1.1. The Dirac Delta

The Dirac delta $\delta(x)$ is a distribution that has the defining property

$$\int_I d\,x\,f(x)\,\delta(x-a) = \begin{cases} f(a) & \text{if } \; a \in I \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

where $f(x)$ is any function regular at $x = 0$. [See §1.2]

Other properties of $\delta(x)$

We'll always assume

$$\int = \int_{-\infty}^{\infty}$$

Integrating by part, we have

$$\int d\,x\,f(x)\,\frac{\partial}{\partial\,x}\,\delta(x-a) = f(x)\,\delta(x-a)\,\Big|_{-\infty}^{\infty} - \int d\,x\,\delta(x-a)\,\frac{\partial\,f}{\partial\,x}$$

$$= -\frac{\partial\,f}{\partial\,x}\Big|_{x=a}$$

Repeating the above n times gives

$$\int d\,x\,f(x)\,\frac{\partial^n}{\partial\,x^n}\,\delta(x-a) = (-)^n\,\frac{\partial^n f}{\partial\,x^n}\Big|_{x=a} \tag{1.2}$$

or

$$\frac{\partial^n}{\partial\,x^n}\,\delta(x-a) = (-)^n\,\delta(x-a)\,\frac{\partial^n}{\partial\,x^n} \tag{1.2a}$$

Let $y = a\,x$, we have

$$\int d\,x = \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} d\,y & \text{for} \;\; a > 0 \\ \frac{1}{a}\int_{\infty}^{-\infty} d\,y & \text{for} \;\; a < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} d\,y & \text{for} \;\; a > 0 \\ -\frac{1}{a}\int_{-\infty}^{\infty} d\,y & \text{for} \;\; a < 0 \end{cases}$$

$$= \frac{1}{|\,a\,|}\int d\,y$$

$$\rightarrow \qquad \int d\,x\,f(x)\,\delta(a\,x-b) = \frac{1}{|\,a\,|}\int d\,y\,f\!\left(\frac{y}{a}\right)\delta(y-b)$$

$$= \frac{1}{|\,a\,|}\,f\!\left(\frac{b}{a}\right) \tag{1.3}$$

or

$$\delta\,(a\,x-b) = \frac{1}{|\,a\,|}\,\delta\!\left(x-\frac{b}{a}\right) \tag{1.3a}$$

Let $y = g(x)$, we have

$$d\,x = \frac{d\,x}{d\,y}\,d\,y \;=\; \frac{1}{g'(x)}\,d\,y \qquad\qquad \text{where} \qquad\qquad g'(x) = \frac{d\,g}{d\,x}$$

$$\to \qquad \int d\,x\, f(x)\,\delta[g(x)] = \int \frac{d\,y}{|\,g'(x)\,|} f(y)\,\delta(y)$$

Let the roots of $g(x)$ be x_i, i.e.,

$$y = g(x_i) = 0 \qquad \text{with} \qquad i = 1, \ \ldots, n$$

$$\to \qquad \int d\,x\, f(x)\,\delta[g(x)] = \sum_{i=1}^{n} \frac{f(x_i)}{|\,g'(x_i)\,|} \tag{1.4}$$

or

$$\delta[g(x)] = \sum_{i=1}^{n} \frac{\delta\,(x - x_i)}{|\,g'(x_i)\,|} \tag{1.4a}$$

Generalizing to $\boldsymbol{x} \in R^n$, eq(1.1) becomes

$$\int_I d^{\,n}\,x\, f(\boldsymbol{x})\,\delta^n(\boldsymbol{x} - \boldsymbol{a}) = \begin{cases} f(\boldsymbol{a}) & \text{if} \quad \boldsymbol{a} \in I \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

or simply

$$\int d\,\boldsymbol{x}\, f(\boldsymbol{x})\,\delta(\boldsymbol{x} - \boldsymbol{a}) = f(\boldsymbol{a}) \tag{1.5a}$$