1.2. Completeness

Let $f(x)$ be real, piecewise continuous & periodic for $x \in (-L, L)$. Then we can write

$$f(x) = \sum_{n=-\infty}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}$$

$$= a_0 + \sum_{n=1}^{\infty} \left\{ (a_n + a_{-n}) \cos\left(\frac{n\pi x}{L}\right) + (b_n - b_{-n}) \sin\left(\frac{n\pi x}{L}\right) \right\} \qquad (1.6)$$

Using

$$\frac{1}{L}\int_{-L}^{L} dx \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = 0 \qquad (1.11)$$

$$\frac{1}{L}\int_{-L}^{L} dx \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \delta_{nm} \qquad (1.12)$$

$$\frac{1}{L}\int_{-L}^{L} dx \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) = \delta_{nm} \qquad (1.13)$$

we have

$$a_n = a_{-n} = \frac{1}{2L}\int_{-L}^{L} dx\, f(x) \cos\left(\frac{n\pi x}{L}\right) \qquad (1.7)$$

$$b_n = -b_{-n} = \frac{1}{2L}\int_{-L}^{L} dx\, f(x) \sin\left(\frac{n\pi x}{L}\right) \qquad (1.8)$$

The set of functions $\left\{ \cos\left(\frac{n\pi x}{L}\right),\ \sin\left(\frac{n\pi x}{L}\right) \right\}$ are said to be complete with respect to the set $\mathcal{F}$ of all real, piecewise continuous & periodic functions over the interval $(-L, L)$.

Treating $\mathcal{F}$ as a vector space, we endow it with an inner product

$$f \cdot g = \langle f \mid g \rangle = \frac{1}{L}\int_{-L}^{L} dx\, f^*(x)\, g(x) \qquad \forall\ f, g \in \mathcal{F} \qquad (1.10)$$

Note that the function f is now taken as a vector $\mid f \rangle$ in $\mathcal{F}$.

The natural basis of $\mathcal{F}$ is the set

$$X = \{ \mid x \rangle \ ;\ x \in (-L, L) \}$$

By definition, X is complete & with δ-function normalization, i.e.

$$\frac{1}{L}\int_{-L}^{L} dx \mid x \rangle \langle x \mid = I \qquad (1.10a)$$

$$\langle x' \mid x'' \rangle = \delta(x', x'') \qquad (1.10b)$$

so that for any $f \in \mathcal{F}$,

$$\mid f \rangle = \frac{1}{L}\int_{-L}^{L} dx \mid x \rangle \langle x \mid f \rangle \qquad (1.10c)$$

$$= \frac{1}{L}\int_{-L}^{L} dx \mid x \rangle f(x)$$

where

$$f(x) = \langle x \mid f \rangle \qquad \text{with} \qquad f^*(x) = \langle x \mid f \rangle^* = \langle f \mid x \rangle \qquad (1.10d)$$

is the component (projection) of f along (onto) $\mid x \rangle$.

Eq(1.10) can be written as

$$\langle f \mid g \rangle = \langle f \mid I \mid g \rangle = \frac{1}{L}\int_{-L}^{L} dx \langle f \mid x \rangle \langle x \mid g \rangle \qquad (1.10b)$$

The Fourier series eq(1.1) can now be interpreted as representing $f(x)$ in terms of another basis

$$\{ \mid c_n \rangle,\ \mid s_n \rangle\ ;\ n = 0, 1, 2, \ldots \infty \}$$

where

$$\langle x \mid c_n \rangle = \cos\left(\frac{n\pi x}{L}\right) \qquad \langle x \mid s_n \rangle = \sin\left(\frac{n\pi x}{L}\right)$$

Using the completeness relation

$$\sum_{n=0}^{\infty} (\mid c_n \rangle\langle c_n \mid + \mid s_n \rangle\langle s_n \mid) = I \tag{1.10c}$$

eq(1.6) becomes

$$\begin{aligned} f(x) &= \langle x \mid f \rangle \\ &= \sum_{n=0}^{\infty} \{\langle x \mid c_n \rangle\langle c_n \mid f \rangle + \langle x \mid s_n \rangle\langle s_n \mid f \rangle\} \\ &= \sum_{n=0}^{\infty} \left\{\cos\left(\frac{n\pi x}{L}\right)\langle c_n \mid f \rangle + \sin\left(\frac{n\pi x}{L}\right)\langle s_n \mid f \rangle\right\} \end{aligned}$$

$\rightarrow \quad a_n + a_{-n} = \langle c_n \mid f \rangle \qquad b_n - b_{-n} = \langle s_n \mid f \rangle$

$\quad a_0 = \langle c_0 \mid f \rangle$

Eqs(1.11-3) represent the orthonormality of the basis:

$$\langle c_n \mid s_m \rangle = 0$$

$$\langle c_n \mid c_m \rangle = \langle s_n \mid s_m \rangle = \delta_{nm}$$

Eqs(1.7-8) are just projections

$$a_n + a_{-n} = \langle c_n \mid f \rangle$$

$$b_n - b_{-n} = \langle s_n \mid f \rangle$$

$\langle x \mid \text{eq}(1.10\,c) \mid y \rangle$ gives

$$\sum_{n=0}^{\infty} (\langle x \mid c_n \rangle\langle c_n \mid y \rangle + \langle x \mid s_n \rangle\langle s_n \mid y \rangle) = \langle x \mid y \rangle$$

$$\rightarrow \quad \sum_{n=0}^{\infty} \left\{\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi y}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)\right\} = \delta(x-y) \tag{1.14}$$

For a periodic function $f(x)$ with period $2L$, the roots of $x - y = 0$ are at

$x_n = y + 2nL \qquad$ where $\quad n \in$ integers

$$\rightarrow \quad \int_{-\infty}^{\infty} dx\, f(x)\, \delta(x-y) = \sum_{n=-\infty}^{\infty} f(y + 2nL) \tag{1.15}$$

For functions with no periodicity, i.e., $L \to \infty$, we have

$$\delta(x-y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-y)} \tag{1.16}$$

$$\rightarrow \quad f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} F(k)\, e^{ikx} \tag{1.17}$$

$$F(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} f(x)\, e^{-ikx} \tag{1.18}$$

In terms of the bra-ket notation:

$$\int_{-\infty}^{\infty} dk \mid k \rangle\langle k \mid = I \qquad \int_{-\infty}^{\infty} dx \mid x \rangle\langle x \mid = I$$

$$\rightarrow \quad \langle x \mid y \rangle = \int_{-\infty}^{\infty} dk \langle x \mid k \rangle\langle k \mid y \rangle = \delta(x-y)$$

$$\therefore \quad \langle x \mid k \rangle = \frac{e^{ikx}}{\sqrt{2\pi}} \qquad \langle k \mid y \rangle = \frac{e^{-iky}}{\sqrt{2\pi}}$$

$$f(x) = \langle x \mid f \rangle = \int_{-\infty}^{\infty} d\,k \langle x \mid k \rangle \langle k \mid f \rangle$$
$$= \int_{-\infty}^{\infty} d\,k \frac{e^{i\,k\,y}}{\sqrt{2\,\pi}} F(k)$$
$$F(k) = \langle k \mid f \rangle = \int_{-\infty}^{\infty} d\,x \langle k \mid x \rangle \langle x \mid f \rangle$$
$$= \int_{-\infty}^{\infty} d\,x \frac{e^{-i\,k\,x}}{\sqrt{2\,\pi}} f(x)$$

Generalizing to n-D starts with

$$\delta^n(\boldsymbol{x} - \boldsymbol{y}) = \int_{-\infty}^{\infty} \frac{d^n k}{(2\,\pi)^n} e^{i\,\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \qquad (1.19)$$

Let $D(x)$ be a hermitian differential operator, then

$$\int d\,x\,[D(x)\,f(x)]^*\,g(x) = \int d\,x\,f^*(x)\,D(x)\,g(x) \qquad (1.20)$$
$$\langle D f \mid g \rangle = \langle f \mid D g \rangle$$

whereas, for an arbitary operator $O(x)$,

$$\int d\,x\,[O^*(x)\,f(x)]^*\,g(x) = \int d\,x\,f^*(x)\,O(x)\,g(x)$$
$$\left\langle O^+ f \mid g \right\rangle = \langle f \mid O g \rangle$$

Let $D(x)$ be a 2nd order hermitian linear differential operator. The solutions to the eigenvalue problem

$$D(x)\,\varphi_n(x) = \lambda_n\,\varphi_n(x) \qquad (1.21)$$

with appropriate boundary conditions (B.C.), form an orthonormal complete set:

$$\int d\,x\,\varphi_m^*(x)\,\varphi_n(x) = \delta_{m\,n} \qquad (1.22)$$
$$\sum_n \varphi_n(x)\,\varphi_n^*(y) = \delta(x - y) \qquad (1.23)$$

or, with $\langle x \mid n \rangle = \varphi_n(x)$,

$$\langle m \mid n \rangle = \int d\,x \langle m \mid x \rangle \langle x \mid n \rangle = \delta_{m\,n} \qquad (1.22a)$$
$$\sum_n \mid n \rangle \langle n \mid = I \qquad (1.23a)$$

For D with a spectrum containing both continuous & discrete parts, the completeness relation takes the form

$$\sum_n \varphi_n(x)\,\varphi_n^*(y) + \int d\,k\,\varphi_k(x)\,\varphi_k^*(y) = \delta(x - y) \qquad (1.24)$$

or

$$\sum_n \mid n \rangle \langle n \mid + \int d\,k \mid k \rangle \langle k \mid = I \qquad (1.24a)$$