3.15. Harmonic Oscillator in Ohmic Heat Bath

For an ohmic heat bath [see (3.424)],

$$\rho_b(\omega) \approx 2 M \gamma \omega$$

so that (3.417) gives

$$\begin{aligned} M \Delta\omega^2 &= -\int_0^\infty \frac{d\omega}{2\pi} \rho_b(\omega) \frac{2}{\omega} \\ &= -4 M \gamma \int_0^\infty \frac{d\omega}{2\pi} \\ &= -\infty \end{aligned}$$

Thus, (3.419) becomes

$$\begin{aligned} V_{\text{ren}}(x) &= V(x) + \frac{1}{2} M \Delta\omega^2 x^2 \\ &= V(x) - \infty \end{aligned}$$

By shifting the energy origin to $-\infty$, we have

$$V_{\text{ren}}(x) = V(x) \tag{3.442a}$$

For a harmonic oscillator in an ohmic heat bath, this give

$$V_{\text{ren}}(x) = \frac{1}{2} M \omega^2 x^2 \tag{3.442}$$

and [see (3.420)]

$$\mathcal{A}_{\text{ren}} = \int_0^{\beta\hbar} d\tau \left(\frac{1}{2} M \dot{x}^2 + \frac{1}{2} M \omega^2 x^2 \right) \tag{3.442a}$$

Similarly, (3.413a) becomes

$$\begin{aligned} g_m &= \int_0^\infty \frac{d\omega}{2\pi} \rho_b(\omega) \frac{2\omega_m^2}{\omega(\omega_m^2 + \omega^2)} \\ &= \int_0^\infty \frac{d\omega}{2\pi} \frac{4 M \gamma \omega_m^2}{(\omega_m^2 + \omega^2)} \\ &= M \gamma \mid \omega_m \mid \end{aligned} \tag{3.442b}$$

(3.412) becomes

$$\begin{aligned} g(\tau - \tau') &= \frac{1}{\beta\hbar} \sum_{m=-\infty}^{\infty} g_m e^{-i\omega_m(\tau-\tau')} \\ &= \frac{M\gamma}{\beta\hbar} \sum_{m=-\infty}^{\infty} \left| \omega_m \right| e^{-i\omega_m(\tau-\tau')} \end{aligned} \tag{3.442c}$$

The heat bath action [see (3.415)] becomes

$$\begin{aligned} \mathcal{A}'_{\text{bath}}[x] &= \frac{1}{2} \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' x(\tau) g(\tau - \tau') x(\tau') \\ &= \frac{M\gamma}{2\beta\hbar} \sum_{m=-\infty}^{\infty} \left| \omega_m \right| \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' e^{-i\omega_m(\tau-\tau')} x(\tau) x(\tau') \\ &= \frac{M\gamma}{2} \beta\hbar \sum_{m=-\infty}^{\infty} \left| \omega_m \right| x_m^* x_m \end{aligned} \tag{3.442d}$$

where

$$x_m = \frac{1}{\beta\hbar} \int_0^{\beta\hbar} d\tau \, e^{i\omega_m\tau} x(\tau)$$

$$x_m^* = \frac{1}{\beta\hbar}\int_0^{\beta\hbar} d\tau\, e^{-i\omega_m\tau}\, x(\tau) = x_{-m} \qquad [\, x(\tau) \text{ is real.}\,]$$

$$\int_0^{\beta\hbar} d\tau\, x^2(\tau) = \sum_{m,m'=-\infty}^{\infty} x_m x_{m'} \int_0^{\beta\hbar} d\tau\, e^{-i(\omega_m+\omega_{m'})\tau}$$
$$= \beta\hbar \sum_{m=-\infty}^{\infty} x_m x_m^* \qquad (3.442e)$$

$$\int_0^{\beta\hbar} d\tau\, \dot{x}^2(\tau) = -\sum_{m,m'=-\infty}^{\infty} \omega_m \omega_{m'} x_m x_{m'} \int_0^{\beta\hbar} d\tau\, e^{-i(\omega_m+\omega_{m'})\tau}$$
$$= \beta\hbar \sum_{m=-\infty}^{\infty} \omega_m^2 x_m x_{-m}$$
$$= \beta\hbar \sum_{m=-\infty}^{\infty} \omega_m^2 x_m x_m^* \qquad (3.442f)$$

Hence, the total action in (3.420) becomes

$$\mathcal{A}_e = \mathcal{A}_{\text{ren}} + \mathcal{A}'_{\text{bath}}$$
$$= \beta\hbar \sum_{m=-\infty}^{\infty} \frac{1}{2} M\left(\omega_m^2 + \omega^2 + \gamma\, |\, \omega_m\,|\,\right) x_m x_m^* \qquad \omega_m = \frac{2\pi}{\beta\hbar} m$$
$$= M\beta\hbar\left[\frac{1}{2}\omega x_0^2 + \sum_{m=1}^{\infty}\left(\omega_m^2 + \omega^2 + \gamma\,\omega_m\right) x_m x_m^*\right] \qquad (3.443)$$

where we’ve used the fact that $\omega_0 = 0$ and

$$x_0 = \frac{1}{\beta\hbar}\int_0^{\beta\hbar} d\tau\, x(\tau)$$

is real.

Using (2.447), we have

$$Z_\omega^{\text{damp}} = \int_{-\infty}^{\infty} \frac{d x_0}{l_e} \oint \mathcal{D}' x(\tau)\; e^{-\mathcal{A}_e/\hbar}$$
$$= \frac{1}{\beta\hbar\omega} \prod_{m=1}^{\infty} \left(\frac{\omega_m^2 + \omega^2 + \gamma\,\omega_m}{\omega_m^2}\right)^{-1} \qquad (3.444)$$

For the Drude dissipation, we have [see (3.426)]

$$g_m = M\gamma\; \frac{\omega_D\,|\,\omega_m\,|}{|\,\omega_m\,| + \omega_D}$$

Comparing with (3.442b), this tantamounts to replacing γ with

$$\gamma_m = \gamma\, \frac{\omega_D}{|\,\omega_m\,| + \omega_D} \qquad (3.444a)$$

Using, for $m > 0$,

$$\omega_m^2 + \omega^2 + \gamma_m\,\omega_m = \frac{\left(\omega_m^2 + \omega^2\right)\left(\omega_m + \omega_D\right) + \gamma\,\omega_D\,\omega_m}{\omega_m + \omega_D}$$
$$= \frac{\omega_m^3 + \omega_D\,\omega_m^2 + \left(\omega^2 + \gamma\,\omega_D\right)\omega_m + \omega_D\,\omega^2}{\omega_m + \omega_D}$$

(3.444) thus becomes

$$Z_\omega^{\text{damp}} = \frac{1}{\beta\hbar\omega} \prod_{m=1}^{\infty} \left(\frac{\omega_m^2\left(\omega_m + \omega_D\right)}{\omega_m^3 + \omega_D\,\omega_m^2 + \left(\omega^2 + \gamma\,\omega_D\right)\omega_m + \omega_D\,\omega^2}\right) \qquad (3.445)$$

$$= \frac{1}{\beta \hbar \omega} \prod_{m=1}^{\infty} \frac{\omega_m^2 (\omega_m + \omega_D)}{(\omega_m + \omega_a)(\omega_m + \omega_b)(\omega_m + \omega_c)}$$

where

$$\omega_m^3 + \omega_D \omega_m^2 + \left(\omega^2 + \gamma \omega_D\right) \omega_m + \omega_D \omega^2$$
$$= (\omega_m + \omega_a)(\omega_m + \omega_b)(\omega_m + \omega_c) \qquad (3.446a)$$

so that $-\omega_a$, $-\omega_b$, $-\omega_c$ are the roots of the cubic polynomial (3.446a) with

$$\omega_a + \omega_b + \omega_c = \omega_D \qquad \omega_a \omega_b \omega_c = \omega_D \omega^2 \qquad (3.449)$$

Note that

$$(\omega_m - \omega_a)(\omega_m - \omega_b)(\omega_m - \omega_c)$$
$$= \omega_m^3 - \omega_D \omega_m^2 + \left(\omega^2 + \gamma \omega_D\right) \omega_m - \omega_D \omega^2 \qquad (3.446)$$

so that ω_a, ω_b, ω_c are the roots of the cubic polynomial (3.446).

(3.445) can be written as

$$Z_\omega^{\text{damp}} = \frac{1}{\beta \hbar \omega} \prod_{m=1}^{\infty} \frac{\omega_m}{\omega_m + \omega_a} \frac{\omega_m}{\omega_m + \omega_b} \frac{\omega_m}{\omega_m + \omega_c} \frac{\omega_m + \omega_D}{\omega_m} \qquad (3.447)$$

$$= \frac{1}{\beta \hbar \omega} \prod_{m=1}^{\infty} \frac{m}{m + \frac{\omega_a}{\omega_1}} \frac{m}{m + \frac{\omega_b}{\omega_1}} \frac{m}{m + \frac{\omega_c}{\omega_1}} \frac{m + \frac{\omega_D}{\omega_1}}{m} \qquad (3.447a)$$

where $\omega_1 = \frac{2\pi}{\beta \hbar}$ so that $\omega_m = m\, \omega_1$.

Using the product representation of the Gamma function [see G & R, Formula 8.322]

$$\Gamma(z) = \lim_{n \to \infty} \frac{n^z}{z} \prod_{m=1}^{\infty} \frac{m}{m+z} \qquad (3.448)$$

(3.447a) becomes

$$Z_\omega^{\text{damp}} = \frac{1}{\beta \hbar \omega} \frac{\omega_a}{\omega_1} \frac{\omega_b}{\omega_1} \frac{\omega_c}{\omega_1} \frac{\omega_1}{\omega_D} \frac{\Gamma\left(\frac{\omega_a}{\omega_1}\right) \Gamma\left(\frac{\omega_b}{\omega_1}\right) \Gamma\left(\frac{\omega_c}{\omega_1}\right)}{\Gamma\left(\frac{\omega_D}{\omega_1}\right)} \lim_{n \to \infty} \frac{n^{\omega_D/\omega_1}}{n^{\omega_a/\omega_1} n^{\omega_b/\omega_1} n^{\omega_c/\omega_1}}$$

$$= \frac{1}{\beta \hbar \omega} \frac{\omega_a \omega_b \omega_c}{\omega_1^2 \omega_D} \frac{\Gamma\left(\frac{\omega_a}{\omega_1}\right) \Gamma\left(\frac{\omega_b}{\omega_1}\right) \Gamma\left(\frac{\omega_c}{\omega_1}\right)}{\Gamma\left(\frac{\omega_D}{\omega_1}\right)} \lim_{n \to \infty} n^{(\omega_D - \omega_a - \omega_b - \omega_c)/\omega_1}$$

Using (3.449), we have

$$Z_\omega^{\text{damp}} = \frac{1}{\beta \hbar} \frac{\omega}{\omega_1^2} \frac{\Gamma\left(\frac{\omega_a}{\omega_1}\right) \Gamma\left(\frac{\omega_b}{\omega_1}\right) \Gamma\left(\frac{\omega_c}{\omega_1}\right)}{\Gamma\left(\frac{\omega_D}{\omega_1}\right)}$$

$$= \frac{1}{2\pi} \frac{\omega}{\omega_1} \frac{\Gamma\left(\frac{\omega_a}{\omega_1}\right) \Gamma\left(\frac{\omega_b}{\omega_1}\right) \Gamma\left(\frac{\omega_c}{\omega_1}\right)}{\Gamma\left(\frac{\omega_D}{\omega_1}\right)} \qquad (3.450)$$

In the ohmic limit, $\omega_D \to \infty$, we write (3.446) as

$$\omega_m^3 + \omega^2 \omega_m - \omega_D \left(\omega_m^2 - \gamma \omega_m + \omega^2\right) = 0 \qquad (3.450a)$$

For roots $\omega_m << \omega_D$, (3.450a) can be approximated by

$$\omega_D \left(\omega_m^2 - \gamma \omega_m + \omega^2\right) = 0$$

$$\to \qquad \omega_a = \frac{1}{2}\left(\gamma + \sqrt{\gamma^2 - 4\omega^2}\right) \qquad (3.451a)$$

$$\omega_b = \frac{1}{2}\left(\gamma - \sqrt{\gamma^2 - 4\omega^2}\right) \qquad (3.451b)$$

The 3[rd] root must then be of order ω_D. Writing $\omega_m = \omega_D + a$, & keeping only terms proportional to the highest remaining power, ω_D^2, (3.450a) becomes

$$[\,3\,a\,-(2\,a\,-\gamma)\,]\,\omega_D^2 = 0$$

→ $a = -\gamma$

so that

$$\omega_c = \omega_D - \gamma \qquad (3.451c)$$

Setting

$$\delta = \sqrt{\omega^2 - \frac{1}{4}\gamma^2} \qquad (3.452)$$

(3.451a-c) become

$$\omega_a = \frac{1}{2}\gamma + i\,\delta \qquad \omega_b = \frac{1}{2}\gamma - i\,\delta \qquad (3.451)$$

Using (3.451c) & (3.451), we have

$$\omega_a + \omega_b + \omega_c = \omega_D$$

in agreement with (3.449).

However, since

$$\omega_a\,\omega_b = \frac{1}{4}\gamma^2 + \delta^2 = \omega^2$$

one can use (3.449) to get

$$\omega_c = \frac{\omega_D\,\omega^2}{\omega_a\,\omega_b\,\omega_c} = \omega_D \qquad (3.451d)$$

(3.450) thus simplies to

$$Z_\omega^{\text{ohmic}} = \frac{1}{2\,\pi}\,\frac{\omega}{\omega_1}\,\Gamma\!\left(\frac{\omega_a}{\omega_1}\right)\Gamma\!\left(\frac{\omega_b}{\omega_1}\right) \qquad (3.452)$$

For vanishing friction $\gamma \to 0$,

$$\omega_a = i\,\delta = i\,\omega = -\omega_b$$

so that

$$Z_\omega^{\text{ohmic}}\Big|_{\gamma=0} = \frac{1}{2\,\pi}\,\frac{\omega}{\omega_1}\,\Gamma\!\left(\frac{i\,\omega}{\omega_1}\right)\Gamma\!\left(-\frac{i\,\omega}{\omega_1}\right) \qquad (3.455a)$$

Using [see G&R, 8.334.3]

$$\Gamma(1-z)\,\Gamma(z) = \frac{\pi}{\sin\pi z} \qquad (3.454)$$

$$= -z\,\Gamma(-z)\,\Gamma(z)$$

we have

$$\Gamma\!\left(\frac{i\,\omega}{\omega_1}\right)\Gamma\!\left(-\frac{i\,\omega}{\omega_1}\right) = \frac{\pi}{-i\,\frac{\omega}{\omega_1}\sin\!\left(\frac{\pi\,i\,\omega}{\omega_1}\right)} = \frac{\omega_1}{\omega}\,\frac{\pi}{\sinh\!\left(\frac{\pi\,\omega}{\omega_1}\right)}$$

$$= \frac{\omega_1}{\omega}\,\frac{\pi}{\sinh\!\left(\frac{1}{2}\,\beta\,\hbar\,\omega\right)} \qquad (3.455)$$

(3.455a) becomes

$$Z_\omega^{\text{ohmic}}\Big|_{\gamma=0} = \frac{1}{2\sinh\!\left(\frac{1}{2}\,\beta\,\hbar\,\omega\right)} \qquad (3.455b)$$

in agreement with the undamped result (3.214).

The Helmholtz free energy is [see (3.450)]

$$F = -\frac{1}{\beta} \ln Z_\omega^{\text{damp}}$$
$$= -\frac{1}{\beta}\left[\ln \frac{\omega}{2\pi\omega_1} + \ln\Gamma\left(\frac{\omega_a}{\omega_1}\right) + \ln\Gamma\left(\frac{\omega_b}{\omega_1}\right) + \ln\Gamma\left(\frac{\omega_c}{\omega_1}\right) - \ln\Gamma\left(\frac{\omega_D}{\omega_1}\right)\right] \tag{3.456}$$

At low temperatures,

$$\frac{1}{\omega_1} = \frac{\beta\hbar}{2\pi} >> 1$$

Using the Stirling formula [G&R, 8.327]

$$\ln\Gamma(z) = \left(z - \frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360 z^3} + O\left[z^{-5}\right] \tag{3.457}$$
$$= z\ln z - z - \frac{1}{2}\ln\frac{z}{2\pi} + \frac{1}{12z} + O\left[z^{-3}\right]$$

we have, for $j = a, b, c$,

$$\ln\Gamma\left(\frac{\omega_j}{\omega_1}\right) = \frac{\omega_j}{\omega_1}\ln\left(\frac{\omega_j}{\omega_1}\right) - \frac{\omega_j}{\omega_1} - \frac{1}{2}\ln\left(\frac{\omega_j}{2\pi\omega_1}\right) + \frac{\omega_1}{12\omega_j} + O\left[\left(\frac{\omega_j}{\omega_1}\right)^{-3}\right]$$

$$\to \sum_{j=a,b,c}\ln\Gamma\left(\frac{\omega_j}{\omega_1}\right) \approx \sum_j\left\{\frac{\beta\hbar}{2\pi}\left[\omega_j\ln\left(\frac{\omega_j}{\omega_1}\right) - \omega_j\right] - \frac{1}{2}\ln\omega_j + \frac{\pi}{6\beta\hbar}\frac{1}{\omega_j}\right\} + \frac{3}{2}\ln(2\pi\omega_1)$$

Using [see (3.449)]

$$\omega_D = \sum_{j=a,b,c}\omega_j = \frac{1}{\omega^2}\prod_j\omega_j$$

we have

$$\ln\Gamma\left(\frac{\omega_D}{\omega_1}\right) \approx \frac{\omega_D}{\omega_1}\ln\left(\frac{\omega_D}{\omega_1}\right) - \frac{\omega_D}{\omega_1} - \frac{1}{2}\ln\left(\frac{\omega_D}{2\pi\omega_1}\right) + \frac{\omega_1}{12\omega_D}$$
$$= \sum_j\left\{\frac{\beta\hbar}{2\pi}\left[\omega_j\ln\left(\frac{\omega_D}{\omega_1}\right) - \omega_j\right] - \frac{1}{2}\ln\omega_j\right\} + \frac{1}{2}\ln(2\pi\omega_1) + \ln\omega + \frac{\pi}{6\beta\hbar}\frac{\omega^2}{\prod_j\omega_j}$$

$$\to \sum_{j=a,b,c}\ln\Gamma\left(\frac{\omega_j}{\omega_1}\right) - \ln\Gamma\left(\frac{\omega_D}{\omega_1}\right)$$
$$\approx \frac{\beta\hbar}{2\pi}\sum_j\omega_j\ln\left(\frac{\omega_j}{\omega_D}\right) + \ln\left(\frac{2\pi\omega_1}{\omega}\right) + \frac{\pi}{6\beta\hbar}\left(\sum_j\frac{1}{\omega_j} - \frac{\omega^2}{\prod_j\omega_j}\right)$$

(3.456) thus becomes

$$F \approx -\frac{\hbar}{2\pi}\sum_j\omega_j\ln\left(\frac{\omega_j}{\omega_D}\right) - \frac{\pi}{6\beta^2\hbar}\left(\sum_j\frac{1}{\omega_j} - \frac{\omega^2}{\prod_j\omega_j}\right)$$
$$= E_0 - \frac{\pi}{6\beta^2\hbar}\left(\frac{1}{\omega_a} + \frac{1}{\omega_b} + \frac{1}{\omega_c} - \frac{\omega^2}{\omega_a\omega_b\omega_c}\right) \tag{3.458}$$

where

$$E_0 = -\frac{\hbar}{2\pi}\sum_j\omega_j\ln\left(\frac{\omega_j}{\omega_D}\right) \tag{3.459}$$

is the ground state energy.

For small friction, (3.459) becomes [see “3.15._Code.nb”]

$$E_0 = \frac{1}{2}\hbar\omega + \frac{\hbar}{2\pi}\gamma\ln\frac{\omega_D}{\omega} - \frac{\hbar}{16\omega}\gamma^2\left(1+\frac{4\omega}{\pi\omega_D}\right) + O[\gamma^3] \qquad (3.460)$$

As $T \to 0$, $\beta \to \infty$ so that $\omega_m = \frac{2\pi}{\beta\hbar} m$ becomes continuous and

$$\frac{1}{\beta\hbar}\sum_{m=-\infty}^{\infty} f(\omega_m) \xrightarrow[\beta\to\infty]{} \frac{1}{\beta\hbar}\int_{-\infty}^{\infty} d\,m\, f(\omega_m) = \int_{-\infty}^{\infty}\frac{d\,\omega}{2\pi} f(\omega) \qquad (3.461)$$

Using (3.445), we have

$$F(\beta) = -\frac{1}{\beta}\ln\left[\frac{1}{\beta\hbar\omega}\prod_{m=1}^{\infty}\left(\left(\omega_m^2(\omega_m+\omega_D)\right)/\left(\omega_m^3 + \omega_D\omega_m^2 + (\omega^2+\gamma\omega_D)\omega_m + \omega_D\omega^2\right)\right)\right]$$
$$= -\frac{1}{\beta}\left\{\ln\frac{1}{\beta\hbar\omega} + \sum_{m=1}^{\infty}\ln\left[\left(\omega_m^2(\omega_m+\omega_D)\right)/\left(\omega_m^3 + \omega_D\omega_m^2 + (\omega^2+\gamma\omega_D)\omega_m + \omega_D\omega^2\right)\right]\right\}$$

(3.641a)

so that

$$E_0 = \lim_{\beta\to\infty} F(\beta)$$
$$= -\hbar\int_0^{\infty}\frac{d\,\omega_m}{2\pi}\ln\left[\left(\omega_m^2(\omega_m+\omega_D)\right)/\left(\omega_m^3 + \omega_D\omega_m^2 + (\omega^2+\gamma\omega_D)\omega_m + \omega_D\omega^2\right)\right]$$
$$= \frac{\hbar}{2\pi}\int_0^{\infty} d\,\omega_m \ln\left[\left(\omega_m^3 + \omega_D\omega_m^2 + (\omega^2+\gamma\omega_D)\omega_m + \omega_D\omega^2\right)/\left(\omega_m^2(\omega_m+\omega_D)\right)\right] \qquad (3.462)$$

where we've used

$$\lim_{x\to 0} x\ln x = 0 \qquad \text{so that} \qquad \frac{1}{\beta}\ln\frac{1}{\beta\hbar\omega} \to 0$$

Since everything is positive in the integrand of (3.462), E_0 increases if the friction coefficient γ increases.

Inverting the Laplace transform (1.579), we obtain the density of states [see (1.580)] as

$$\rho(\varepsilon) = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} d\,\beta\, e^{\beta\varepsilon} Z_\omega^{\text{damp}}(\beta) \qquad (3.463)$$

where η is an adjustable parameter inserted to make sure the integral converges.

In the absence of friction,

$$Z_\omega(\beta) = \sum_n e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)}$$

$$\to \qquad \rho(\varepsilon) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{\eta-i\infty}^{\eta+i\infty} d\,\beta\,\exp\left\{\beta\left[\varepsilon - \hbar\omega\left(n+\frac{1}{2}\right)\right]\right\}$$
$$= \frac{1}{2\pi}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} d\,\beta\,\exp\left\{i\beta\left[\varepsilon - \hbar\omega\left(n+\frac{1}{2}\right)\right]\right\} \qquad (\beta \to i\beta;\ \eta = 0)$$
$$= \sum_{n=0}^{\infty}\delta\left[\varepsilon - \hbar\omega\left(n+\frac{1}{2}\right)\right] \qquad (3.464)$$

In the absence of friction, we insert (3.450) into (3.463) to get

$$\rho(\varepsilon) = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} d\,\beta\, e^{\beta\varepsilon}\frac{1}{2\pi}\frac{\omega}{\omega_1}\frac{\Gamma\left(\frac{\omega_a}{\omega_1}\right)\Gamma\left(\frac{\omega_b}{\omega_1}\right)\Gamma\left(\frac{\omega_c}{\omega_1}\right)}{\Gamma\left(\frac{\omega_D}{\omega_1}\right)}$$
$$= \frac{1}{(2\pi)^2 i}\int_{\eta-i\infty}^{\eta+i\infty} d\,\beta\, e^{\beta\varepsilon}\frac{\beta\hbar}{2\pi}\omega\frac{\Gamma\left(\frac{\beta\hbar\omega_a}{2\pi}\right)\Gamma\left(\frac{\beta\hbar\omega_b}{2\pi}\right)\Gamma\left(\frac{\beta\hbar\omega_c}{2\pi}\right)}{\Gamma\left(\frac{\beta\hbar\omega_D}{2\pi}\right)} \qquad (3.464a)$$

For the ohmic damping, ω_j are given by (3.451) & (3.451c). The corresponding poles are shown as dots in Fig.3.5.

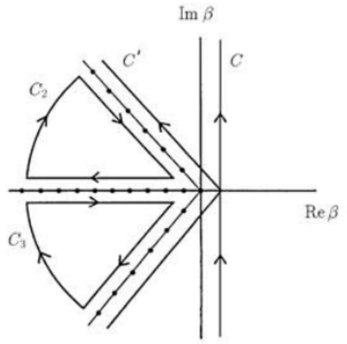


Fig.3.5.

Due to the pole at $n = 0$, we need to set $\eta = 0_+$ in (3.463) so that all poles are to the left of the integraion path C [see Fig.3.5].

Using the representation [see G&R, 8.314]

$$\Gamma[z] = \int_1^\infty d\,t\,t^{z-1}\,e^{-t} + \sum_{n=0}^{\infty} \frac{(-)^n}{n!\,(z+n)} \tag{3.465}$$

we see that the residue of $\Gamma(z)$ at $z = -n$ is simply $\frac{(-)^n}{n!}$.

Writing (3.464a) as

$$\rho(\varepsilon) = \frac{1}{2\,\pi\,i}\,\frac{\omega}{\hbar\,\omega_j^2} \int_{\eta - i\infty}^{\eta + i\infty} d\,z_j\; e^{z_j\,\varepsilon_j}\,z_j\,\frac{\Gamma(z_a)\,\Gamma(z_b)\,\Gamma(z_c)}{\Gamma(z_D)} \tag{3.465a}$$

where

$$z_j = \frac{\beta\,\hbar\,\omega_j}{2\,\pi} \qquad z_D = \frac{\beta\,\hbar\,\omega_D}{2\,\pi} \qquad \varepsilon_j = \frac{2\,\pi\,\varepsilon}{\hbar\,\omega_j} \qquad j = a,\,b,\,c \tag{3.464b}$$

we have

$$\rho(\varepsilon) = \sum_{n=0}^{\infty} \sum_j e^{-n\,\varepsilon_j}\,R_{n,j}$$

$$= \sum_{n=0}^{\infty} \sum_j e^{-2\,\pi\,n\,\varepsilon/\hbar\,\omega_j}\,R_{n,j} \tag{3.466a}$$

where

$$R_{n,j} = \frac{\omega}{\hbar\,\omega_j^2}\,\mathrm{Res}\!\left[\,z_j\,\frac{\Gamma(z_a)\,\Gamma(z_b)\,\Gamma(z_c)}{\Gamma(z_D)}\,;\;z_j = -\,n\right] \tag{3.466b}$$

For example,

$$R_{n,a} = \frac{\omega}{\hbar\,\omega_a^2}\,\mathrm{Res}\!\left[\,z_a\,\frac{\Gamma(z_a)\,\Gamma(z_b)\,\Gamma(z_c)}{\Gamma(z_D)}\,;\;z_a = -\,n\right]$$

$$= \frac{\omega}{\hbar\,\omega_a^2}\,\frac{(-)^{n+1}}{n!}\left[\,z_a\,\frac{\Gamma(z_b)\,\Gamma(z_c)}{\Gamma(z_D)}\right]_{z_a = -\,n}$$

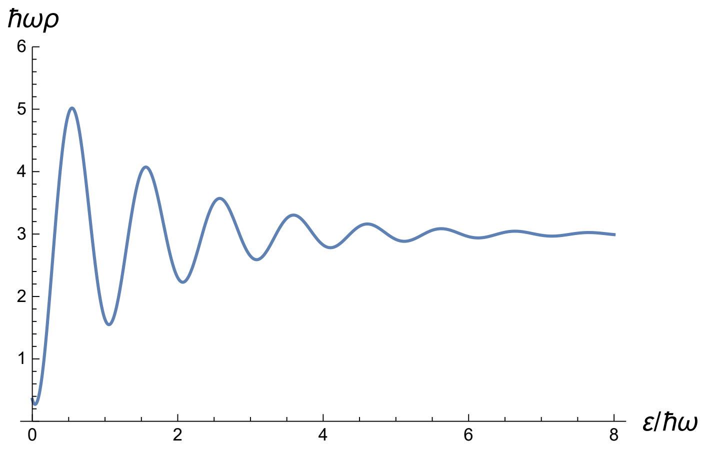
$$= \frac{\omega}{\hbar\,\omega_a^2}\,\frac{(-)^{n+1}}{(n-1)!}\,\frac{\Gamma\!\left(-\frac{n\,\omega_b}{\omega_a}\right)\Gamma\!\left(-\frac{n\,\omega_c}{\omega_a}\right)}{\Gamma\!\left(-\frac{n\,\omega_D}{\omega_a}\right)} \tag{3.467}$$

Note that the term $n = 0$ needs to be calculated by a limiting process, which gives [see 3.15._-Code.nb"]

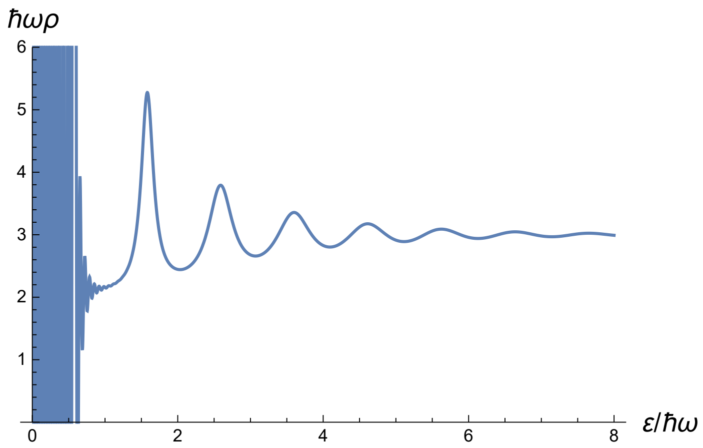
$$\sum_j R_{0,j} = \frac{3\,\omega\,\omega_D}{\hbar\,\omega_a\,\omega_b\,\omega_c} = \frac{3}{\hbar\,\omega} \tag{3.467a}$$

Plots of $\rho(\varepsilon)$ with $n_{\max} = 20$ are given below [see 3.15._Code.nb"].

Underdamped case: $\frac{\gamma}{\omega} = 0.2$, $\frac{\omega_D}{\omega} = 10$



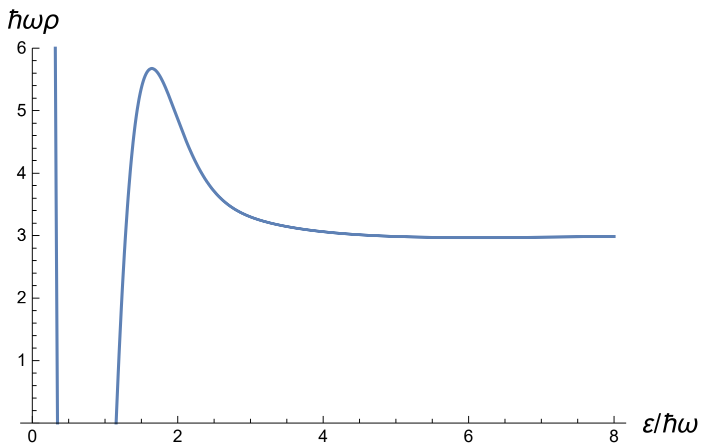
$n_{\max} = 1$



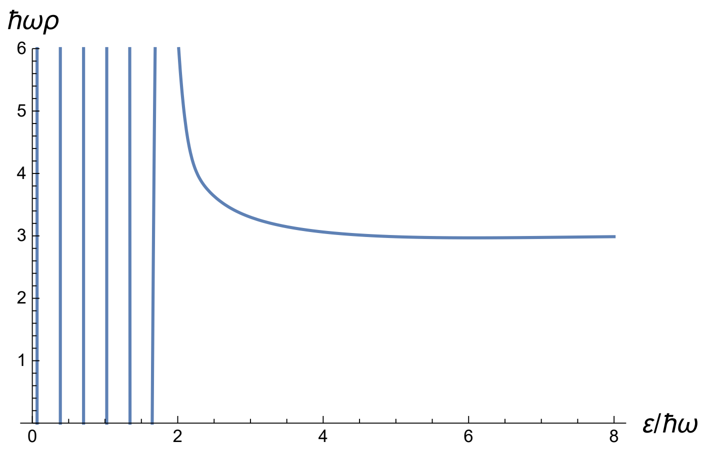
$n_{\max} = 15$

Note that the numerical result is unstable in the low energy region. (3.466a) should be treated like an asymptotic expansion.

Overdamped case: $\frac{\gamma}{\omega} = 5$, $\frac{\omega_D}{\omega} = 10$



$n_{\max} = 5$



$n_{\max} = 15$

The plots in Kleinert's Fig.3.6 corresponds to replacing the $n = 0$ term (3.467a) with $\frac{1}{\hbar\,\omega}$.