Appendix 3B. Energy Shifts for g x^4 / 4 -Interaction

For an oscillator,

$$H = \frac{1}{2M} p^2 + \frac{1}{2} M \omega^2 x^2 = \hbar \omega \left(a^+ a + \frac{1}{2} \right)$$

where

$$x = \gamma \, (a^+ + a) \qquad\qquad \gamma = \sqrt{\frac{\hbar}{2 M \omega}}$$

with

$$[a, a^+] = 1$$

$$a \mid n \rangle = \sqrt{n} \left| n - 1 \right\rangle \qquad\qquad a^+ \mid n \rangle = \sqrt{n+1} \left| n + 1 \right\rangle \qquad \text{(A)}$$

The eq. of motion for $a(t)$ is

$$i \hbar \partial_t a(t) = [a(t), H] = \hbar \omega \, a(t)$$

$$\rightarrow \qquad a(t) = a \, e^{-i \omega t} = a \, z^{-1} \qquad\qquad a^+ (t) = a^+ e^{i \omega t} = a^+ z \qquad \text{(B)}$$

where

$$z = e^{i \omega t}$$

For the level n, we introduce the simplified notation

$$\langle f(z) \rangle_\omega \equiv \langle n \mid f(z) \mid n \rangle \qquad \text{(C)}$$

For example, for the perturbation

$$V = \frac{1}{4} g \, x^4 \qquad \text{(D)}$$

we have

$$\langle V \rangle_\omega = \frac{1}{4} g \left\langle x^4(z) \right\rangle_\omega = \frac{1}{4} g \, \gamma^4 \left\langle n \left| \left(a^+ z + a \, z^{-1} \right)^4 \right| n \right\rangle \qquad \text{(E)}$$

Since $\langle n \mid m \rangle = \delta_{n m}$, only monomials with equal powers in a & a^+ survive. Hence, the z's cancel out. Thus,

$$\left\langle x^4(z) \right\rangle_\omega = \gamma^4 \left\langle n \left| \left(a^+ a^+ a \, a + a^+ a \, a^+ a + a^+ a \, a \, a^+ \right.\right.\right.$$

$$\left.\left.\left. + a \, a^+ a^+ a + a \, a^+ a \, a^+ + a \, a \, a^+ a^+ \right) \right| n \right\rangle$$

$$= \gamma^4 \left[n \, (n-1) + n^2 + n \, (n+1) + (n+1) \, n + (n+1)^2 + (n+1) \, (n+2) \right]$$

$$= \gamma^4 \left(6 \, n^2 + 6 \, n + 3 \right) \qquad \text{(3B.4)}$$

For $n = 0$, this simplifies to

$$\left\langle x^4(z) \right\rangle_\omega = \gamma^4 \langle 0 \mid a \, (a^+ a + a \, a^+) \, a^+ \mid 0 \rangle$$

$$= \gamma^4 \langle 1 \mid (2 \, a^+ a + 1) \mid 1 \rangle$$

$$= 3 \, \gamma^4 \qquad \text{(3B.1a)}$$

The cumulants are defined as

$$\langle f \, (V) \rangle_{\omega, c} = \left\langle \left(f(V) - f[\langle V \rangle_\omega] \right) \right\rangle_\omega$$

$$= \left\langle \left(f[x^4(z)] - f[\langle x^4(z) \rangle_\omega] \right) \right\rangle_\omega$$

For example,

$$\left\langle x^4(z) \right\rangle_{\omega, c} = \left\langle \left(x^4(z) - \left\langle x^4(z) \right\rangle_\omega \right) \right\rangle_\omega = 0$$

Such calculations are straight-forward but quickly become tedious and preferably done by machines. Other results listed in Kleinert's Appendix 3B can be found in "A3B._Code.nb" and will not be listed here.