3.21. Perturbative Definition of Interacting Path Integrals

Consider again the perturbation expansion [see (3.487)]

$$Z[j] = \exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau\, V\left[\frac{\delta}{\delta j(\tau)}\right]\right\} Z_\omega[j] \tag{3.555}$$

based on the generating functional

$$Z_\omega[j] = \oint \mathcal{D}x \exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau\left[\frac{1}{2}M\left(\dot{x}^2+\omega^2 x^2\right)-j(x)\right]\right. \tag{3.553}$$

After a quadratic completion, we have

$$Z_\omega[j] = Z_\omega[0]\exp\left\{\frac{1}{2M\hbar}\int_0^{\beta\hbar} d\tau\int_0^{\beta\hbar} d\tau'\, j(\tau)\, G^p_{\omega^2,e}(\tau-\tau')\, j(\tau')\right\} \tag{3.552}$$

where

$$G^p_{\omega^2,e}(\tau) = \frac{\cosh\omega\left(\frac{1}{2}\beta\hbar-|\tau|\right)}{2\omega\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} \qquad \tau\in(-\beta\hbar,\beta\hbar) \tag{3.554}$$

and

$$Z_\omega[0] = \frac{1}{2\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} \tag{3.552}$$

can be obtained using the analytic regularization described in §2.15.

Thus, the perturbation expansion can be achieved using only functional derivatives and analytic regularization, without evaluating any path integrals. Obviously, the procedure also applies to other systems with arbitrary unperturbed Hamiltonian, and is widely used in quantum field theory to study particle physics and critical phenomena.

Note that there are many important problems that cannot be studied in terms of perturbation owing to divergence difficulties, tunneling phenomena being an example to be discussed in Chapter 17. Sometimes re-summation techniques can provide partial solutions. This will be discussed in Chapter 5.