The Fourier expansion of the vector potential in a volume V can be written as

$$\boldsymbol{A}(\boldsymbol{x},\, t) = \sum_{\boldsymbol{k}} c_{\boldsymbol{k}}(\boldsymbol{x})\, \boldsymbol{X}_{\boldsymbol{k}}(t) \tag{3.430}$$

where

$$c_{\boldsymbol{k}}(\boldsymbol{x}) = \frac{1}{\sqrt{V}}\, e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} \qquad \sum_{\boldsymbol{k}} = \frac{V}{(2\,\pi)^3} \int d^3 k \tag{3.430a}$$

The EM field action is

$$\mathcal{A}_{\text{EM}} = \frac{1}{8\,\pi} \int_0^{\beta\hbar} d\,\tau \int d^3 x \left(\boldsymbol{B}^2 - \boldsymbol{E}^2\right) \tag{3.430b}$$

where $\boldsymbol{E}$ & $\boldsymbol{B}$ are the electric & magnetic fields, respectively.

Using

$$\boldsymbol{E} = \frac{1}{c}\frac{\partial\,\boldsymbol{A}}{\partial\,t} = \frac{1}{c} \sum_{\boldsymbol{k}} c_{\boldsymbol{k}}(\boldsymbol{x})\, \frac{d\boldsymbol{X}_k(t)}{d\,t} = \frac{i}{c} \sum_{\boldsymbol{k}} c_{\boldsymbol{k}}(\boldsymbol{x})\, \boldsymbol{\partial}_\tau \boldsymbol{X}_k(\tau)$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \sum_{\boldsymbol{k}} \nabla c_{\boldsymbol{k}}(\boldsymbol{x}) \times \boldsymbol{X}_{\boldsymbol{k}}(\tau) = i \sum_{\boldsymbol{k}} c_{\boldsymbol{k}}(\boldsymbol{x})\, \boldsymbol{k}\times\boldsymbol{X}_{\boldsymbol{k}}(\tau)$$

where c is the speed of light, we have

$$\int d^3 x\, \boldsymbol{E}^2 = -\frac{1}{c^2} \sum_{\boldsymbol{k},\,\boldsymbol{k}'} \int d^3 x\;\; c_{\boldsymbol{k}}(\boldsymbol{x})\, c_{\boldsymbol{k}'}(\boldsymbol{x})\, \partial_\tau \boldsymbol{X}_k(\tau) \cdot \partial_\tau \boldsymbol{X}_{\boldsymbol{k}'}(\tau)$$

$$= -\frac{1}{c^2} \sum_{\boldsymbol{k},\,\boldsymbol{k}'} \frac{1}{V} \int d^3 x\;\; e^{i\,(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}}\, \partial_\tau \boldsymbol{X}_k(\tau) \cdot \partial_\tau \boldsymbol{X}_{\boldsymbol{k}'}(\tau)$$

$$= -\frac{1}{c^2} \sum_{\boldsymbol{k}} \partial_\tau \boldsymbol{X}_{\boldsymbol{k}} \cdot \partial_\tau \boldsymbol{X}_{-\boldsymbol{k}} \tag{3.430c}$$

where

$$\frac{1}{V} \int d^3 x\;\; e^{i\,(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}} = \delta_{\boldsymbol{k},\,-\boldsymbol{k}'}$$

Similarly,

$$\int d^3 x\, \boldsymbol{B}^2 = -\sum_{\boldsymbol{k},\,\boldsymbol{k}'} \int d^3 x\;\; c_{\boldsymbol{k}}(\boldsymbol{x})\, c_{\boldsymbol{k}'}(\boldsymbol{x}) \left[\, \boldsymbol{k}\times\boldsymbol{X}_k(\tau) \,\right] \cdot \left[\, \boldsymbol{k}'\times\boldsymbol{X}_{\boldsymbol{k}'}(\tau) \,\right]$$

$$= -\sum_{\boldsymbol{k},\,\boldsymbol{k}'} \frac{1}{V} \int d^3 x\;\; e^{i\,(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}} \left[\, \boldsymbol{k}\times\boldsymbol{X}_k(\tau) \,\right] \cdot \left[\, \boldsymbol{k}'\times\boldsymbol{X}_{\boldsymbol{k}'}(\tau) \,\right]$$

$$= \sum_{\boldsymbol{k}} (\,\boldsymbol{k}\times\boldsymbol{X}_k) \cdot (\,\boldsymbol{k}\times\boldsymbol{X}_{-\boldsymbol{k}})$$

$$= \sum_{\boldsymbol{k}} \left[\, \boldsymbol{k}^2\, \boldsymbol{X}_k \cdot \boldsymbol{X}_{-\boldsymbol{k}} - (\boldsymbol{k}\cdot\boldsymbol{X}_{\boldsymbol{k}})\,(\boldsymbol{k}\cdot\boldsymbol{X}_{-\boldsymbol{k}}) \right]$$

In the Coulomb gauge

$$\nabla\cdot\boldsymbol{A} = 0 \qquad \rightarrow \qquad \boldsymbol{k}\cdot\boldsymbol{X}_{\boldsymbol{k}} = 0 \tag{3.430d}$$

we have

$$\int d^3 x\, \boldsymbol{B}^2 = \sum_{\boldsymbol{k}} \boldsymbol{k}^2\, \boldsymbol{X}_k \cdot \boldsymbol{X}_{-\boldsymbol{k}} \tag{3.430e}$$

(3.430b) becomes

$$\mathcal{A}_{\text{EM}} = \int_0^{\beta\hbar} d\,\tau \sum_{\boldsymbol{k}} \frac{1}{c^2} \left[\, \partial_\tau \boldsymbol{X}_{\boldsymbol{k}} \cdot \partial_\tau \boldsymbol{X}_{-\boldsymbol{k}} + c^2\, \boldsymbol{k}^2\, \boldsymbol{X}_k \cdot \boldsymbol{X}_{-\boldsymbol{k}} \right]$$

Since $\boldsymbol{A}$ is real, we have

$$\boldsymbol{X}_{-\boldsymbol{k}} = \boldsymbol{X}_{\boldsymbol{k}}^* \qquad \boldsymbol{X}_k \cdot \boldsymbol{X}_{-\boldsymbol{k}} = |\, \boldsymbol{X}_{\boldsymbol{k}} \,|^2 \qquad \partial_\tau \boldsymbol{X}_{\boldsymbol{k}} \cdot \partial_\tau \boldsymbol{X}_{-\boldsymbol{k}} = |\, \partial_\tau \boldsymbol{X}_{\boldsymbol{k}} \,|^2$$

so that

$$\mathcal{A}_{\text{EM}} = \int_0^{\beta\hbar} d\tau \sum_{\boldsymbol{k}} \frac{1}{c^2} \left[|\partial_\tau \boldsymbol{X}_{\boldsymbol{k}}|^2 + \Omega_{\boldsymbol{k}}^2 |\boldsymbol{X}_{\boldsymbol{k}}|^2 \right] \qquad (3.430f)$$

where

$$\Omega_{\boldsymbol{k}} = c\,|\,\boldsymbol{k}\,| \qquad (3.430g)$$

Thus, the EM field consists of a collection of harmonic oscillators with unit mass & frequencies $\Omega_{\boldsymbol{k}}$. A photon of wave vector $\boldsymbol{k}$ is therefore a quantum of the oscillator $\boldsymbol{X}_{\boldsymbol{k}}(t)$, while n such photons form the n^{th} excited state of $\boldsymbol{X}_{\boldsymbol{k}}(t)$.

The statistical distribution of these photons describes the black-body radiation in terms of the Planck’s formula.

Using the imaginary-time version of the Lagrangian (2.640a), the Euclidean action for a charged particle in an EM field is

$$\begin{aligned}\mathcal{A}_e &= \int_0^{\beta\hbar} d\tau \left(\frac{1}{2} m \dot{\boldsymbol{x}}^2 - i \frac{e}{c} \dot{\boldsymbol{x}} \cdot \boldsymbol{A} \right) \\ &= \int_0^{\beta\hbar} d\tau \left(\frac{1}{2} m \dot{\boldsymbol{x}}^2 - i \frac{e}{c} \sum_{\boldsymbol{k}} c_{\boldsymbol{k}}(\boldsymbol{x})\, \dot{\boldsymbol{x}} \cdot \boldsymbol{X}_{\boldsymbol{k}} \right) \qquad (3.430h)\end{aligned}$$

Moving the interaction term to the photon bath, we have [cf.(3.396a)]

$$\mathcal{A}_{\text{EM}}[c_{\boldsymbol{k}}] = \int_0^{\beta\hbar} d\tau \sum_{\boldsymbol{k}} \left[\frac{1}{c^2} \left(|\partial_\tau \boldsymbol{X}_{\boldsymbol{k}}|^2 + \Omega_{\boldsymbol{k}}^2 |\boldsymbol{X}_{\boldsymbol{k}}|^2 \right) - i \frac{e}{c} c_{\boldsymbol{k}}(\boldsymbol{x})\, \dot{\boldsymbol{x}} \cdot \boldsymbol{X}_{\boldsymbol{k}} \right] \qquad (3.430i)$$

Since $\boldsymbol{X}_{\boldsymbol{k}}$ must obey the transverse condition (3.430d), we define the Green function matrix as

$$\left(-\partial_\tau^2 + \Omega_{\boldsymbol{k}}^2\right) \mathbb{G}_{\boldsymbol{k},\boldsymbol{k}'}(\tau, \tau') = \delta_{\boldsymbol{k},-\boldsymbol{k}'}\, \delta(\tau - \tau')\, \mathbb{I}_T(\boldsymbol{k}) \qquad (3.430j)$$

where

$$\mathbb{I}_T(\boldsymbol{k}) = \mathbb{I} - \frac{1}{\boldsymbol{k}^2} \boldsymbol{k}\,\boldsymbol{k}^T \qquad (3.430k)$$

is the projector onto the plane perpendicular to $\boldsymbol{k}$. Thus, for any vector $\boldsymbol{p}$, we have

$$\boldsymbol{k}^T\, \mathbb{I}_T(\boldsymbol{k})\, \boldsymbol{p} = \boldsymbol{k}^T \boldsymbol{p} - \frac{1}{\boldsymbol{k}^2} \boldsymbol{k}^T \boldsymbol{k}\,\boldsymbol{k}^T \boldsymbol{p} = \boldsymbol{k}^T \boldsymbol{p} - \boldsymbol{k}^T \boldsymbol{p} = 0$$

The Green function is related to the correlation function by [cf. (3.301)]

$$\begin{aligned}\mathbb{G}_{\boldsymbol{k},\boldsymbol{k}'}(\tau, \tau') &= \frac{1}{\hbar} \left\langle \boldsymbol{X}_{\boldsymbol{k}}(\tau)\, \boldsymbol{X}_{\boldsymbol{k}'}^T(\tau') \right\rangle \\ &= \delta_{\boldsymbol{k},-\boldsymbol{k}'}\, G^p_{\Omega_{\boldsymbol{k}}^2,e}(\tau, \tau')\, \mathbb{I}_T(\boldsymbol{k}) \qquad (3.430j)\end{aligned}$$

where [see (3.219)]

$$G^p_{\Omega_{\boldsymbol{k}}^2,e}(\tau, \tau') = \frac{\cosh\left[\Omega_{\boldsymbol{k}}\left(\frac{1}{2}\beta\hbar - |\,\tau - \tau'\,|\right)\right]}{2\,\Omega_{\boldsymbol{k}} \sinh(\beta\hbar\Omega_{\boldsymbol{k}}/2)} \qquad (3.435)$$

The photon bath action [cf. (3.399)] is therefore

$$\mathcal{A}_{\text{bath}} = \frac{1}{2} \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, \dot{\boldsymbol{x}}^T(\tau)\, \boldsymbol{\alpha}(\boldsymbol{x}, \tau; \boldsymbol{x}', \tau')\, \dot{\boldsymbol{x}}(\tau') \qquad (3.431)$$

where

$$\begin{aligned}\boldsymbol{\alpha}(\boldsymbol{x}, \tau; \boldsymbol{x}', \tau') &= e^2 \sum_{\boldsymbol{k}} c_{-\boldsymbol{k}}(\boldsymbol{x})\, c_{\boldsymbol{k}}(\boldsymbol{x}')\, \mathbb{G}_{\boldsymbol{k},\boldsymbol{k}'}(\tau, \tau') \\ &= \frac{e^2}{\hbar} \sum_{\boldsymbol{k}} c_{-\boldsymbol{k}}(\boldsymbol{x})\, c_{\boldsymbol{k}}(\boldsymbol{x}') \left\langle \boldsymbol{X}_{-\boldsymbol{k}}(\tau)\, \boldsymbol{X}_{\boldsymbol{k}}^T(\tau') \right\rangle \qquad (3.432)\end{aligned}$$

$$= \frac{e^2}{V} \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \frac{\cosh\left[\Omega_{\mathbf{k}}\left(\frac{1}{2}\beta\hbar - |\tau-\tau'|\right)\right]}{2\,\Omega_{\mathbf{k}}\sinh(\beta\hbar\Omega_{\mathbf{k}}/2)} \mathbb{I}_T(\mathbf{k})$$

$$= e^2 \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \frac{\cosh\left[\Omega_{\mathbf{k}}\left(\frac{1}{2}\beta\hbar - |\tau-\tau'|\right)\right]}{2\,\Omega_{\mathbf{k}}\sinh(\beta\hbar\Omega_{\mathbf{k}}/2)} \mathbb{I}_T(\mathbf{k}) \quad (3.436)$$

At zero temperature, $\beta \to \infty$ and

$$\frac{\cosh\left[\Omega_{\mathbf{k}}\left(\frac{1}{2}\beta\hbar - |\tau-\tau'|\right)\right]}{\sinh(\beta\hbar\Omega_{\mathbf{k}}/2)}$$

$$= \left(\cosh\left(\frac{1}{2}\beta\hbar\Omega_{\mathbf{k}}\right)\cosh(\Omega_{\mathbf{k}}|\tau-\tau'|) - \sinh\left(\frac{1}{2}\beta\hbar\Omega_{\mathbf{k}}\right)\sinh(\Omega_{\mathbf{k}}|\tau-\tau'|)\right) \Big/ \sinh(\beta\hbar\Omega_{\mathbf{k}}/2)$$

$$= \coth\left(\frac{1}{2}\beta\hbar\Omega_{\mathbf{k}}\right)\cosh(\Omega_{\mathbf{k}}|\tau-\tau'|) - \sinh(\Omega_{\mathbf{k}}|\tau-\tau'|)$$

$$\to \cosh(\Omega_{\mathbf{k}}|\tau-\tau'|) - \sinh(\Omega_{\mathbf{k}}|\tau-\tau'|)$$

$$= \cos(i\Omega_{\mathbf{k}}|\tau-\tau'|) + i\sin(i\Omega_{\mathbf{k}}|\tau-\tau'|)$$

$$= \exp(-\Omega_{\mathbf{k}}|\tau-\tau'|)$$

(3.436) becomes

$$\boldsymbol{\alpha}(\mathbf{x},\tau;\mathbf{x}',\tau') \to e^2 \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \frac{\exp(-\Omega_{\mathbf{k}}|\tau-\tau'|)}{2\,\Omega_{\mathbf{k}}} \mathbb{I}_T(\mathbf{k})$$

$$= \frac{e^2}{c} \int \frac{d^3k}{(2\pi)^3\, 2\,|\mathbf{k}|} \exp\left\{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - c\,|\mathbf{k}|\,\big||\tau-\tau'|\big|\right\} \mathbb{I}_T(\mathbf{k}) \quad (3.437)$$

Now,

$$\int \frac{d^3k}{(2\pi)^3\, 2\,|\mathbf{k}|} \exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - c\,|\mathbf{k}|\,\big||\tau-\tau'|\big|\right]$$

$$= \frac{1}{2(2\pi)^2} \int_0^\infty dk \int_{-1}^1 d\cos\theta \; k \exp\left[-ik\,|\mathbf{x}-\mathbf{x}'|\cos\theta - ck\,|\tau-\tau'|\right]$$

$$= \frac{1}{2(2\pi)^2} \int_0^\infty dk \, \frac{e^{ik|\mathbf{x}-\mathbf{x}'|} - e^{-ik|\mathbf{x}-\mathbf{x}'|}}{i\,|\mathbf{x}-\mathbf{x}'|} \exp(-ck\,|\tau-\tau'|)$$

$$= \frac{1}{2(2\pi)^2 i\,|\mathbf{x}-\mathbf{x}'|} \left\{ \frac{-1}{i\,|\mathbf{x}-\mathbf{x}'| - c\,|\tau-\tau'|} - \frac{-1}{-i\,|\mathbf{x}-\mathbf{x}'| - c\,|\tau-\tau'|)} \right\}$$

$$= \frac{1}{2(2\pi)^2 i\,|\mathbf{x}-\mathbf{x}'|} \left\{ \frac{2i\,|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|^2 + c^2\,|\tau-\tau'|^2} \right\}$$

$$= \frac{1}{(2\pi)^2} \left\{ \frac{1}{|\mathbf{x}-\mathbf{x}'|^2 + c^2\,|\tau-\tau'|^2} \right\}$$

$$= G_e^R(\mathbf{x},\tau;\mathbf{x}',\tau') \quad (3.438)$$

which is the imaginary-time version of the retarded Green function.

If the dimension of the system is much smaller than the average wavelength of the bath, then

$$|\mathbf{x}-\mathbf{x}'| << \frac{2\pi}{\langle k\rangle} = \frac{2\pi c}{\langle \Omega_k \rangle} = \langle\, |\tau-\tau'|\,\rangle c$$

and (3.436) reduces to

$$\boldsymbol{\alpha}(\mathbf{x},\tau;\mathbf{x}',\tau') \approx e^2 \int \frac{d^3k}{(2\pi)^3} \frac{\cosh\left[\Omega_{\mathbf{k}}\left(\frac{1}{2}\beta\hbar - |\tau-\tau'|\right)\right]}{2\,\Omega_{\mathbf{k}}\sinh(\beta\hbar\Omega_{\mathbf{k}}/2)} \mathbb{I}_T(\mathbf{k})$$

$$= e^2 \int \frac{d\,\omega}{(2\,\pi\,c)^3}\,\omega\,\frac{\cosh\left[\,\omega\left(\frac{1}{2}\,\beta\,\hbar - \left|\,\tau - \tau'\,\right|\right)\right]}{2\sinh(\beta\,\hbar\,\omega\,/2\,)}\,\langle\,\mathbb{I}_T(\boldsymbol{k})\,\rangle \;(3.439)$$

where we've used

$$k = \frac{\Omega_{\boldsymbol{k}}}{c} = \frac{\omega}{c}$$

and

$$\langle\,\mathbb{I}_T(\boldsymbol{k})\,\rangle = \int_0^{2\,\pi} d\,\phi \int_{-1}^{1} d\cos\theta\;\mathbb{I}_T(\boldsymbol{k})$$
$$= \int_0^{2\,\pi} d\,\phi \int_{-1}^{1} d\cos\theta\left(\mathbb{I} - \frac{1}{k^2}\,\boldsymbol{k}\,\boldsymbol{k}^T\right)$$

Using

$$\frac{\boldsymbol{k}\,\boldsymbol{k}^T}{k^2} = \begin{pmatrix} \sin^2\theta\cos^2\phi & \sin^2\theta\cos\phi\sin\phi & \sin\theta\cos\phi\cos\theta \\ \sin^2\theta\cos\phi\sin\phi & \sin^2\theta\sin^2\phi & \sin\theta\sin\phi\cos\theta \\ \sin\theta\cos\phi\cos\theta & \sin\theta\sin\phi\cos\theta & \cos^2\theta \end{pmatrix}$$

we have [see “3.14._Code.nb”]

$$\left\langle\frac{\boldsymbol{k}\,\boldsymbol{k}^T}{k^2}\right\rangle = \frac{4\,\pi}{3}\,\mathbb{I}$$

so that

$$\langle\,\mathbb{I}_T(\boldsymbol{k})\,\rangle = 4\,\pi\,\mathbb{I} - \frac{4\,\pi}{3}\,\mathbb{I} = \frac{8\,\pi}{3}\,\mathbb{I}$$

and (3.439) becomes

$$\boldsymbol{\alpha}\,(\boldsymbol{x},\,\tau\,;\boldsymbol{x}',\,\tau') \approx \frac{e^2}{6\,\pi^2\,c^3}\,\mathbb{I}\int d\,\omega\;\omega\,\frac{\cosh\left[\,\omega\left(\frac{1}{2}\,\beta\,\hbar - \left|\,\tau - \tau'\,\right|\right)\right]}{\sinh(\beta\,\hbar\,\omega\,/2\,)} \qquad (3.439)$$

which corresponds to a spectral density [cf.(3.407)]

$$\rho_{\text{pb}}(\omega) = \frac{e^2}{3\,\pi\,c^3}\,\omega \qquad (3.440)$$

Although this takes the Ohmic form (3.424), the photon bath action (3.431) couples with $\dot{\boldsymbol{x}}$ instead of $\boldsymbol{x}$. This gives rise to an extra factor ω^2 in (3.424) so that we may define a spectral density as

$$\rho_{\text{pb}}(\omega) = \frac{e^2}{3\,\pi\,c^3}\,\omega^3 = 2\,M\,\gamma\,\omega^3 \qquad (3.441)$$

where

$$\gamma = \frac{e^2}{6\,\pi\,c^3\,M} \qquad (3.441\text{a})$$