3.20. Calculation of Perturbation Series via Feynman Diagrams

Consider the perturbation potential

$$V(x) = \frac{1}{4} g\, x^4 \tag{3.540}$$

which is the classical version of the potential used in φ^4 quantum field theories.

The 1st order energy shift [see (3.484)] is proportional to

$$\begin{aligned} \langle \mathcal{A}_{\text{int},\,e} \rangle_\omega &= \int_0^{\beta\hbar} d\tau \left\langle V[x(\tau)] \right\rangle_\omega \\ &= \frac{1}{4} g \int_0^{\beta\hbar} d\tau \left\langle x^4(\tau) \right\rangle_\omega \\ &= \frac{1}{4} g \int_0^{\beta\hbar} d\tau\, G^{(4)}_{\omega^2}(\tau_1, \tau_2, \tau_3, \tau_4) \Big|_{\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau} \end{aligned} \tag{3.541}$$

where

$$G^{(4)}_{\omega^2}(\tau_1, \tau_2, \tau_3, \tau_4) = \langle x(\tau_1)\, x(\tau_2)\, x(\tau_3)\, x(\tau_4) \rangle_\omega$$

is the 4^{th} order correlation function with periodic B.C.

According to the Wick's rule (3.302), the number of terms in the Wick expansion for $G^{(4)}_{\omega^2}(\tau_1, \tau_2, \tau_3, \tau_4)$ is

$$\frac{1}{2!} C^4_2\, C^2_2 = \frac{1}{2!} \frac{4!}{2! \times 2!} \frac{2!}{2! \times 0!} = 3$$

[There are C^4_2 ways to pick the arguments of the 1st $G^{(2)}_\omega$ and C^2_2 for the 2nd. Finally, the order of the $G^{(2)}_\omega$'s is immaterial.]

Thus,

$$\begin{aligned} &G^{(4)}_{\omega^2}(\tau_1, \tau_2, \tau_3, \tau_4) \\ &= \langle x(\tau_1)\, x(\tau_2) \rangle_\omega \langle x(\tau_3)\, x(\tau_4) \rangle_\omega + \langle x(\tau_1)\, x(\tau_3) \rangle_\omega \langle x(\tau_2)\, x(\tau_4) \rangle_\omega \\ &\quad + \langle x(\tau_1)\, x(\tau_4) \rangle_\omega \langle x(\tau_2)\, x(\tau_3) \rangle_\omega \\ &= G^{(2)}_\omega(\tau_1, \tau_2)\, G^{(2)}_\omega(\tau_3, \tau_4) + G^{(2)}_\omega(\tau_1, \tau_3)\, G^{(2)}_\omega(\tau_2, \tau_4) + G^{(2)}_\omega(\tau_1, \tau_4)\, G^{(2)}_\omega(\tau_2, \tau_3) \end{aligned}$$

For $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau$, all 3 terms are the same so that (3.541)

$$\langle \mathcal{A}_{\text{int},\,e} \rangle_\omega = \frac{3}{4} g \int_0^{\beta\hbar} d\tau \left[G^{(2)}_\omega(\tau, \tau) \right]^2 \tag{3.542}$$

In a Feynman diagram, a Green function (or propagator) $G(\tau_1, \tau_2)$ is represented by an arrow going from τ_2 to τ_1. However, since

$$G^{(2)}_\omega(\tau_1, \tau_2) = G^{(2)}_\omega(\,|\,\tau_1 - \tau_2\,|\,)$$

it can be represented by a line connecting points τ_1 & τ_2. This means $G^{(2)}_\omega(\tau, \tau)$ is a circle through τ.

For the present case, the rules for constructing the diagrams are

1. Each $V(\tau)$ is represented by a dot (or vertex).
2. There are exactly 4 lines at each vertex since $V \propto x^4 \propto (a + a^+)^4$.
3. Owing to the periodic B.C., there are no open-ended lines, i.e., each line must start & end at some vertex. This kind of diagrams are called vacuum diagrams.

Since each vertex comes with a coupling constant γ, and each line a $G^{(2)}_{\omega^2}$, a diagram Γ^L_V with L lines & V vertices represents a value

$$\Gamma^L_V = \mu\, \gamma^V c^L_V \tag{3.542a}$$

where μ is the multiplicity of the diagram resulted from setting some of the τ_k's equal and

$$c_V^L = \prod_{k=1}^{V} \int_0^{\beta\hbar} d\,\tau_k\, f\left(G_{\omega^2}^{(2)}\right) \qquad (3.542b)$$

where $f\left(G_{\omega^2}^{(2)}\right)$ is a product of L $G_{\omega^2}^{(2)}$'s dictated by the topology of Γ.

In terms of the dimensionless quantities,

$$x = \beta\hbar\omega \qquad s = \omega\tau$$

$$\mathcal{G}(s) = \frac{\cosh\left(\frac{1}{2}x - s\right)}{2\sinh\left(\frac{1}{2}x\right)} \qquad (3.542c)$$

we have

$$G_\omega^{(2)}(\tau_1, \tau_2) = \frac{1}{\omega}\mathcal{G}(\,|\, s_1 - s_2\,|\,) \qquad (3.542d)$$

$$c_V^L(x) = \left(\frac{\hbar}{M\omega}\right)^L \frac{1}{\omega^V} \chi_V^L(x) \qquad (3.542e)$$

$$\chi_V^L(x) = \prod_{k=1}^{V} \int_0^x d\,s_k\, f(\mathcal{G}) \qquad (3.542f)$$

where $f(\mathcal{G})$ is a product of L $\mathcal{G}$'s. Note that χ_V^L is dimensionless and related to Kleinert's α_V^{2L} by

$$\chi_V^L(x) = x\,\alpha_V^{2L}(x) \qquad (3.542g)$$

Using

$$a_V^{2L} = \left(\frac{\hbar}{M\omega}\right)^L \alpha_V^{2L}(x) \qquad (3.548)$$

we also have

$$a_V^{2L} = \frac{1}{x}\left(\frac{\hbar}{M\omega}\right)^L \chi_V^L = \frac{1}{x}\,\omega^V c_V^L \qquad (3.548a)$$

Henceforth, we shall denote a diagram either as c_V^L or χ_V^L with

$$\Gamma_V^L = \mu\,\gamma^V \left(\frac{\hbar}{M\omega}\right)^L \frac{1}{\omega^V}\chi_V^L(x) \qquad (3.548b)$$

$$= \mu\,\gamma^V c_V^L$$

There is a useful property of these diagrams. Let χ be composed of two subdiagrams $\chi^{(1)}$ & $\chi^{(2)}$ that join at a single vertex. Then χ can be factorized as

$$\chi = \frac{1}{x}\chi^{(1)}\chi^{(2)} \qquad (3.548c)$$

Proof is as follows. Label the time at the vertex at the junction as τ_1. The Green fuctions attached to it are of the form $\mathcal{G}(s_k - s_1)$. Setting $\tilde{s}_k = s_k - s_1$ and using the periodicity property that

$$\int_a^{x+a} d\,s\; f(\mathcal{G}) = \int_0^x d\,s\; f(\mathcal{G})$$

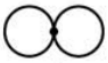
we can do the integral s_1 to give x, while the $\tilde{s}_k$ integral retains the same form as s_k. Hence

$$\chi = x\,\tilde{\chi}^{(1)}\,\tilde{\chi}^{(2)}$$

where $\tilde{\chi}^{(j)}$ is $\chi^{(j)}$ without vertex τ_1 Restoring the vertex τ_1 to $\tilde{\chi}^{(j)}$ and resetting $\tilde{s}_k$ to s_k gives $\chi^{(j)} = x\,\tilde{\chi}^{(j)}$. QED.

To calculate the multiplitícty μ for an n^{th} order diagram, we divide the $4n$ arguments into n groups. Let the number α of distinct ways to assign these groups to the vertices of the diagram be α. Next, we count the number N of distinct ways to assign $4n$ arguments to the correlation functions in the diagram. Finally, $\mu = \alpha N$.

For the 1st order term (3.542), there is only 1 possible diagram:

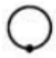


3

The number 3 underneath is μ of the diagram as calculated in (3.542). We now re-calculate it in terms of the rules discribed above. First of all, it is obvious that $\alpha = 1$. For the loop on the left, there are C_2^4 ways to choose its arguments. This leaves C_2^2 ways for the one on the right. Since the left-right distinction is immaterial, we have

$$\mu = \alpha N = N = \frac{1}{2!} C_2^4 C_2^2 = 3$$

Furthermore,

 $= \frac{1}{x}$  ^{2}

where

 is $\quad \chi_1^1 = \int_0^x d\,s\,\mathcal{G}(0) = x\,\mathcal{G}(0) = \frac{1}{2} x \coth \frac{x}{2} = x\,\alpha_1^2$

or $\quad c_1^1 = \int_0^{\beta\hbar} d\,\tau G_{\omega^2}^{(2)}(0) = \frac{\hbar}{M\omega} \frac{1}{\omega} x\,\alpha_1^2 = \frac{x}{\omega} a^2 \qquad (3.547a)$

Note that Kleinert denote

$$\alpha_1^2 = a_1^2 = a^2 = \frac{1}{2} \coth \frac{x}{2} \qquad (3.543a)$$

The 2nd order energy shift is proportional to

$$\left\langle \mathcal{A}_{\text{int}, e}^2 \right\rangle_\omega = \int_0^{\beta\hbar} d\,\tau_2 \int_0^{\beta\hbar} d\,\tau_1 \left\langle V[x(\tau_2)]\, V[x(\tau_1)] \right\rangle_\omega$$

$$= \left(\frac{1}{4} g\right)^2 \int_0^{\beta\hbar} d\,\tau_2 \int_0^{\beta\hbar} d\,\tau_1 \left\langle x^4(\tau_2)\; x^4(\tau_1) \right\rangle_\omega \qquad (3.543)$$

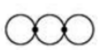
$$= \left(\frac{1}{4} g\right)^2 \int_0^{\beta\hbar} d\,\tau_2 \int_0^{\beta\hbar} d\,\tau_1\; G_{\omega^2}^{(8)}\left(\tau_2, \tau_2, \tau_2, \tau_2, \tau_1, \tau_1, \tau_1, \tau_1\right)$$

The number of diagrams in $G_{\omega^2}^{(8)}(\tau_1, \ldots, \tau_8)$ is

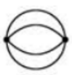
$$\frac{1}{4!} C_2^8 C_2^6 C_2^4 C_2^2 = \frac{1}{4!} \frac{8!}{6! \times 2!} \frac{6!}{4! \times 2!} \frac{4!}{2! \times 2!} \frac{2!}{2! \times 0!} = \frac{1}{4!} \frac{8!}{(2!)^4} = 105$$

[There're C_2^8 ways to pick the arguments of the 1st $G_\omega^{(2)}$, C_2^6 ways for the 2nd $G_\omega^{(2)}$, ...etc. Finally, the order of picking these four $G_\omega^{(2)}$'s does not matter.]

Applying the construction rules to (3.543), we get only 3 possibilities:



72

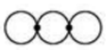


24

9

In these diagrams, the two vertices are equivalent. Therefore, they all have

$$\alpha = C_2^2 = 1$$

For ,

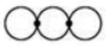
$$\mu = N = C_2^4 \left(\frac{1}{2!} \left(C_1^2 C_1^2\right) \left(C_1^1 C_1^1\right) \right) C_2^4 = 72$$

[From the left group, there're C_2^4 ways to pick the arguments for the bubble on the left. Same for the bubble on the right. There are now 2 arguments each left in the vertices. For the upper line joining the vertices, there're $C_1^2 C_1^2$ ways to pick one argument each from the left and right groups. This leaves $C_1^1 C_1^1$ ways for the lower line. Finally, the distinction between upper & lower lines is irrelevant.]

Alternatively, counting from left to right in the diagram, we get

$$\mu = N = C_2^4 \left[\frac{(C_1^2 \times 4)(C_1^1 \times 3)}{2!} \right] C_2^2 = 72$$

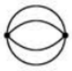
Alternatively, we can also write

 $= \frac{1}{\chi^2}$  $\times$  ^{2}

where

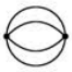
 is $\quad \chi_2^2 = \int_0^x d\,s_2 \int_0^x d\,s_1 \left[\mathcal{G}(\,|\,s_2 - s_1\,|\,) \right]^2 = x\,\alpha_2^4$

or $\quad c_2^2 = \int_0^{\beta\hbar} d\,\tau_2 \int_0^{\beta\hbar} d\,\tau_1 \left[G_{\omega^2}^{(2)}(\tau_2,\,\tau_1) \right]^2 = \frac{x}{\omega^2}\,a_2^4 \qquad$ (3.547b)

For , counting down from the top, we have

$$\mu = N = \frac{1}{4!}(C_1^4\,C_1^4)(C_1^3\,C_1^3)(C_1^2\,C_1^2)(C_1^1\,C_1^1) = \frac{1}{4!}\,4^2\,3^2\,2^2\,1^2 = 24$$

with

 is $\quad \chi_4^4 = \int_0^x d\,s_2 \int_0^x d\,s_1 \left[\mathcal{G}(\,|\,s_2 - s_1\,|\,) \right]^4 = x\,\alpha_4^8$

or $\quad c_2^4 = \int_0^{\beta\hbar} d\,\tau_2 \int_0^{\beta\hbar} d\,\tau_1 \left[G_{\omega^2}^{(2)}(\tau_2,\,\tau_1) \right]^4 = \frac{x}{\omega^2}\,a_2^8 \qquad$ (3.547c)

For ,

$$\mu = N = \left(\frac{1}{2!}\,C_2^4\,C_2^2 \right)\left(\frac{1}{2!}\,C_2^4\,C_2^2 \right) = 9$$

so that

 $= \left\{ \int_0^x d\,s_1 \left[\mathcal{G}(0) \right]^2 \right\}^2 \qquad$ (3.547c)

(3.543) thus becomes

$$\left\langle \mathcal{A}_{\text{int},\,e}^2 \right\rangle_\omega = \left(\frac{1}{4}\,g \right)^2 \left(\frac{\hbar}{M\,\omega} \right)^4 \frac{1}{\omega^2} \int_0^x d\,s_2 \int_0^x d\,s_1 \qquad (3.544a)$$

$$\times \Big\{ 72 \left[\mathcal{G}(0) \right]^2 \left[\mathcal{G}(\,|\,s_2 - s_1\,|\,) \right]^2 + 24 \left[\mathcal{G}(\,|\,s_2 - s_1\,|\,) \right]^4$$

$$+ 9 \left[\mathcal{G}(0) \right]^2 \left[\mathcal{G}(0) \right]^2 \Big\}$$

$$= \left(\frac{1}{4}\,g \right)^2 \int_0^{\beta\hbar} d\,\tau_2 \int_0^{\beta\hbar} d\,\tau_1 \qquad (3.544)$$

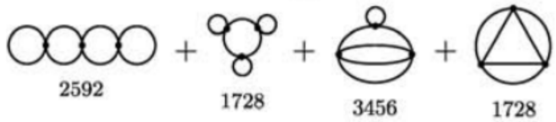
$$\times \Big\{ 72\,G_\omega^{(2)}(\tau_2,\,\tau_2) \left[G_\omega^{(2)}(\tau_2,\,\tau_1) \right]^2 G_\omega^{(2)}(\tau_1,\,\tau_1) + 24 \left[G_\omega^{(2)}(\tau_2,\,\tau_1) \right]^4$$

$$+ 9 \left[G_\omega^{(2)}(\tau_2,\,\tau_2) \right]^2 \left[G_\omega^{(2)}(\tau_1,\,\tau_1) \right]^2 \Big\}$$

Diagrams like  are called disconnected for obvious reasons. In fact, for each connected diagram K, there is a series of disconnected diagrams of the form $\frac{1}{n!} K^n$ for the n^{th} order perturbation. Summing them up give a factor $e^{iK/\hbar}$ to the path integral, and hence K to the energy shift. Therefore, only connected diagrams need be considered in a pertubation caculation. This is called the linked cluster theorem or Goldstone's theorem [see A.L.Fetter & J.D.Walecka, "Quantum Theory of Many-Particle Systems", p.111.].

It will be understood that in calculating the cumulant $\langle \mathcal{A}^2_{\text{int},\,e} \rangle_{\omega,\,c}$, only the connected diagrams are used.

The 3rd order perturbation involves $G^{(12)}_{\omega^2}(\tau_1, \ ..., \ \tau_{12})$ for which the only connected diagrams are



For , since the left & right vertices are equivalent,

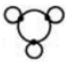
$$\alpha = \frac{3!}{2!} = 3$$

Going from left to right, we have

$$N = C_2^4 \left[\frac{(C_1^2 \times 4)(C_1^2 \times 3)}{2!} \right] \left[\frac{(C_1^2 \times 4)(C_1^2 \times 3)}{2!} \right] C_2^2 = 864$$

$$\mu = 2592$$

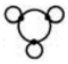
 $= \frac{1}{x^3}$  $^2 \times$  ^{2}

For , since all vertices are equivalent,

$$\alpha = 1$$

Going counterclockwise, we have

$$\mu = N = C_2^4 (C_1^2 \times 4) C_2^3 (C_1^1 \times 4) C_2^3 (C_1^1 \times C_1^1) = 1728$$

 $= \frac{1}{x^3}$  $\times$  ^{3}

where

is $\quad \chi_3^3 = \int_0^x d\,s_3 \int_0^x d\,s_2 \int_0^x d\,s_1 \ \mathcal{G}(\,|\,s_1 - s_2\,|\,)\,\mathcal{G}(\,|\,s_2 - s_3\,|\,)\,\mathcal{G}(\,|\,s_3 - s_1\,|\,)$

$$= x\,\alpha_3^6$$

or $\quad c_3^3 = \int_0^{\beta\hbar} d\,\tau_3 \int_0^{\beta\hbar} d\,\tau_2 \int_0^{\beta\hbar} d\,\tau_1 \, G^{(2)}_{\omega^2}(\tau_1, \ \tau_2)\, G^{(2)}_{\omega^2}(\tau_2, \ \tau_3)\, G^{(2)}_{\omega^2}(\tau_3, \ \tau_1)$

$$= \frac{x}{\omega^3} a_3^6 \qquad (3.547e)$$

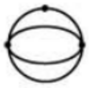
For , since the left & right vertices are equivalent,

$$\alpha = \frac{3!}{2!} = 3$$

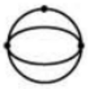
Going from top to bottom, we have

$$N = C_2^4 \left[(C_1^2 \times 4)(C_1^1 \times 4) \right] \left[\frac{(C_1^3\,C_1^3)(C_1^2\,C_1^2)(C_1^1\,C_1^1)}{3!} \right] = 1152$$

$\mu = 3456$

 $= \dfrac{1}{x}$  $\times$ 

where, calling the top vertex s_3,



is $\quad \chi_3^5 = \int_0^x d\, s_3 \int_0^x d\, s_2 \int_0^x d\, s_1 \left[\mathcal{G}(\,|\, s_1 - s_2\,|\,)\right]^3 \mathcal{G}(\,|\, s_1 - s_3\,|\,)\, \mathcal{G}(\,|\, s_2 - s_3\,|\,)$

$\quad = x\, \alpha_3^{10}$

or $\quad c_3^5 = \int_0^{\beta\hbar} d\,\tau_3 \int_0^{\beta\hbar} d\,\tau_2 \int_0^{\beta\hbar} d\,\tau_1 \left[G_{\omega^2}^{(2)}(\tau_1,\, \tau_2)\right]^3 G_{\omega^2}^{(2)}(\tau_1,\, \tau_3)\, G_{\omega^2}^{(2)}(\tau_2,\, \tau_3)$

$$= \frac{x}{\omega^3}\, a_3^{10} \qquad (3.547f)$$

For , since all vertices are equivalent,

$\alpha = 1$

Going counterclockwise, we have

$$\mu = N = \left[\frac{(C_1^4\, C_1^4)\,(C_1^3\, C_1^3)}{2!}\right]\left[\frac{(C_1^2\, C_1^4)\,(C_1^1\, C_1^3)}{2!}\right]\left[\frac{(C_1^2\, C_1^2)\,(C_1^1\, C_1^1)}{2!}\right] = 1728$$

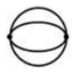
is $\chi_3^6 = \int_0^x d\, s_3 \int_0^x d\, s_2 \int_0^x d\, s_1 \left[\mathcal{G}(\,|\, s_1 - s_2\,|\,)\right]^2 \left[\mathcal{G}(\,|\, s_2 - s_3\,|\,)\right]^2 \left[\mathcal{G}(\,|\, s_3 - s_1\,|\,)\right]^2$

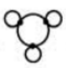
$\quad = x\, \alpha_3^{12}$

or $\quad c_3^6 = \int_0^{\beta\hbar} d\,\tau_3 \int_0^{\beta\hbar} d\,\tau_2 \int_0^{\beta\hbar} d\,\tau_1 \left[G_{\omega^2}^{(2)}(\tau_1,\, \tau_2)\right]^2 \left[G_{\omega^2}^{(2)}(\tau_1,\, \tau_3)\right]^2 \left[G_{\omega^2}^{(2)}(\tau_2,\, \tau_3)\right]^2$

$$= \frac{x}{\omega^3}\, a_3^{12} \qquad (3.547g)$$

Thus, we have [see (3.484)]

$\beta\,(F - F_\omega) = \dfrac{1}{\hbar}$  3 $- \dfrac{1}{2!\,\hbar^2}\Big($  72 +  24 $\Big)$

$+ \dfrac{1}{3!\,\hbar^3}\Big($  2592 +  1728 +  3456 +  1728 $\Big)$ (3.545a)

where the diagrams are treated as Γ's [see (3.548b)] and [see (3.324) or (2.524)]

$$F_\omega = \frac{1}{\beta} \ln\left[2 \sinh\left(\frac{1}{2}\, \beta\, \hbar\, \omega\right)\right] \qquad (3.545)$$

is sometimes represented as the 1-loop diagram as

$$\beta\, F_\omega = -\frac{1}{2} \operatorname{Tr} \ln G_{\omega^2}^{(2)}$$

$$= -\frac{1}{2\,\beta\,\hbar} \int_0^{\beta\hbar} d\,\tau \ln\left[G_{\omega^2}^{(2)}(\tau,\, \tau)\right]$$

$$= -\frac{1}{2} \ln\left[\frac{1}{\omega}\, \mathcal{G}(0)\right]$$

$$= -\frac{1}{2}$$  (3.546)

Note that the integrand in  involves the logarithm of $\mathcal{G}$.

Using (3.546a-g), we can write (3.545a) as

$$\beta(F - F_\omega) = \frac{1}{\hbar}\left(\frac{1}{4}g\right)\left(\frac{\hbar}{M\omega}\right)^2 \frac{1}{\omega}\left[3\,\frac{1}{x}(\chi_1^1)^2\right] \tag{3.549a}$$

$$-\frac{1}{2\,\hbar^2}\left(\frac{1}{4}g\right)^2\left(\frac{\hbar}{M\omega}\right)^4 \frac{1}{\omega^2}\left[72\,\frac{1}{x^2}\,\chi_2^2\,(\chi_1^1)^2 + 24\,\chi_2^4\right]$$

$$+\frac{1}{3!\,\hbar^3}\left(\frac{1}{4}g\right)^3\left(\frac{\hbar}{M\omega}\right)^6 \frac{1}{\omega^3}\left[2592\,\frac{1}{x^3}\,(\chi_2^2)^2\,(\chi_1^1)^2\right.$$

$$\left. + 1728\,\frac{1}{x^3}\,\chi_3^3(\chi_1^1)^3 + 3456\,\frac{1}{x}\,\chi_3^5\,\chi_1^1 + 1728\,\chi_3^6\right] + \ldots$$

Using (3.548a), we have

$$F - F_\omega = \frac{1}{4}g\left[3\,(a_1^2)^2\right] - \frac{1}{2\,\hbar\,\omega}\left(\frac{1}{4}g\right)^2\left[72\,a_2^4\,(a_1^2)^2 + 24\,a_2^8\right] \tag{3.549}$$

$$+\frac{1}{3!\,\hbar^2\,\omega^2}\left(\frac{1}{4}g\right)^3\left[2592\,(a_2^4)^2\,(a_1^2)^2 + 1728\,a_3^6(a_1^2)^3\right.$$

$$\left. + 3456\;a_3^{10}\,a_1^2 + 1728\,a_3^{12}\right] + \ldots$$

Note that Kleinert denote a_1^2 as a^2 so that $(a_1^2)^2 = a^4$.

Values of χ_V^L & a_V^{2L} can be found in Appendix 3D.

In the limit $T \to 0$, we can use (3D.13) of Appendix 3D to get

$$a_2^4 \to \frac{1}{4}\left(\frac{\hbar}{M\omega}\right)^2 = \frac{4}{4}(a_1^2)^2 \;= a^4$$

$$a_3^6 \to \frac{3}{16}\left(\frac{\hbar}{M\omega}\right)^3 = \frac{3\times 8}{16}(a_1^2)^3 \;= \frac{3}{2}a^6$$

$$a_2^8 \to \frac{1}{32}\left(\frac{\hbar}{M\omega}\right)^4 = \frac{16}{32}(a_1^2)^4 \;= \frac{1}{2}a^8$$

$$a_3^{10} \to \frac{5}{256}\left(\frac{\hbar}{M\omega}\right)^5 = \frac{5\times 32}{256}(a_1^2)^5 \;= \frac{5}{8}a^{10}$$

$$a_3^{12} \to \frac{3}{512}\left(\frac{\hbar}{M\omega}\right)^6 = \frac{3\times 64}{512}(a_1^2)^6 \;= \frac{3}{8}a^{12} \tag{3.550}$$

With

$$F_\omega \to \frac{1}{\beta}\ln\left[\exp\left(\frac{1}{2}\beta\,\hbar\,\omega\right)\right] = \frac{1}{2}\hbar\,\omega$$

(3.549) becomes

$$F \to \frac{1}{2}\hbar\,\omega + \frac{1}{4}g\left[3\,a^4\right] - \frac{1}{2\,\hbar\,\omega}\left(\frac{1}{4}g\right)^2\left[72 + 24\times\frac{1}{2}\right]a^8$$

$$+\frac{1}{3!\,\hbar^2\,\omega^2}\left(\frac{1}{4}g\right)^3\left[2592\; + 1728\times\frac{3}{2} + 3456\;\frac{5}{8} + 1728\times\frac{3}{8}\right]a^{12} + \ldots$$

$$= \frac{1}{2}\hbar\,\omega + \frac{1}{4}g\left[3\,a^4\right] - \frac{1}{\hbar\,\omega}\left(\frac{1}{4}g\right)^2(42\,a^8) + \frac{1}{\hbar^2\,\omega^2}\left(\frac{1}{4}g\right)^3(\,1332\,a^{12}) + \ldots \tag{3.551}$$

See last paragraph of Kleinert’s text on tips on evaluating a_V^{2L} in this limit.