3.16. Harmonic Oscillator in Photon Heat Bath

For a photon heat bath, the spectral density is given by (3.441) as

$$\rho_{pb}(\omega) \approx 2\, M\, \gamma\, \omega^3 \qquad\qquad \gamma = \frac{e^2}{6\,\pi\, c^3\, M} \tag{3.468a}$$

which can be taken as an ohmic heat bath [see (3.424)] with friction coefficient $\gamma\, \omega^2$.

The partition function for an oscillator coupled to the heat bath is therefore [see (3.444)]

$$Z_\omega^{\text{damp}} = \frac{1}{\beta\, \hbar\, \omega} \prod_{m=1}^{\infty} \left(\frac{\omega_m^2 + \omega^2 + \gamma\, \omega_m^3}{\omega_m^2} \right)^{-1} \tag{3.468}$$

For Drude friction model, we replace γ with [see (3.428)]

$$\gamma_m = \gamma\ \frac{\omega_D}{|\ \omega_m\ | + \omega_D} \tag{3.468b}$$

and (3.468) becomes

$$\begin{aligned} Z_\omega^{\text{damp}} &= \frac{1}{\beta\, \hbar\, \omega} \prod_{m=1}^{\infty} \frac{\omega_m^2\, (\omega_m + \omega_D)}{(\omega_m^2 + \omega^2)\, (\omega_m + \omega_D) + \gamma\, \omega_D\, \omega_m^3} \\ &= \frac{1}{\beta\, \hbar\, \omega} \prod_{m=1}^{\infty} \frac{\omega_m^2\, (\omega_m + \omega_D)}{(1 + \gamma\, \omega_D)\, \omega_m^3 + \omega_D\, \omega_m^2 + \omega^2\, \omega_m + \omega_D\, \omega^2} \qquad (3.469) \\ &= \frac{1}{\beta\, \hbar\, \omega} \prod_{m=1}^{\infty} \frac{\omega_m^2\, (\omega_m + \omega_D)}{(\omega_m + \omega_a)(\omega_m + \omega_b)(\omega_m + \omega_c)} \qquad (3.469\text{a}) \end{aligned}$$

where [cf. (3.446)]

$$\begin{aligned} &(\omega_m - \omega_a)(\omega_m - \omega_b)(\omega_m - \omega_c) \\ &= (1 + \gamma\, \omega_D)\, \omega_m^3 - \omega_D\, \omega_m^2 + \omega^2\, \omega_m - \omega_D\, \omega^2 \end{aligned} \tag{3.470}$$

Thus [see (3.450)]

$$Z_\omega^{\text{damp}} = \frac{1}{2\,\pi}\, \frac{\omega}{\omega_1}\, \frac{\Gamma\left(\frac{\omega_a}{\omega_1}\right) \Gamma\left(\frac{\omega_b}{\omega_1}\right) \Gamma\left(\frac{\omega_c}{\omega_1}\right)}{\Gamma\left(\frac{\omega_D}{\omega_1}\right)} \tag{3.470a}$$

In the ohmic limit, $\omega_D \to \infty$, we write (3.470) as

$$\omega_m^3 + \omega^2\, \omega_m + \omega_D \left(\gamma\, \omega_m^3 - \omega_m^2 - \omega^2 \right) = 0 \tag{3.470a}$$

For roots $\omega_m << \omega_D$, (3.470a) can be approximated by

$$\omega_D \left(\gamma\, \omega_m^3 - \omega_m^2 - \omega^2 \right) = 0 \tag{3.470b}$$

Since EM couplings are weak, it suffices to solve (3.470b) only to the lowest order in γ. Writing it as

$$(1 - \gamma\, \omega_m)\, \omega_m^2 = -\omega^2$$

$$\begin{aligned} \to \qquad \omega_m &= \pm i\, \omega\, (1 - \gamma\, \omega_m)^{-1/2} \\ &\approx \pm i\, \omega \left(1 + \frac{1}{2}\, \gamma\, \omega_m \right) \\ &\approx \pm i\, \omega \left(1 \pm \frac{1}{2}\, i\, \gamma\, \omega \right) \\ &= \pm i\, \omega - \frac{1}{2}\, \gamma\, \omega^2 \end{aligned}$$

Hence,

$$\omega_a = i\, \omega - \frac{1}{2}\, \gamma\, \omega^2 \ = i\, \omega - \frac{1}{2}\, \gamma_{pb}^{\text{eff}} \tag{3.471a}$$

$$\omega_b = -i\,\omega - \frac{1}{2}\,\gamma\,\omega^2 \;= -i\,\omega - \frac{1}{2}\,\gamma_{\rm pb}^{\rm eff} \qquad (3.471b)$$

where

$$\gamma_{\rm pb}^{\rm eff} = \gamma\,\omega^2 = \frac{e^2}{6\,\pi\,c^3\,M}\,\omega^2 \qquad \text{[(3.468a) used.]} \qquad (3.472)$$

Using $[X]$ to denote the dimensions of X, we have, in the Gaussian units,

$$\left[e^2\right] = \text{erg}\cdot\text{cm} \qquad \left[m\,c^2\right] = \text{erg} \qquad [c] = \text{cm}\cdot s^{-1} \qquad [\omega] = s^{-1}$$

$$\rightarrow \qquad \left[\gamma_{\rm pb}^{\rm eff}\right] = \frac{\text{erg}\cdot\text{cm}}{\text{erg}\cdot\text{cm}\cdot s^{-1}}\,s^{-1} = s^{-1}$$

which is the same as the ordinary ohmic friction coefficient (3.451).

The 3rd root must then be of order ω_D. Writing $\omega_m = \alpha\,\omega_D$, & keeping only terms with the highest power of ω_D, (3.470) becomes

$$(1 + \gamma\,\omega_D)\,\alpha^3\,\omega_D^3 - \alpha^2\,\omega_D^3 = 0$$

$$\rightarrow \qquad \alpha = \frac{1}{1 + \gamma\,\omega_D}$$

so that

$$\omega_c = \frac{\omega_D}{1 + \gamma\,\omega_D} \;= \frac{\omega_D}{1 + \gamma_{\rm pb}^{\rm eff}\,\frac{\omega_D}{\omega^2}} \qquad (3.451c)$$