Appendix 3D. Feynman Integrals for $T \neq 0$

As described in §3.20, a diagram with L lines & V vertices represents a value

$$\Gamma_V^L = \mu\,\gamma^V \left(\frac{\hbar}{M\omega}\right)^L \frac{1}{\omega^V}\chi_V^L(x) \qquad = \mu\,\gamma^V c_V^L$$

$$= \mu\,\gamma^V \left(\frac{\hbar}{M\omega}\right)^L \frac{1}{\omega^V}\, x\,\alpha_V^{2L}(x) = \mu\,\gamma^V \frac{x}{\omega^V} a_V^{2L} \qquad (3.D1a)$$

where μ is the multiplicity of the diagram, γ the coupling constant and χ_V^L is the dimensionless V-fold τ-integral over the product of L $\mathcal{G}$'s with

$$x = \beta\,\hbar\,\omega \qquad\qquad s = \omega\,\tau \qquad (3D.2a)$$

$$\mathcal{G}(s) = \frac{\cosh\left(\frac{1}{2}x - s\right)}{2\sinh\left(\frac{1}{2}x\right)} \qquad (3D.2b)$$

$$G_{\omega^2}^{(2)}(\tau_1,\ \tau_2) = \frac{\hbar}{M\omega}\mathcal{G}(\,|\,s_1 - s_2\,|\,) \qquad (3D.2)$$

Evaluation of χ_V^L is straight-forward but tedious. It is best handled by symbolic manipulation soft-wares such as *Mathematica.*

Since *Mathematica* does not process $|\,s\,|$ unless s is a numerical value, we need to break up every multiple s-integral into ones for which all $|\,s_j - s_k\,|$'s have definite signs. Consider for example the integral

$$\mathcal{I} = \int_0^x d\,s_3 \int_0^x d\,s_2 \int_0^x d\,s_1\, f(\,|\,s_1 - s_2\,|\,,\ |\,s_2 - s_3\,|\,,\ |\,s_3 - s_1\,|\,) \qquad (3D.1b)$$

where f is a product of three $\mathcal{G}$'s. Owing to the periodic B.C., f is periodic in s_j with period x so that

$$\int_a^{x+a} d\,s_j\, f = \int_0^x d\,s_j\, f \qquad (3D1c)$$

With the substitution

$$s_1 - s_3 \to s_1 \qquad\qquad s_2 - s_3 \to s_2$$

and the help of (3D1c), we can write (3D.1b) as

$$\mathcal{I} = \int_0^x d\,s_3 \int_0^x d\,s_2 \int_0^x d\,s_1\, f(\,|\,s_1 - s_2\,|\,,\, s_2,\, s_1\,)$$

$$= x\int_0^x d\,s_2 \int_0^x d\,s_1\, f(\,|\,s_1 - s_2\,|\,,\, s_2,\, s_1\,)$$

$$= x\int_0^x d\,s_2 \left[\int_0^{s_2} d\,s_1\, f(\,s_2 - s_1,\, s_2,\, s_1\,) + \int_{s_2}^x d\,s_1\, f(\,s_1 - s_2,\, s_2,\, s_1\,)\right] \qquad (3D.1)$$

Mathematica codes for the calculation of $\chi_V^L(x) = x\alpha_V^{2L}$ can be found in "A3D._Code.nb". Here, we list those results that are used in §3.20.

[Format of the results may look awkful since they are copied from the *Mathematica* output.]

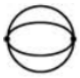
$$\chi_1^1 = x\,\alpha_1^2 = x\,a^2 = \int_0^x d\,s\,\mathcal{G}(0)$$

$$= \frac{1}{2}x\coth\left(\frac{x}{2}\right) \qquad (3D.3)$$

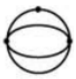
$$\chi_2^2 = x\,\alpha_2^4 = \int_0^x d\,s_2 \int_0^x d\,s_1\ \mathcal{G}(\,|\,s_1 - s_2\,|\,)^2$$

$$= \frac{1}{8}x\,(x + \sinh(x))\,\text{csch}^2\left(\frac{x}{2}\right) \qquad (3D.4)$$

$$\chi_3^3 = x\,\alpha_3^6 = \int_0^x d\,s_3 \int_0^x d\,s_2 \int_0^x d\,s_1\, \mathcal{G}(\,|\,s_1 - s_2\,|\,)\, \mathcal{G}(\,|\,s_2 - s_3\,|\,)\, \mathcal{G}(\,|\,s_3 - s_1\,|\,)$$
$$= x \int_0^x d\,s_2 \int_0^x d\,s_1\, \mathcal{G}(\,|\,s_1 - s_2\,|\,)\, \mathcal{G}(s_2)\, \mathcal{G}(s_1)$$
$$= \frac{1}{32}\, x \left(6 \coth\left(\frac{x}{2}\right) + x \left(x \coth\left(\frac{x}{2}\right) + 3\right) \operatorname{csch}^2\left(\frac{x}{2}\right)\right) \qquad (3D.5)$$

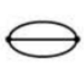


$$\chi_2^4 = x\,\alpha_2^8 = \int_0^x d\,s_2 \int_0^x d\,s_1\, \mathcal{G}(\,|\,s_1 - s_2\,|\,)^4$$
$$= x \int_0^x d\,s_1\, \mathcal{G}\,(s_1)^4$$
$$= \frac{1}{256}\, x\,(6\,x + 8 \sinh(x) + \sinh(2\,x)) \operatorname{csch}^4\left(\frac{x}{2}\right) \qquad (3D.6)$$

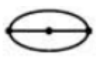


$$\chi_3^5 = x\,\alpha_3^{10} = \int_0^x d\,s_3 \int_0^x d\,s_2 \int_0^x d\,s_1\, \mathcal{G}(\,|\,s_1 - s_2\,|\,)^3\, \mathcal{G}(\,|\,s_1 - s_3\,|\,)\, \mathcal{G}(\,|\,s_2 - s_3\,|\,)$$
$$= x \int_0^x d\,s_2 \int_0^x d\,s_1\, \mathcal{G}(\,|\,s_1 - s_2\,|\,)^3\, \mathcal{G}(s_1)\, \mathcal{G}(s_2)$$
$$= \frac{1}{4096}\, x \operatorname{csch}^5\left(\frac{x}{2}\right) \left(8\,(3\,x^2 - 5) \cosh\left(\frac{x}{2}\right) + 72\,x \sinh\left(\frac{x}{2}\right) + 12\,x \sinh\left(\frac{3\,x}{2}\right) + 35 \cosh\left(\frac{3\,x}{2}\right) + 5 \cosh\left(\frac{5\,x}{2}\right)\right) \qquad (3D.7)$$

$$\chi_3^6 = x\,\alpha_3^{12} = \int_0^x d\,s_3 \int_0^x d\,s_2 \int_0^x d\,s_1\, \mathcal{G}(\,|\,s_1 - s_2\,|\,)^2\, \mathcal{G}(\,|\,s_1 - s_3\,|\,)^2\, \mathcal{G}(\,|\,s_2 - s_3\,|\,)^2$$
$$= x \int_0^x d\,s_2 \int_0^x d\,s_1\, \mathcal{G}(\,|\,s_1 - s_2\,|\,)^2\, \mathcal{G}(s_1)^2\, \mathcal{G}(s_2)^2$$
$$= \frac{1}{16\,384} \left(x \operatorname{csch}^6\left(\frac{x}{2}\right) \left(32\,x^2 + \left(8\,x^2 - 3\right) \cosh(x) + 108\,x \sinh(x) + 48 \cosh(2\,x) + 3 \cosh(3\,x) - 48\right)\right) \qquad (3D.8)$$



$$\chi_2^3 = x\,\alpha_2^6 = \int_0^x d\,s_2 \int_0^x d\,s_1\, \mathcal{G}(\,|\,s_1 - s_2\,|\,)^3$$
$$= x \int_0^x d\,s_1\, \mathcal{G}\,(s_1)^3$$
$$= \frac{1}{24}\, x\,(\cosh(x) + 5) \operatorname{csch}^2\left(\frac{x}{2}\right) \qquad (3D.9)$$



$$\chi_3^4 = x\,\alpha_3^8 = \int_0^x d\,s_3 \int_0^x d\,s_2 \int_0^x d\,s_1\, \mathcal{G}(\,|\,s_1 - s_2\,|\,)^2\, \mathcal{G}(\,|\,s_1 - s_3\,|\,)\, \mathcal{G}(\,|\,s_2 - s_3\,|\,)$$
$$= x \int_0^x d\,s_2 \int_0^x d\,s_1\, \mathcal{G}(\,|\,s_1 - s_2\,|\,)^2\, \mathcal{G}(s_1)\, \mathcal{G}(s_2)$$
$$= \frac{1}{72}\, x \operatorname{csch}^3\left(\frac{x}{2}\right) \left(9 \sinh\left(\frac{x}{2}\right) + \sinh\left(\frac{3\,x}{2}\right) + 3\,x \cosh\left(\frac{x}{2}\right)\right) \qquad (3D.10)$$

$$\chi_{3'}^5 = x\,\alpha_{3'}^{10} = \int_0^x d\,s_3 \int_0^x d\,s_2 \int_0^x d\,s_1\, \mathcal{G}(\,|\,s_1 - s_2\,|\,)\, \mathcal{G}(\,|\,s_1 - s_3\,|\,)^2\, \mathcal{G}(\,|\,s_2 - s_3\,|\,)^2$$
$$= x \int_0^x d\,s_2 \int_0^x d\,s_1\, \mathcal{G}(\,|\,s_1 - s_2\,|\,)\, \mathcal{G}(s_1)^2\, \mathcal{G}(s_2)^2$$

$$= \frac{x \operatorname{csch}^4\left(\frac{x}{2}\right)(15\,x + \sinh(x)\,(5\cosh(x) + 52))}{1152} \tag{3D.11}$$

In the low-temperature limit, $x \to \infty$, we have

$$\begin{aligned}
&\{\alpha_1^2, \alpha_2^4, \alpha_3^6, \alpha_2^8, \alpha_3^{10}, \alpha_3^{12}, \alpha_2^6, \alpha_3^8, \alpha_{3'}^{10}\} \\
&= \frac{1}{x}\{\chi_1^1, \chi_2^2, \chi_3^3, \chi_2^4, \chi_3^5, \chi_3^6, \chi_2^3, \chi_3^4, \chi_{3'}^5\} \\
&\xrightarrow[x\to\infty]{} \left\{\frac{1}{2}, \frac{1}{4}, \frac{3}{16}, \frac{1}{32}, \frac{5}{256}, \frac{3}{512}, \frac{1}{12}, \frac{1}{18}, \frac{5}{288}\right\}
\end{aligned} \tag{3D.13}$$

respectively [see “A3D._Code.nb”].

Using [see (3.548)]

$$a_V^{2L} = \left(\frac{\hbar}{M\omega}\right)^L \alpha_V^{2L}(x)$$

we deduce that

$$\begin{aligned}
&a_1^2 = a^2 \;\to\; \frac{1}{2}\left(\frac{\hbar}{M\omega}\right) \\
\therefore\quad &a_2^6 \to \frac{1}{12}\left(\frac{\hbar}{M\omega}\right)^3 = \frac{8}{12}\left(a_1^2\right)^3 = \frac{2}{3}a^6 \\
&a_3^8 \to \frac{1}{18}\left(\frac{\hbar}{M\omega}\right)^4 = \frac{16}{18}\left(a_1^2\right)^4 = \frac{8}{9}a^8 \\
&a_{3'}^{10} \to \frac{5}{288}\left(\frac{\hbar}{M\omega}\right)^5 = \frac{5\times 32}{288}\left(a_1^2\right)^5 = \frac{5}{9}a^{10}
\end{aligned} \tag{3D.14}$$

In the high temperature limit, $x \to 0$,

$$\mathcal{G}(s_j - s_k) \to \mathcal{G}(0) \to \frac{1}{x}$$

so that

$$a^2 \to \frac{1}{x}\left(\frac{\hbar}{M\omega}\right)$$

$$\chi_V^L \;\to\; x^V \mathcal{G}(0)^L = x^{V-L}$$

$$\therefore\quad \alpha_V^{2L} \;\to\; x^{V-L-1} \tag{3D.15a}$$

$$a_V^{2L} \;\to\; \left(\frac{\hbar}{M\omega}\right)^L x^{V-L-1} \tag{3D.15}$$

For small x, (3D.3-11) give [see “A3D._Code.nb”]

$$\alpha_1^2 = \frac{1}{x} + \frac{x}{12} - \frac{x^3}{720} + \frac{x^5}{30\,240} - \frac{x^7}{1\,209\,600} + \frac{x^9}{47\,900\,160} + \ldots$$

$$\alpha_2^4 = \frac{1}{x} + \frac{x^3}{720} - \frac{x^5}{15\,120} + \frac{x^7}{403\,200} - \frac{x^9}{11\,975\,040} + \ldots$$

$$\alpha_3^6 = \frac{1}{x} + \frac{x^5}{30\,240} - \frac{x^7}{403\,200} + \frac{x^9}{7\,983\,360} + \ldots$$

$$\alpha_2^8 = \frac{1}{x^3} + \frac{x}{120} - \frac{x^3}{3780} + \frac{x^5}{80\,640} - \frac{x^7}{1\,663\,200} + \frac{7213\,x^9}{261\,534\,873\,600} + \ldots$$

$$\alpha_3^{10} = \frac{1}{x^3} + \frac{x}{240} - \frac{x^3}{15\,120} + \frac{x^7}{6\,652\,800} - \frac{7213\,x^9}{523\,069\,747\,200} + \ldots$$

$$\alpha_3^{12} = \frac{1}{x^4} + \frac{1}{240} + \frac{x^2}{15\,120} - \frac{x^6}{4\,989\,600} + \frac{701\,x^8}{34\,871\,316\,480} + \ldots$$

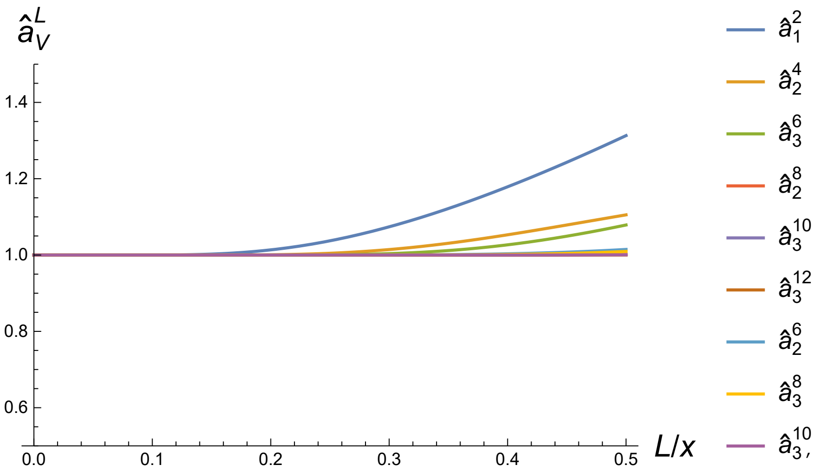
$$\alpha_2^6 = \frac{1}{x^2} + \frac{x^2}{240} - \frac{x^4}{6048} + \frac{x^6}{172\,800} - \frac{x^8}{5\,322\,240} + \ldots$$
$$\alpha_3^8 = \frac{1}{x^2} + \frac{x^2}{720} - \frac{x^6}{518\,400} + \frac{x^8}{7\,983\,360} + \ldots$$
$$\alpha_{3'}^{10} = \frac{1}{x^3} + \frac{x}{360} - \frac{x^5}{1\,209\,600} + \frac{629\,x^9}{261\,534\,873\,600} + \ldots \tag{3D.16}$$

Define the reduced quantities

$$\hat{a}_V^{2L}(x) \equiv \frac{a_V^{2L}(x)}{a_{V,\infty}^{2L}(x)}$$

where $a_{V,\infty}^{2L}(x)$ is the low temperature limit of $a_V^{2L}(x)$ expressed as a function of $a_1^2(x)$ and listed in (3.550) & (3D.14). They are plotted below

Out[75]=



Unfortunately, except for $\hat{a}_1^2$, they look nothing like Kleinert's fig.3.16.

Using the spectral representation [see (3.114e) & (3.301)]

$$G_{\omega^2}^{(2)}(\tau, \tau') = \frac{\hbar}{M} \frac{1}{\beta\hbar} \sum_{m=-\infty}^{\infty} e^{-i\omega_m(\tau-\tau')} \frac{1}{\omega_m^2 + \omega^2} \tag{3D.18}$$

we have

$$\begin{aligned}
G_{\omega^2+\delta\omega^2}^{(2)}(\tau, \tau') &= \frac{1}{M\beta\hbar} \sum_{m=-\infty}^{\infty} e^{-i\omega_m(\tau-\tau')} \frac{\hbar}{\omega_m^2 + \omega^2 + \delta\omega^2} \\
&= \frac{1}{M\beta\hbar} \sum_{m=-\infty}^{\infty} e^{-i\omega_m(\tau-\tau')} \frac{\hbar}{\omega_m^2 + \omega^2} \left(1 + \frac{\delta\omega^2}{\omega_m^2 + \omega^2}\right)^{-1} \\
&= \frac{1}{M\beta\hbar} \sum_{m=-\infty}^{\infty} e^{-i\omega_m(\tau-\tau')} \frac{\hbar}{\omega_m^2 + \omega^2} \sum_{n=0}^{\infty} \left(-\frac{\delta\omega^2}{\omega_m^2 + \omega^2}\right)^n \\
&= \frac{1}{M\beta\hbar} \sum_{m=-\infty}^{\infty} e^{-i\omega_m(\tau-\tau')} \left\{ \frac{\hbar}{\omega_m^2 + \omega^2} - \frac{\delta\omega^2}{\hbar} \left(\frac{\hbar}{\omega_m^2 + \omega^2}\right)^2 \right. \\
&\qquad \left. + \left(\frac{\delta\omega^2}{\hbar}\right)^2 \left(\frac{\hbar}{\omega_m^2 + \omega^2}\right)^3 + \ldots \right\}
\end{aligned} \tag{3D.19}$$

Now, using

$$\int_0^{\beta\hbar} d\tau\, e^{-i(\omega_m - \omega_{m'})\tau} = \beta\hbar\,\delta_{m m'} \qquad \omega_m = \frac{2\pi}{\beta\hbar} m$$

we have

$$\int_0^{\beta\hbar} d\tau_1\, G^{(2)}_{\omega^2}(\tau, \tau_1)\, G^{(2)}_{\omega^2}(\tau_1, \tau')$$
$$= \left(\frac{1}{M\beta\hbar}\right)^2 \sum_{m,m'=-\infty}^{\infty} \int_0^{\beta\hbar} d\tau_1\, e^{-i\omega_m(\tau-\tau_1)}\, e^{-i\omega_{m'}(\tau_1-\tau')} \frac{\hbar}{\omega_m^2+\omega^2} \frac{\hbar}{\omega_{m'}^2+\omega^2}$$
$$= \frac{1}{M^2\beta\hbar} \sum_{m=-\infty}^{\infty} e^{-i\omega_m(\tau-\tau')} \left(\frac{\hbar}{\omega_m^2+\omega^2}\right)^2 \qquad \text{(3D.19a)}$$

$$\int_0^{\beta\hbar} d\tau_2 \int_0^{\beta\hbar} d\tau_1\, G^{(2)}_{\omega^2}(\tau, \tau_2)\, G^{(2)}_{\omega^2}(\tau_2, \tau_1)\, G^{(2)}_{\omega^2}(\tau_1, \tau')$$
$$= \left(\frac{1}{M\beta\hbar}\right)^3 \sum_{m,m',m''=-\infty}^{\infty} \int_0^{\beta\hbar} d\tau_2 \int_0^{\beta\hbar} d\tau_1$$
$$\times e^{-i\omega_m(\tau-\tau_2)}\, e^{-i\omega_{m'}(\tau_2-\tau_1)}\, e^{-i\omega_{m''}(\tau_1-\tau')} \frac{\hbar}{\omega_m^2+\omega^2} \frac{\hbar}{\omega_{m'}^2+\omega^2} \frac{\hbar}{\omega_{m''}^2+\omega^2}$$
$$= \frac{1}{M^3\beta\hbar} \sum_{m=-\infty}^{\infty} e^{-i\omega_m(\tau-\tau')} \left(\frac{\hbar}{\omega_m^2+\omega^2}\right)^3 \qquad \text{(3D.19b)}$$

Using (3D.19a-b) on (3D.19), we have

$$G^{(2)}_{\omega^2+\delta\omega^2}(\tau, \tau') = G^{(2)}_{\omega^2}(\tau, \tau') - \left(\frac{M\delta\omega^2}{\hbar}\right) \int_0^{\beta\hbar} d\tau_1\, G^{(2)}_{\omega^2}(\tau, \tau_1)\, G^{(2)}_{\omega^2}(\tau_1, \tau') \qquad \text{(3D.20)}$$
$$+ \left(\frac{M\delta\omega^2}{\hbar}\right)^2 \int_0^{\beta\hbar} d\tau_2 \int_0^{\beta\hbar} d\tau_1\, G^{(2)}_{\omega^2}(\tau, \tau_2)\, G^{(2)}_{\omega^2}(\tau_2, \tau_1)\, G^{(2)}_{\omega^2}(\tau_1, \tau') + \ldots$$

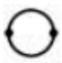
On the other hand, Taylor expension of $G^{(2)}_{\omega^2+\delta\omega^2}(\tau, \tau')$ gives

$$G^{(2)}_{\omega^2+\delta\omega^2}(\tau, \tau') = \sum_{n=0}^{\infty} \frac{(\delta\omega^2)^n}{n!} \left(\frac{\partial}{\partial\omega^2}\right)^n G^{(2)}_{\omega^2}(\tau, \tau')\Big|_{\delta\omega^2=0} \qquad \text{(3D.20a)}$$

Comparing with (3D.20) gives

$$\mathcal{I}_n = \int_0^{\beta\hbar} d\tau_n \ldots \int_0^{\beta\hbar} d\tau_1\, G^{(2)}_{\omega^2}(\tau, \tau_n)\, G^{(2)}_{\omega^2}(\tau_n, \tau_{n-1}) \ldots G^{(2)}_{\omega^2}(\tau_2, \tau_1)\, G^{(2)}_{\omega^2}(\tau_1, \tau')$$
$$= \frac{1}{n!}\left(-\frac{\hbar}{M}\frac{\partial}{\partial\omega^2}\right)^n G^{(2)}_{\omega^2}(\tau, \tau')\Big|_{\delta\omega^2=0} \qquad \text{(3D.17)}$$
$$= \frac{1}{n!}\left(-\frac{\hbar}{M}\frac{\partial}{2\omega\,\partial\omega}\right)^n G^{(2)}_{\omega^2}(\tau, \tau')\Big|_{\delta\omega^2=0}$$

Note that $\mathcal{I}_n$ corresponds to the graph obtained by inserting n points to the line joining τ & τ'. This operation is called a mass insertion. Thus, we have

 $= \left(-\frac{\hbar}{M}\frac{\partial}{\partial\omega^2}\right)$  $\rightarrow \quad a_2^4 = \left(-\frac{\hbar}{M}\frac{\partial}{\partial\omega^2}\right) a_1^2$

 $= \frac{1}{2!}\left(-\frac{\hbar}{M}\frac{\partial}{\partial\omega^2}\right)^2$  $\rightarrow \quad a_2^6 = \frac{1}{2!}\left(-\frac{\hbar}{M}\frac{\partial}{\partial\omega^2}\right)^2 a_1^2$

From (2.489), we have

$$F_\omega = \frac{1}{2\beta} \sum_{m=-\infty}^{\infty} \ln\left(\omega_m^2 + \omega^2\right)$$
$$F_{\sqrt{\omega^2+\delta\omega^2}} = \frac{1}{2\beta} \sum_{m=-\infty}^{\infty} \ln\left(\omega_m^2 + \omega^2 + \delta\omega^2\right)$$

$$= \frac{1}{2\beta}\sum_{m=-\infty}^{\infty}\ln\left(\omega_m^2+\omega^2\right)+\frac{1}{2\beta}\sum_{m=-\infty}^{\infty}\ln\left(1+\frac{\delta\omega^2}{\omega_m^2+\omega^2}\right)$$

$$= F_\omega+\frac{1}{2\beta}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\left(\frac{\delta\omega^2}{\omega_m^2+\omega^2}\right)^n \qquad \text{(3D.20b)}$$

From (3D.19a-b), we can deduce that

$$\frac{1}{\beta\hbar}\sum_{m=-\infty}^{\infty}e^{-i\,\omega_m(\tau-\tau')}\left(\frac{1}{\omega_m^2+\omega^2}\right)^n$$

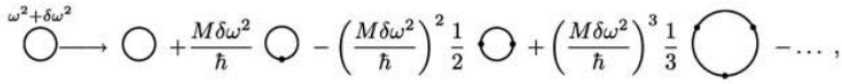
$$= \left(\frac{M}{\hbar}\right)^n\int_0^{\beta\hbar}d\,\tau_{n-1}\ \ldots\int_0^{\beta\hbar}d\,\tau_1\ G^{(2)}_{\omega^2}(\tau,\ \tau_{n-1})\ldots G^{(2)}_{\omega^2}(\tau_1,\ \tau')$$

$$= \left(\frac{M}{\hbar}\right)^n\frac{1}{n!}\left(-\frac{\hbar}{M}\frac{\partial}{\partial\,\omega^2}\right)^n G^{(2)}_{\omega^2}(\tau,\ \tau')\Big|_{\delta\omega^2=0} \qquad \text{[(3D.17) used.]} \qquad \text{(3D.20c)}$$

so that (3D.20b) becomes

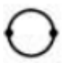
$$F_{\sqrt{\omega^2+\delta\omega^2}} = F_\omega+\frac{\hbar}{2}\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\left(\frac{M\,\delta\,\omega^2}{\hbar}\right)^n\frac{1}{n!}\left(-\frac{\hbar}{M}\frac{\partial}{\partial\,\omega^2}\right)^n G^{(2)}_{\omega^2}(\tau,\ \tau')\Big|_{\delta\omega^2=0}$$

or, graphically,



where the diagrams are to be taken as c_V^L's.

For example, using

 $= c_2^2 = \dfrac{x}{\omega^2}\,a_2^4$

$$-\frac{1}{2}\frac{x}{\omega^2}\,a_2^4 = \frac{1}{2}\left(-\frac{\hbar}{M}\frac{\partial}{\partial\,\omega^2}\right)^2(-2\,\beta\,F_\omega) \qquad \text{(3D.21)}$$