{1}{2} M\omega \coth\left(\frac{1}{2} \beta\hbar\omega \right)$$

$$G^p_{xp}(0_+) = \frac{i}{2} \tag{3.375b}$$

but there is an ambiguity

$$G^p_{xp}(0) = \pm\frac{i}{2} = \pm G^p_{xp}(0_+)$$

Thus, (3.375a) can be written as

$$\begin{aligned}
\mathcal{A}_0 &= \frac{1}{2}\left\{-\frac{G^{p\,2}_{xp}(0_+)}{G^p_{xx}(0)}\left(\pm x_b + x_a\right)^2 + (x_b - x_a)^2\, G^p_{pp}(0)\right\} \\
&= \frac{1}{2}\left\{\frac{M\omega}{2\coth\left(\frac{1}{2}\beta\hbar\omega\right)}\left(\pm x_b + x_a\right)^2 + \frac{1}{2}M\omega\coth\left(\frac{1}{2}\beta\hbar\omega\right)(x_b - x_a)^2\right\} \\
&= \frac{1}{4}M\omega\left\{\left[\frac{1}{\coth\left(\frac{1}{2}\beta\hbar\omega\right)} + \coth\left(\frac{1}{2}\beta\hbar\omega\right)\right]\left(x_b^2 + x_a^2\right)\right. \\
&\qquad\left. + 2x_b x_a\left[\pm\frac{1}{\coth\left(\frac{1}{2}\beta\hbar\omega\right)} - \coth\left(\frac{1}{2}\beta\hbar\omega\right)\right]\right\}
\end{aligned}$$

Using

$$\begin{aligned}
\pm\frac{1}{\coth\theta} - \coth\theta &= \pm\frac{\sinh\theta}{\cosh\theta} - \frac{\cosh\theta}{\sinh\theta} = \frac{\pm\sinh^2\theta - \cosh^2\theta}{\cosh\theta\sinh\theta} \\
&= \frac{2}{\sinh 2\theta}\begin{cases} -1 \\ -\cosh 2\theta \end{cases}
\end{aligned}$$

we have

$$\begin{aligned}
\mathcal{A}_0 &= \frac{1}{4}M\omega\left\{\frac{2\cosh(\beta\hbar\omega)}{\sinh(\beta\hbar\omega)}\left(x_b^2 + x_a^2\right)\right. \\
&\qquad\left. + 2x_b x_a\left[\pm\frac{1}{\coth\left(\frac{1}{2}\beta\hbar\omega\right)} - \coth\left(\frac{1}{2}\beta\hbar\omega\right)\right]\right\}
\end{aligned}$$

We choose the upper sign so that

$$G^p_{xp}(0) = \frac{i}{2} = G^p_{xp}(0_+) \tag{3.375c}$$

and

$$\mathcal{A}_0 = \frac{M\omega}{2\sinh(\beta\hbar\omega)}\left\{\left(x_b^2 + x_a^2\right)\cosh(\beta\hbar\omega) - 2x_b x_a\right\} \tag{3.375d}$$

is just the classical action [see (2.409)].

Using

$$Z_\omega = \frac{1}{2\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} \tag{3.214}$$

we have

$$\begin{aligned}
\frac{Z_\omega}{\sqrt{2\pi\hbar G^p_{xx}(0)}} &= \frac{1}{2\sinh\left(\frac{1}{2}\beta\hbar\omega\right)}\sqrt{\frac{M\omega\sinh\left(\frac{1}{2}\beta\hbar\omega\right)}{\pi\hbar\cosh\left(\frac{1}{2}\beta\hbar\omega\right)}} \\
&= \sqrt{\frac{M\omega}{2\pi\hbar\sinh(\beta\hbar\omega)}}
\end{aligned} \tag{3.375e}$$

(3.375) thus becomes

$$(x_b\,,\,\beta\hbar\mid x_a\,0)[0,\,0]=\sqrt{\frac{M\,\omega}{2\,\pi\,\hbar\,\sinh(\beta\,\hbar\,\omega)}}\;\exp\left\{-\frac{1}{\hbar}\mathcal{A}_0\right\} \tag{3.377}$$

in agreement with (2.409).

$(x_b\,,\,\beta\hbar\mid x_a\,0)[j,\,k]$

From (3.370j), we get the terms in $\mathcal{A}_e^J[\boldsymbol{J},\,\boldsymbol{\varphi}]$ that are not involved in the p_a-integral

$$C\,[\boldsymbol{J},\,\boldsymbol{\varphi}]=-i\,(x_b-x_a)\int_0^{\beta\hbar}d\,\tau\left\{G^p_{x\,p}(\tau)\,j(\tau)+G^p_{p\,p}(\tau)\,k(\tau)\right\} \tag{3.377b}$$

On the other hand, all terms in $\mathcal{A}_e^J[\boldsymbol{\varphi}]$ not involved in the p_a-integral are already included in $(x_b\,,\,\beta\hbar\mid x_a\,0)[0,\,0]$.

The terms in Q that are not included in $(x_b\,,\,\beta\hbar\mid x_a\,0)[0,\,0]$ are denoted as [see (3.374c)]

$$Q\,'=\frac{1}{2\,G^p_{x\,x}(0)}\left\{-2\,i\left(x_b+x_a\right)G^p_{x\,p}(0_+)\int_0^{\beta\hbar}d\,\tau\left[G^p_{x\,x}(\tau)\,j(\tau)\,-G^p_{x\,p}(\tau)\,k(\tau)\right]\right.$$
$$\left.-\left(\int_0^{\beta\hbar}d\,\tau\left[G^p_{x\,x}(\tau)\,j(\tau)\,-G^p_{x\,p}(\tau)\,k(\tau)\right]\right)^2\right\} \tag{3.377c}$$

Thus,

$$C\,[\boldsymbol{J},\,\boldsymbol{\varphi}]-Q\,'=\int_0^{\beta\hbar}d\,\tau\left[c_j(\tau)\,j(\tau)+c_k(\tau)\,k(\tau)\right]$$
$$-\frac{1}{2}\int_0^{\beta\hbar}d\,\tau\int_0^{\beta\hbar}d\,\tau'\,\boldsymbol{J}^T(\tau)\,\mathbb{X}(\tau,\,\tau')\,\boldsymbol{J}(\tau') \tag{3.377d}$$

where

$$c_j(\tau)=-i\left[(x_b-x_a)\,G^p_{x\,p}(\tau)-(x_b+x_a)\,\frac{G^p_{x\,p}(0_+)}{G^p_{x\,x}(0)}\,G^p_{x\,x}(\tau)\right] \tag{3.377e}$$

$$c_k(\tau)=-i\left[(x_b-x_a)\,G^p_{p\,p}(\tau)+(x_b+x_a)\,\frac{G^p_{x\,p}(0_+)}{G^p_{x\,x}(0)}\,G^p_{x\,p}(\tau)\right] \tag{3.377f}$$

$$\mathbb{X}(\tau,\,\tau')=-\frac{1}{G^p_{x\,x}(0)}\begin{pmatrix}G^p_{x\,x}(\tau)\,G^p_{x\,x}(\tau') & -G^p_{x\,x}(\tau)\,G^p_{x\,p}(\tau')\\ -G^p_{x\,p}(\tau)\,G^p_{x\,x}(\tau') & G^p_{x\,p}(\tau)\,G^p_{x\,p}(\tau')\end{pmatrix} \tag{3.377g}$$

Using (3.345b), we have, for $\tau\in[0,\,\beta\hbar)$,

$$G^p_{x\,p}(\tau)\;=\frac{i}{2}\,\frac{\sinh\omega\left(\frac{1}{2}\,\beta\,\hbar-\tau\right)}{\sinh\left(\frac{1}{2}\,\beta\,\hbar\,\omega\right)} \tag{3.377h}$$

$$G^p_{x\,x}(\tau)=\frac{1}{2\,M\,\omega}\,\frac{\cosh\omega\left(\frac{1}{2}\,\beta\,\hbar-\tau\right)}{\sinh\left(\frac{1}{2}\,\beta\,\hbar\,\omega\right)}=\frac{1}{(M\,\omega)^2}\,G^p_{p\,p}(\tau)$$

Together with (3.375b-c), (3.377e) becomes

$$c_j(\tau)\;=\frac{1}{2}\left[(x_b-x_a)\,\frac{\sinh\omega\left(\frac{1}{2}\,\beta\,\hbar-\tau\right)}{\sinh\left(\frac{1}{2}\,\beta\,\hbar\,\omega\right)}-(x_b+x_a)\,\frac{\cosh\omega\left(\frac{1}{2}\,\beta\,\hbar-\tau\right)}{\cosh\left(\frac{1}{2}\,\beta\,\hbar\,\omega\right)}\right]$$

Using

$$\frac{\sinh\omega\left(\frac{1}{2}\,\beta\,\hbar-\tau\right)}{\sinh\left(\frac{1}{2}\,\beta\,\hbar\,\omega\right)}\pm\frac{\cosh\omega\left(\frac{1}{2}\,\beta\,\hbar-\tau\right)}{\cosh\left(\frac{1}{2}\,\beta\,\hbar\,\omega\right)}$$

$$= \left(\sinh\omega\left(\frac{1}{2}\beta\hbar - \tau\right)\cosh\left(\frac{1}{2}\beta\hbar\omega\right) \pm \cosh\omega\left(\frac{1}{2}\beta\hbar - \tau\right)\sinh\left(\frac{1}{2}\beta\hbar\omega\right)\right) \Big/ \left(\sinh\left(\frac{1}{2}\beta\hbar\omega\right)\cosh\left(\frac{1}{2}\beta\hbar\omega\right)\right)$$

$$= \frac{2}{\sinh(\beta\hbar\omega)} \begin{cases} \sinh\omega(\beta\hbar - \tau) \\ -\sinh\omega\tau \end{cases}$$

we have

$$c_j(\tau) = -\frac{1}{\sinh(\beta\hbar\omega)}\left[x_b \sinh\omega\tau + x_a \sinh\omega(\beta\hbar - \tau)\right] \tag{3.380}$$

$$= -x_{\rm cl}^0(\tau)$$

where $x_{\rm cl}^0(\tau)$ is the source-free classical path.

Similarly, (3.377f) becomes

$$c_k(\tau) = -i\frac{M\omega}{2}\left[(x_b - x_a)\frac{\cosh\omega\left(\frac{1}{2}\beta\hbar - \tau\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} - (x_b + x_a)\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar - \tau\right)}{\cosh\left(\frac{1}{2}\beta\hbar\omega\right)}\right]$$

Using

$$\frac{\cosh\omega\left(\frac{1}{2}\beta\hbar - \tau\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} \pm \frac{\sinh\omega\left(\frac{1}{2}\beta\hbar - \tau\right)}{\cosh\left(\frac{1}{2}\beta\hbar\omega\right)}$$

$$= \left(\cosh\omega\left(\frac{1}{2}\beta\hbar - \tau\right)\cosh\left(\frac{1}{2}\beta\hbar\omega\right) \pm \sinh\omega\left(\frac{1}{2}\beta\hbar - \tau\right)\sinh\left(\frac{1}{2}\beta\hbar\omega\right)\right) \Big/ \left(\sinh\left(\frac{1}{2}\beta\hbar\omega\right)\cosh\left(\frac{1}{2}\beta\hbar\omega\right)\right)$$

$$= \frac{2}{\sinh(\beta\hbar\omega)} \begin{cases} \cosh\omega(\beta\hbar - \tau) \\ \cosh\omega\tau \end{cases}$$

we have

$$c_k(\tau) = -\frac{iM\omega}{\sinh(\beta\hbar\omega)}\left[x_b \cosh\omega\tau - x_a \cosh\omega(\beta\hbar - \tau)\right] \tag{3.381}$$

$$= -iM\partial_\tau x_{\rm cl}^0(\tau)$$

$$= -p_{\rm cl}^0(\tau)$$

where

$$p_{\rm cl}^0(\tau) = \frac{d\,x_{\rm cl}^0}{d\,t} = i\,\partial_\tau x_{\rm cl}^0(\tau)$$

is the source-free classical momentum.

(3.374a) thus becomes

$$(x_b, \beta\hbar \mid x_a\, 0)[j, k]$$

$$= (x_b, \beta\hbar \mid x_a\, 0)[0, 0]\, \exp\left\{-\frac{1}{\hbar}\left(C[\boldsymbol{J}, \boldsymbol{\varphi}] + Q' + \mathcal{A}_e^J[\boldsymbol{J}]\right)\right\}$$

$$= (x_b, \beta\hbar \mid x_a\, 0)[0, 0]\, \exp\left\{\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau\left[x_{\rm cl}^0(\tau)\, j(\tau) + i\, p_{\rm cl}^0(\tau)\, k(\tau)\right]\right\} \tag{3.374}$$

$$\times \exp\left\{\frac{1}{2\hbar}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, \boldsymbol{J}^T(\tau)\, \mathbb{G}_{\omega^2, e}^{D}(\tau, \tau')\, \boldsymbol{J}(\tau')\right\}$$

where

$$\mathbb{G}_{\omega^2, e}^{D}(\tau, \tau') = \mathbb{G}_{\omega^2, e}^{p}(\tau, \tau') + \mathbb{X}(\tau, \tau')$$

$$= \begin{pmatrix} G^p_{xx}(\tau,\,\tau') - \dfrac{G^p_{xx}(\tau)\,G^p_{xx}(\tau')}{G^p_{xx}(0)} & G^p_{xp}(\tau,\,\tau') + \dfrac{G^p_{xx}(\tau)\,G^p_{xp}(\tau')}{G^p_{xx}(0)} \\ G^p_{px}(\tau,\,\tau') + \dfrac{G^p_{xp}(\tau)\,G^p_{xx}(\tau')}{G^p_{xx}(0)} & G^p_{pp}(\tau,\,\tau') - \dfrac{G^p_{xp}(\tau)\,G^p_{xp}(\tau')}{G^p_{xx}(0)} \end{pmatrix} \quad (3.382\text{-}5)$$

$$\equiv \begin{pmatrix} G^D_{xx}(\tau,\,\tau') & G^D_{xp}(\tau,\,\tau') \\ G^D_{px}(\tau,\,\tau') & G^D_{pp}(\tau,\,\tau') \end{pmatrix}$$

$$\frac{G^p_{xx}(\tau)\,G^p_{xx}(\tau')}{G^p_{xx}(0)} = \frac{1}{2M\omega}\,\frac{\cosh\omega\left(\frac{1}{2}\beta\hbar-\tau\right)\cosh\omega\left(\frac{1}{2}\beta\hbar-\tau'\right)}{\cosh\left(\frac{1}{2}\beta\hbar\omega\right)\sinh\left(\frac{1}{2}\beta\hbar\omega\right)}$$

$$= \frac{1}{2M\omega}\,\frac{\cosh\omega(\beta\hbar-\tau-\tau') + \cosh\omega(\tau-\tau')}{\sinh(\beta\hbar\omega)}$$

$$G^p_{xx}(\tau,\,\tau') = \frac{1}{2M\omega}\,\frac{\cosh\omega\left(\frac{1}{2}\beta\hbar-|\tau-\tau'|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)}$$

$$= \frac{1}{M\omega}\,\frac{\cosh\omega\left(\frac{1}{2}\beta\hbar-|\tau-\tau'|\right)\cosh\left(\frac{1}{2}\beta\hbar\omega\right)}{\sinh(\beta\hbar\omega)}$$

$$= \frac{1}{2M\omega}\,\frac{\cosh\omega(\beta\hbar-|\tau-\tau'|) + \cosh\omega(\tau-\tau')}{\sinh(\beta\hbar\omega)}$$

$$\rightarrow \qquad G^D_{xx}(\tau,\,\tau') = \frac{1}{2M\omega}\left((\cosh\omega(\beta\hbar-|\tau-\tau'|) - \cosh\omega(\beta\hbar-\tau-\tau'))/\sinh(\beta\hbar\omega)\right) \qquad (3.386)$$

$$\frac{G^p_{xx}(\tau)\,G^p_{xp}(\tau')}{G^p_{xx}(0)} = \frac{\cosh\omega\left(\frac{1}{2}\beta\hbar-|\tau|\right)}{\cosh\left(\frac{1}{2}\beta\hbar\omega\right)}\,\frac{i\,\epsilon(\tau')}{2}\,\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar-|\tau'|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)}$$

$$= \frac{i\,\epsilon(\tau')}{2}\left((\sinh\omega(\beta\hbar-|\tau|-|\tau'|) + \sinh\omega(|\tau|-|\tau'|))/\sinh(\beta\hbar\omega)\right)$$

$$= \frac{i}{2}\,\frac{\sinh\omega(\beta\hbar-\tau-\tau') + \sinh\omega(\tau-\tau')}{\sinh(\beta\hbar\omega)} \qquad \tau,\,\tau' \in [0,\,\beta\hbar)$$

$$G^p_{xp}(\tau,\,\tau') = \frac{i\,\epsilon(\tau-\tau')}{2}\,\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar-|\tau-\tau'|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)}$$

$$= i\,\epsilon(\tau-\tau')\,\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar-|\tau-\tau'|\right)\cosh\left(\frac{1}{2}\beta\hbar\omega\right)}{\sinh(\beta\hbar\omega)}$$

$$= \frac{i}{2}\,\epsilon(\tau-\tau')\,\frac{\sinh\omega(\beta\hbar-|\tau-\tau'|) - \sinh(\omega\,|\tau-\tau'|)}{\sinh(\beta\hbar\omega)}$$

$$\rightarrow \qquad G^D_{xp}(\tau,\,\tau') = \frac{i}{2\sinh(\beta\hbar\omega)}\Big\{\epsilon(\tau-\tau')\Big[\sinh\omega(\beta\hbar-|\tau-\tau'|) - \sinh(\omega\,|\tau-\tau'|)\Big] + \sinh\omega(\beta\hbar-\tau-\tau') + \sinh\omega(\tau-\tau')\Big\}$$

For $\tau > \tau'$,

$$G^D_{xp}(\tau,\,\tau') = \frac{i}{2\sinh(\beta\hbar\omega)}\Big\{\sinh\omega(\beta\hbar-\tau+\tau') + \sinh\omega(\beta\hbar-\tau-\tau')\Big\}$$

For $\tau < \tau'$,

$$G^D_{xp}(\tau,\ \tau') = \frac{i}{2\sinh(\beta\hbar\omega)}\Big\{ -\sinh\omega\,(\beta\hbar - \tau' + \tau) + \sinh\omega\,(\beta\hbar - \tau - \tau')\Big\}$$

$$\therefore \quad G^D_{xp}(\tau,\ \tau') = \frac{i}{2\sinh(\beta\hbar\omega)}\Big\{\ \theta(\tau-\tau')\sinh\omega\big[\beta\hbar - (\tau-\tau')\big] - \theta(\tau'-\tau)\sinh\omega\big[\beta\hbar - (\tau'-\tau)\big] + \sinh\omega\,(\beta\hbar - \tau - \tau')\Big\} \qquad (3.387)$$

$$G^D_{px}(\tau,\ \tau') = G^D_{xp}(\tau',\ \tau) = \frac{i}{2\sinh(\beta\hbar\omega)}\Big\{\ \theta(\tau'-\tau)\sinh\omega\big[\beta\hbar - (\tau'-\tau)\big] - \theta(\tau-\tau')\sinh\omega\big[\beta\hbar - (\tau-\tau')\big] + \sinh\omega\,(\beta\hbar - \tau - \tau')\Big\} \qquad (3.388)$$

$$\frac{G^p_{xp}(\tau)\, G^p_{xp}(\tau')}{G^p_{xx}(0)} = -\frac{M\omega}{2}\,\epsilon(\tau)\,\epsilon(\tau')\,\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar - |\tau|\right)}{\cosh\left(\frac{1}{2}\beta\hbar\omega\right)}\,\frac{\sinh\omega\left(\frac{1}{2}\beta\hbar - |\tau'|\right)}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)}$$

$$= -\frac{M\omega}{2\sinh(\beta\hbar\omega)}\big[\cosh\omega(\beta\hbar - \tau - \tau') - \cosh\omega\,(\tau-\tau')\big]$$

$$G^p_{pp}(\tau,\ \tau') = (M\omega)^2\, G^p_{xx}(\tau,\ \tau') = \frac{M\omega}{2\sinh(\beta\hbar\omega)}\big[\cosh\omega\,(\beta\hbar - |\tau-\tau'|) + \cosh\omega(\tau-\tau')\big]$$

$$\to\ G^D_{pp}(\tau,\ \tau') = \frac{M\omega}{2\sinh(\beta\hbar\omega)}\big[\cosh\omega\,(\beta\hbar - |\tau-\tau'|) + \cosh\omega(\beta\hbar - \tau - \tau')\big] \qquad (3.389)$$

From (3.386-9), we have

$$G^D_{xx}(0,\ \tau') = \frac{1}{2M\omega}\,\frac{\cosh\omega\,(\beta\hbar - |\tau'|) - \cosh\omega\,(\beta\hbar - \tau')}{\sinh(\beta\hbar\omega)}$$

$$= 0 \quad = G^D_{xx}(\tau,\ 0) \qquad \text{for } \tau,\ \tau' \in [0,\ \beta\hbar)$$

$$G^D_{xp}(0,\ \tau') = \frac{i}{2\sinh(\beta\hbar\omega)}\Big\{\ \theta(-\tau')\sinh\omega\,(\beta\hbar + \tau') - \theta(\tau')\sinh\omega\,(\beta\hbar - \tau') + \sinh\omega\,(\beta\hbar - \tau')\Big\}$$

$$= 0 \qquad \text{for } \tau,\ \tau' \in [0,\ \beta\hbar)$$

$$G^D_{px}(\tau,\ 0) = G^D_{xp}(\tau,\ 0) = 0$$

However,

$$G^D_{xp}(\tau,\ 0) = \frac{i}{2\sinh(\beta\hbar\omega)}\Big\{\ \theta(\tau)\sinh\omega\,(\beta\hbar - \tau) - \theta(-\tau)\sinh\omega\,(\beta\hbar + \tau) + \sinh\omega\,(\beta\hbar - \tau)\Big\}$$

$$= \frac{i}{\sinh(\beta\hbar\omega)}\sinh\omega\,(\beta\hbar - \tau) \qquad \text{for } \tau,\ \tau' \in [0,\ \beta\hbar)$$

$$= G^D_{px}(0,\ \tau) \neq 0$$

$$G^D_{pp}(0,\ \tau') = \frac{M\omega}{2\sinh(\beta\hbar\omega)}\big[\cosh\omega\,(\beta\hbar - |\tau'|) + \cosh\omega(\beta\hbar - \tau')\big]$$

$$= \frac{M\omega}{\sinh(\beta\hbar\omega)}\cosh\omega(\beta\hbar - \tau') \qquad \text{for } \tau,\ \tau' \in [0,\ \beta\hbar)$$

$$\neq 0$$

Hence, only the spatial x-components of $\mathbb{G}^{D}_{\omega^2,e}(\tau,\ \tau')$ obey the Dirichlet B.C.

Either from the symmetry properties of $\mathbb{G}^{p}_{\omega^2,e}(\tau,\ \tau')$ or more directly from (3.386-9), we have

$$G^{D}_{xx}(\tau,\ \tau') = G^{D}_{xx}(\tau',\ \tau) \qquad G^{D}_{pp}(\tau,\ \tau') = G^{D}_{pp}(\tau',\ \tau)$$

$$G^{D}_{xp}(\tau,\ \tau') = -G^{D}_{xp}(\tau',\ \tau) = G^{D}_{px}(\tau',\ \tau) = -G^{D}_{px}(\tau,\ \tau') \qquad (3.390\text{-}2)$$

Using [see (3.345)]

$$G^{p}_{xp}(\tau,\ \tau') = -i\,\partial_\tau G^{p}_{\omega^2,e}(\tau,\ \tau') = -i\,M\,\partial_\tau G^{p}_{xx}(\tau,\ \tau')$$

$$G^{p}_{px}(\tau,\ \tau') = i\,\partial_\tau G^{p}_{\omega^2,e}(\tau,\ \tau') = i\,M\,\partial_\tau G^{p}_{xx}(\tau,\ \tau')$$

we get

$$\begin{aligned}
-i\,M\,\partial_\tau G^{D}_{xx}(\tau,\ \tau') &= -i\,M\,\partial_\tau G^{p}_{xx}(\tau,\ \tau') - \frac{-i\,M\,\partial_\tau G^{p}_{xx}(\tau)\,G^{p}_{xx}(\tau')}{G^{p}_{xx}(0)} \\
&= G^{p}_{xp}(\tau,\ \tau') - \frac{G^{p}_{xp}(\tau)\,G^{p}_{xx}(\tau')}{G^{p}_{xx}(0)} \\
&= -G^{p}_{xp}(\tau',\ \tau) - \frac{G^{p}_{xx}(\tau')\,G^{p}_{xp}(\tau)}{G^{p}_{xx}(0)} \\
&= -G^{D}_{xp}(\tau',\ \tau)
\end{aligned}$$

$$\therefore \quad G^{D}_{xp}(\tau,\ \tau') = i\,M\,\partial_{\tau'} G^{D}_{xx}(\tau',\ \tau) = i\,M\,\partial_{\tau'} G^{D}_{xx}(\tau,\ \tau') \qquad (3.393)$$

$$G^{D}_{px}(\tau,\ \tau') = G^{D}_{xp}(\tau',\ \tau) = i\,M\,\partial_\tau G^{D}_{xx}(\tau,\ \tau') \qquad (3.394)$$

Caution: Owing to the presence of the term $\cosh\omega(\beta\hbar - \tau - \tau')$ in (3.386),

$$\partial_\tau G^{D}_{xx}(\tau,\ \tau') \neq -\partial_{\tau'} G^{D}_{xx}(\tau,\ \tau')$$

so that parts of Kleinert's version of (3.393-4) are not valid.

Finally,

$$\begin{aligned}
\partial_\tau \partial_{\tau'} G^{D}_{xx}(\tau,\ \tau') &= \partial_\tau \partial_{\tau'} G^{p}_{xx}(\tau,\ \tau') - \frac{\partial_\tau G^{p}_{xx}(\tau)\,\partial_{\tau'} G^{p}_{xx}(\tau')}{G^{p}_{xx}(0)} \\
&= \frac{1}{M}\,\partial_\tau \partial_{\tau'} G^{p}_{\omega^2,e}(\tau,\ \tau') - \frac{1}{M^2}\,\frac{\partial_\tau G^{p}_{\omega^2,e}(\tau)\,\partial_{\tau'} G^{p}_{\omega^2,e}(\tau')}{G^{p}_{xx}(0)} \\
&= -\frac{1}{M}\,\partial_\tau^2 G^{p}_{\omega^2,e}(\tau,\ \tau') + \frac{1}{M^2}\,\frac{G^{p}_{xp}(\tau)\,G^{p}_{xp}(\tau')}{G^{p}_{xx}(0)} \\
&= -\frac{1}{M}\,\omega^2\,G^{p}_{\omega^2,e}(\tau,\ \tau') + \frac{1}{M}\,\delta(\tau - \tau') + \frac{1}{M^2}\,\frac{G^{p}_{xp}(\tau)\,G^{p}_{xp}(\tau')}{G^{p}_{xx}(0)} \\
&= -\frac{1}{M^2}\,G^{p}_{pp}(\tau,\ \tau') + \frac{1}{M}\,\delta(\tau - \tau') + \frac{1}{M^2}\,\frac{G^{p}_{xp}(\tau)\,G^{p}_{xp}(\tau')}{G^{p}_{xx}(0)} \\
&= -\frac{1}{M^2}\,G^{D}_{pp}(\tau,\ \tau') + \frac{1}{M}\,\delta(\tau - \tau')
\end{aligned} \qquad (3.395)$$