1.4. Matrices

A matrix $\boldsymbol{M}$ is an $m \times n$ array of numbers.

The element at the i^{th} row & j^{th} column of $\boldsymbol{M}$ is denoted by M_{ij}.

An $n \times 1$ matrix is usually called a vector.

Basic matrix operations:

1. Linear combinations:

$$(a\boldsymbol{M} + b\boldsymbol{N})_{ij} = a M_{ij} + b N_{ij} \qquad (1.51)$$

where a & b are numbers.

2. Multiplication:

$$(\boldsymbol{M}\boldsymbol{N})_{ij} = M_{ik} N_{kj} \qquad (1.52)$$

3. Commutator:

$$[\boldsymbol{M}, \boldsymbol{N}] = \boldsymbol{M}\boldsymbol{N} - \boldsymbol{N}\boldsymbol{M} \qquad (1.53)$$

$\boldsymbol{M}$ & $\boldsymbol{N}$ commute if $[\boldsymbol{M}, \boldsymbol{N}] = 0$.

4. Transpose:

$$(\boldsymbol{M}^T)_{ij} = M_{ji} \qquad (1.53a)$$ Hermitian adjoint:

$$(\boldsymbol{M}^+)_{ij} = M^*_{ji} \qquad (1.53b)$$

→ $\left[(\boldsymbol{M}\boldsymbol{N})^T\right]_{ij} = (\boldsymbol{M}\boldsymbol{N})_{ji} = M_{jk} N_{ki}$

$(\boldsymbol{N}^T \boldsymbol{M}^T)_{ij} = (\boldsymbol{N}^T)_{ik} (\boldsymbol{M}^T)_{kj} = N_{ki} M_{jk}$

∴ $(\boldsymbol{M}\boldsymbol{N})^T = \boldsymbol{N}^T \boldsymbol{M}^T$

Taking the complex conjugate on both sides, we have

$$(\boldsymbol{M}\boldsymbol{N})^+ = \boldsymbol{N}^+ \boldsymbol{M}^+$$

The inner product of 2 vectors $\boldsymbol{x}$ & $\boldsymbol{y}$ can be written in matrix form as

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^+ \boldsymbol{y} = x_i^* y_i \qquad (1.53c)$$

$\boldsymbol{x}$ & $\boldsymbol{y}$ are orthogonal if $\boldsymbol{x} \cdot \boldsymbol{y} = 0$.

The Levi-Civita symbol is defined as

$$\varepsilon^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n} = \begin{cases} 1 & \text{if } \alpha_1 \ldots \alpha_n = \text{even permutation of } \beta_1 \ldots \beta_n \\ -1 & \text{if } \alpha_1 \ldots \alpha_n = \text{odd permutation of } \beta_1 \ldots \beta_n \\ 0 & \text{otherwise} \end{cases} \qquad (1.53d)$$

For the special case $\beta_1 \ldots \beta_n = 1 \ldots n$, the notation is simplified to

$$\varepsilon^{\alpha_1 \ldots \alpha_n} = \begin{cases} 1 & \text{if } \alpha_1 \ldots \alpha_n = \text{even permutation of } 1 \ldots n \\ -1 & \text{if } \alpha_1 \ldots \alpha_n = \text{odd permutation of } 1 \ldots n \\ 0 & \text{otherwise} \end{cases} \qquad (1.53e)$$

$$= \varepsilon_{\alpha_1 \ldots \alpha_n}$$

It's easy to check that

$$\varepsilon^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n} = \varepsilon^{\alpha_1 \ldots \alpha_n} \varepsilon_{\beta_1 \ldots \beta_n}$$

$$= \frac{1}{n!} \varepsilon^{\alpha_1 \ldots \alpha_n}_{\gamma_1 \ldots \gamma_n} \varepsilon^{\gamma_1 \ldots \gamma_n}_{\beta_1 \ldots \beta_n} \qquad (1.53f)$$

The determinant of an $n \times n$ matrix $\boldsymbol{M}$ is defined as

$$\det \boldsymbol{M} = \varepsilon^{\alpha_1 \ldots \alpha_n} M_{1\alpha_1} \ldots M_{n\alpha_n} \qquad (1.54)$$

$$= \varepsilon^{\alpha_1 \ldots \alpha_n} M_{\alpha_1 1} \ldots M_{\alpha_n n} \qquad (1.55)$$

$$= (-)^{\beta}\, \varepsilon^{\alpha_1 \ldots \alpha_n}\, M_{\beta_1 \alpha_1} \ldots M_{\beta_n \alpha_n} \qquad \text{where} \qquad (-)^{\beta} = \varepsilon_{\beta_1 \ldots \beta_n}$$

$$= \frac{1}{n!}\, \varepsilon^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n}\, M_{\beta_1 \alpha_1} \ldots M_{\beta_n \alpha_n}$$

M is unimodular if $\det \boldsymbol{M} = 1$.

$$\det(\boldsymbol{M}\,\boldsymbol{N}) = \frac{1}{n!}\, \varepsilon^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n}\, M_{\alpha_1 \gamma_1} N_{\gamma_1 \beta_1} \ldots M_{\alpha_n \gamma_n} N_{\gamma_n \beta_n}$$

$$= \frac{1}{n!}\, \varepsilon^{\alpha_1 \ldots \alpha_n}\, \varepsilon_{\beta_1 \ldots \beta_n}\, M_{\alpha_1 \gamma_1} N_{\gamma_1 \beta_1} \ldots M_{\alpha_n \gamma_n} N_{\gamma_n \beta_n}$$

$$= \frac{1}{n!}\, \varepsilon^{\gamma_1 \ldots \gamma_n}\, \varepsilon_{\gamma_1 \ldots \gamma_n} \det \boldsymbol{M} \det \boldsymbol{N}$$

$$= \det \boldsymbol{M} \det \boldsymbol{N} \qquad (1.55a)$$

The trace of a square ***M*** is defined as

$$\mathrm{Tr}\, \boldsymbol{M} = M_{i\,i} \qquad (1.56)$$

Cyclic property of the trace:

$$\mathrm{Tr}\,(\boldsymbol{M}_1 \boldsymbol{M}_2 \ldots \boldsymbol{M}_n) = \mathrm{Tr}\,(\boldsymbol{M}_{\alpha_1} \boldsymbol{M}_{\alpha_2} \ldots \boldsymbol{M}_{\alpha_n}) \qquad (1.56a)$$

where $\alpha_1\, \alpha_2 \ldots \alpha_n$ is any cyclic permutation of $12 \ldots n$.

The identity matrix ***I*** is a square matrix with elements $I_{i\,j} = \delta_{i\,j}$.

For any square matrix ***M***,

$$(\boldsymbol{I}\,\boldsymbol{M})_{i\,j} = \delta_{i\,k}\, M_{k\,j} = M_{i\,j}$$

$$(\boldsymbol{M}\,\boldsymbol{I})_{i\,j} = M_{i\,k}\, \delta_{k\,j} = M_{i\,j}$$

→ $\boldsymbol{I}\,\boldsymbol{M} = \boldsymbol{M}\,\boldsymbol{I}$

i.e., ***I*** commutes with all square matrix ***M***.

Obviously,

$$\det \boldsymbol{I} = 1 \qquad \mathrm{Tr}\, \boldsymbol{I} = n$$

The left & right inverse of ***M*** are defined by

$$\boldsymbol{M}_L^{-1}\, \boldsymbol{M} = \boldsymbol{I} \qquad \boldsymbol{M}\, \boldsymbol{M}_R^{-1} = \boldsymbol{I}$$

Note that $\boldsymbol{M}_L^{-1} = \boldsymbol{M}_R^{-1} \equiv \boldsymbol{M}^{-1}$ only if ***M*** is a square matrix.
In which case,

$$\det\left(\boldsymbol{M}\,\boldsymbol{M}^{-1}\right) = \det \boldsymbol{M} \det \boldsymbol{M}^{-1} = \det \boldsymbol{I} = 1$$

→ $\det \boldsymbol{M}^{-1} = \dfrac{1}{\det \boldsymbol{M}}$

which means $\boldsymbol{M}^{-1}$ exists iff $\det \boldsymbol{M} \neq 0$.

An eigenvector ***x*** of a square matrix ***M*** satisfies

$$\boldsymbol{M}\,\boldsymbol{x} = \lambda\, \boldsymbol{x}$$

where λ is the associated eigenvalue.

Obviously, ***x*** is determined only up to a constant so that only independent eigenvectors are significant.

The eigenvalues of ***M*** are given by the secular eq.

$$\det(\boldsymbol{M} - \lambda\, \boldsymbol{I}) = 0 \qquad (1.58)$$

For an $n \times n$ matrix, eq(1.58) has exactly n roots so that there are always n eigenvalues, although some of them may share the same values. If the eigenvalues are degenerate, it is possible that there are less than n independent eigenvectors.

A similarity transform of a square matrix ***M*** is defined as

$$\boldsymbol{M}' = \boldsymbol{S}^{-1}\,\boldsymbol{M}\,\boldsymbol{S}$$

where $\boldsymbol{S}$ is any non-singular (i.e., $\det \boldsymbol{S} \neq 0$) matrix.

The condition for $\boldsymbol{M}$ to have a deficit of eigenvectors is that its Jordan canonical form is non-diagonal. [see, e.g., J.H.Wilkinson, “The Algebraic Eigenvalue Problem”, Chap.1.]

For example,

$$\boldsymbol{J} = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$$

has three eigenvalues a , a & b but only 2 independent eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. $\boldsymbol{J}$ itself is already in the Jordan canonical form. Any matrix related to $\boldsymbol{J}$ by a similarity transform share the same Jordan canonical form & hence the same eigenvalues & deficit of eigenvectors.

Special (square) matrices:

1. $\boldsymbol{O}$ is orthogonal if $\boldsymbol{O}^{-1} = \boldsymbol{O}^{T}$.
2. $\boldsymbol{S}$ is symmetric if $\boldsymbol{S}^{T} = \boldsymbol{S}$.
3. $\boldsymbol{A}$ is anti-symmetric if $\boldsymbol{A}^{T} = -\boldsymbol{A}$.
4. $\boldsymbol{U}$ is unitary if $\boldsymbol{U}^{-1} = \boldsymbol{U}^{+}$.
5. $\boldsymbol{H}$ is hermitian if $\boldsymbol{H}^{+} = \boldsymbol{H}$.

It is easy to show that a similarity transform preserves the features of a special matrix & leave both $\det \boldsymbol{M}$ & $\operatorname{Tr} \boldsymbol{M}$ invariant.

A special case of the similarity transform is the unitary transform defined for a square matrix $\boldsymbol{M}$ as

$$\boldsymbol{M}' = \boldsymbol{U}^{+}\,\boldsymbol{M}\,\boldsymbol{U} \tag{1.59}$$

Let $\boldsymbol{H}$ be hermitian & consider

$\boldsymbol{H}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$ (a)

→ $\boldsymbol{x}^{+}\,\boldsymbol{H} = \lambda^{*}\,\boldsymbol{x}^{+}$

$\boldsymbol{x}^{+}\,\boldsymbol{H}\,\boldsymbol{x} = \lambda^{*}\,\boldsymbol{x}^{+}\,\boldsymbol{x}$

Eq(a) → $\boldsymbol{x}^{+}\,\boldsymbol{H}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}^{+}\,\boldsymbol{x}$

∴ $\lambda = \lambda^{*}$

i.e., all eigenvalues of $\boldsymbol{H}$ are real.

Let $\boldsymbol{H}\,\boldsymbol{y} = \mu\,\boldsymbol{y}$

→ $\boldsymbol{x}^{+}\,\boldsymbol{H}\,\boldsymbol{y} = \mu\,\boldsymbol{x}^{+}\,\boldsymbol{y}$

Eq(a) → $\boldsymbol{y}^{+}\,\boldsymbol{H}\,\boldsymbol{x} = \lambda\,\boldsymbol{y}^{+}\,\boldsymbol{x}$

$\boldsymbol{x}^{+}\,\boldsymbol{H}\,\boldsymbol{y} = \lambda\,\boldsymbol{x}^{+}\,\boldsymbol{y}$

∴ $\mu\,\boldsymbol{x}^{+}\,\boldsymbol{y} = \lambda\,\boldsymbol{x}^{+}\,\boldsymbol{y}$

→ Either $\mu = \lambda$

or $\boldsymbol{x}^{+}\,\boldsymbol{y} = 0$ if $\mu \neq \lambda$

i.e., eigenvectors belonging to different eigenvalues of $\boldsymbol{H}$ are orthogonal.

Since hermiticity is preserved by similarity transform, the Jordan canonical form of $\boldsymbol{H}$ must be diagonal & there is no deficit in eigenvectors. We can therefore always construct a set of n orthonormalized eigenvectors for $\boldsymbol{H}$.

In fact, the matrix $\boldsymbol{U}$ whose columns are the orthonormalized eigenvectors of $\boldsymbol{H}$ is unitary.

Let $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ be the orthonormalized eigenvectors of $\boldsymbol{H}$ with corresponding eigenvalues $(\lambda_1, \ldots, \lambda_n)$. Consider the matrix $\boldsymbol{U}$ with these eigenvectors as columns.

$$\boldsymbol{U} = (\boldsymbol{x}_1, \ ..., \ \boldsymbol{x}_n) \qquad \to \qquad \boldsymbol{U}^+ = \begin{pmatrix} \boldsymbol{x}_1^+ \\ \vdots \\ \boldsymbol{x}_n^+ \end{pmatrix}$$

$$\boldsymbol{U}^+ \boldsymbol{U} = \begin{pmatrix} \boldsymbol{x}_1^+ \\ \vdots \\ \boldsymbol{x}_n^+ \end{pmatrix} (\boldsymbol{x}_1, \ ..., \ \boldsymbol{x}_n) = \boldsymbol{I} \qquad (\ \boldsymbol{U} \text{ is unitary. })$$

$$\boldsymbol{H}\,\boldsymbol{U} = (\ \lambda_1 \boldsymbol{x}_1, \ ..., \ \lambda_n \boldsymbol{x}_n)$$

$$\boldsymbol{U}^+ \boldsymbol{H}\,\boldsymbol{U} = \begin{pmatrix} \boldsymbol{x}_1^+ \\ \vdots \\ \boldsymbol{x}_n^+ \end{pmatrix} (\ \lambda_1 \boldsymbol{x}_1, \ ..., \ \lambda_n \boldsymbol{x}_n) = \text{diag}(\lambda_1, \ ..., \ \lambda_n\ ) \qquad (1.60)$$

i.e., $\boldsymbol{H}$ can always be diagonalized by a unitary transform.

$$\det(\boldsymbol{U}^+ \boldsymbol{H}\,\boldsymbol{U}) = \det \boldsymbol{U}^+ \det \boldsymbol{H} \det \boldsymbol{U} \ = \det \boldsymbol{H}$$

$$= \det[\text{diag}(\lambda_1, \ ..., \ \lambda_n\ )] \ = \prod_{i=1}^{n} \lambda_i \qquad (1.61)$$

$$\text{Tr}(\boldsymbol{U}^+ \boldsymbol{H}\,\boldsymbol{U}) = \text{Tr}\,(\boldsymbol{U}\,\boldsymbol{U}^+ \boldsymbol{H}) = \text{Tr}\,\boldsymbol{H}$$

$$= \text{Tr}\,[\text{diag}(\lambda_1, \ ..., \ \lambda_n\ )] \ = \sum_{i=1}^{n} \lambda_i \qquad (1.62)$$

$$\det \boldsymbol{H} = \exp\left[\ \ln\left(\prod_{i=1}^{n} \lambda_i\right)\right] \ = \exp\left(\sum_{i=1}^{n} \ln \lambda_i\right) = \exp\,(\text{Tr} \ln \boldsymbol{H}) \qquad (1.65)$$

More generally, these eqs still hold if we replace the unitary with the similarity transform & $\boldsymbol{H}$ with an arbitrary matrix.

Functions of matrices can be defined via the Taylor series. For example

$$\exp \boldsymbol{M} = \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{M}^n \qquad (1.63)$$

$$\ln \boldsymbol{M} = \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} (\ \boldsymbol{M} - \boldsymbol{I}\ )^n \qquad (1.64)$$

The series converges only if every element of $\boldsymbol{M}$ converges.

Baker-Campbell-Hausdorff theorem:

Let

$$\boldsymbol{C} = [\ \boldsymbol{A}, \ \boldsymbol{B}\ ]$$

$$\& \qquad [\ \boldsymbol{A}, \ \boldsymbol{C}\ ] = [\ \boldsymbol{B}, \ \boldsymbol{C}\ ] = 0 \qquad \text{(b)}$$

$$\to \qquad e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}} e^{\boldsymbol{B}} e^{-\boldsymbol{C}/2} \qquad (1.66)$$

Proof:

Using

$$[\ \boldsymbol{A}\,\boldsymbol{B}, \ \boldsymbol{D}\ ] = \boldsymbol{A}\,\boldsymbol{B}\,\boldsymbol{D} - \boldsymbol{D}\,\boldsymbol{A}\,\boldsymbol{B}$$

$$= \boldsymbol{A}\,\boldsymbol{B}\,\boldsymbol{D} - \boldsymbol{A}\,\boldsymbol{D}\,\boldsymbol{B} + \boldsymbol{A}\,\boldsymbol{D}\,\boldsymbol{B} - \boldsymbol{D}\,\boldsymbol{A}\,\boldsymbol{B}$$

$$= \boldsymbol{A}\,[\ \boldsymbol{B}, \ \boldsymbol{D}\ ] + [\ \boldsymbol{A}\ , \ \boldsymbol{D}\ ]\,\boldsymbol{B}$$

we have

$$[\ \boldsymbol{A}^n, \ \boldsymbol{C}\ ] = [\ \boldsymbol{A}\,\boldsymbol{A}^{n-1}, \ \boldsymbol{C}\ ] = \boldsymbol{A}\,[\ \boldsymbol{A}^{n-1}, \ \boldsymbol{C}\ ] \ = \boldsymbol{A}^2\,[\ \boldsymbol{A}^{n-2}, \ \boldsymbol{C}\ ]$$

$$\vdots$$

$$= \boldsymbol{A}^{n-1}\,[\ \boldsymbol{A}, \ \boldsymbol{C}\ ] = 0 \qquad \text{(c)}$$

$$\& \qquad [\ \boldsymbol{A}^n, \ \boldsymbol{B}\ ] = [\ \boldsymbol{A}\,\boldsymbol{A}^{n-1}, \ \boldsymbol{B}\ ] \ = \boldsymbol{A}\,[\boldsymbol{A}^{n-1}, \ \boldsymbol{B}\ ] + [\boldsymbol{A}, \ \boldsymbol{B}\ ]\,\boldsymbol{A}^{n-1}$$

$$= \boldsymbol{A}\left(\boldsymbol{A}\,[\boldsymbol{A}^{n-2}, \ \boldsymbol{B}\ ] + [\boldsymbol{A}, \ \boldsymbol{B}\ ]\,\boldsymbol{A}^{n-2}\right) + [\boldsymbol{A}, \ \boldsymbol{B}\ ]\,\boldsymbol{A}^{n-1}$$

$$= A^2 \left[A^{n-2}, B\right] + A\,[A, B]\,A^{n-2} + [A, B]\,A^{n-1}$$

$$= A^2 \left[A^{n-2}, B\right] + 2\,[A, B]\,A^{n-1} \qquad \text{[Eq(b) used.]}$$

$$= A^3 \left[A^{n-3}, B\right] + 3\,[A, B]\,A^{n-1}$$

$$\vdots$$

$$= A^{n-1}\,[A, B] + (n-1)\,[A, B]\,A^{n-1}$$

$$= n\,[A, B]\,A^{n-1} \qquad \text{[Eq(c) used.]}$$

$$= n\,C\,A^{n-1} \qquad (1.67)$$

$$e^{-\alpha A}\,B\,e^{\alpha A} = \sum_{m,n=0}^{\infty} \frac{(-)^m\,\alpha^{m+n}}{m!\,n!}\,A^m\,B\,A^n$$

$$= \sum_{m,n=0}^{\infty} \frac{(-)^m\,\alpha^{m+n}}{m!\,n!}\left(B\,A^{m+n} + m\,[A, B]\,A^{m+n-1}\right) \qquad \text{[Eq(1.67) used.]}$$

$$= B \sum_{m,n=0}^{\infty} \frac{(-\alpha\,A)^m\,(\alpha\,A)^n}{m!\,n!} - \alpha\,[A, B] \sum_{m=1,\,n=0}^{\infty} \frac{(-\alpha\,A)^{m-1}\,(\alpha\,A)^n}{(m-1)!\,n!}\Bigg)$$

$$= B\,e^{-\alpha A}\,e^{\alpha A} - \alpha\,[A, B]\,e^{-\alpha A}\,e^{\alpha A}$$

$$= B - \alpha\,[A, B] \qquad (1.68)$$

Note that for $\alpha = 0$, eq(1.68) becomes the identity $B = B$.

Let

$$F(\alpha) = e^{-\alpha B}\,e^{-\alpha A}\,e^{A+B} \qquad (1.69)$$

then

$$\frac{d\,F(\alpha)}{d\,\alpha} = -B\,F(\alpha) - e^{-\alpha B}\,A\,e^{-\alpha A}\,e^{A+B}$$

$$= -B\,F(\alpha) - e^{-\alpha B}\,A\,e^{\alpha B}\,F(\alpha)$$

$$= -B\,F(\alpha) - (\,A - \alpha\,[\,B, A\,]\,)\,F(\alpha) \qquad \text{[Eq(1.68) used.]}$$

$$= -(\,A + B + \alpha\,[\,A, B\,]\,)\,F(\alpha) \qquad (1.70)$$

$$\rightarrow \qquad F(\alpha) = \beta \exp\left\{-(\,A + B\,)\,\alpha - \frac{1}{2}\,\alpha^2\,[\,A, B\,]\right\}$$

Since

$$F(0) = e^{A+B}$$

we have

$$F(\alpha) = \exp\left\{-(\,A + B\,)\,\alpha - \frac{1}{2}\,\alpha^2\,[\,A, B\,]\right\} e^{A+B} \qquad (1.71)$$

$$\therefore \qquad F(1) = e^{-B}\,e^{-A}\,e^{A+B} = \exp\left\{-(\,A + B\,) - \frac{1}{2}\,[\,A, B\,]\right\} e^{A+B}$$

$$\rightarrow \qquad e^{-B}\,e^{-A} = \exp\left\{-(\,A + B\,) - \frac{1}{2}\,[\,A, B\,]\right\} \qquad (1.71a)$$

Using

$$[\,A + B,\ C\,] = 0$$

eq(1.71a) becomes [see note below]

$$e^{-B}\,e^{-A} = \exp\left\{-\frac{1}{2}\,[\,A, B\,]\right\} e^{-(A+B)}$$

$$e^{-B}\,e^{-A}\,e^{(A+B)} = \exp\left(-\frac{1}{2}\,C\right) \qquad \text{QED.}$$

Note: We've used the result that if $[A, B] = 0$, then

$$e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}}\, e^{\boldsymbol{B}} = e^{\boldsymbol{B}}\, e^{\boldsymbol{A}}$$

which can be proved by Taylor expansion or using the BCH theorem itself.

The inner product of 2 functions f & g of 1 variable is defined as

$$\begin{aligned} f \cdot g &= \langle f \mid g \rangle \\ &= \int_{-L}^{L} dx \langle f \mid x \rangle \langle x \mid g \rangle \qquad \int_{-L}^{L} dx \mid x \rangle \langle x \mid = I \\ &= \int_{-L}^{L} dx\, f^*(x)\, g(x) \\ &= \lim_{N\to\infty} \sum_{j=1}^{N} \left(\frac{2L}{N}\right) f^*(x_j)\, g(x_j) \qquad x_j = -L + \frac{2L}{N} j \quad (1.73) \end{aligned}$$

which resembles the inner product of 2 vectors.

For functions of 2 variables, the inner product results in another function:

$$h(x, z) = \int_{-L}^{L} dy\, f(x, y)\, g(y, z) \qquad (1.74)$$

$$= \lim_{N\to\infty} \sum_{j=1}^{N} \left(\frac{2L}{N}\right) f(x, y_j)\, g(y_j, z) \qquad (1.75)$$

which resembles the product of 2 matrices.

Eqs(1.73-5) give the guidelines for studying the properties of functionals as analog of matrix analysis.

Thus, $f(x, y)$ is a matrix of continuous indices. $\delta(x-y)$ stands for the identity matrix so that

$$\int_{-L}^{L} dy\, f(x, y)\, \delta(y-z) = f(x, z) \qquad (1.76)$$

The trace of f^n is

$$\operatorname{Tr} f^n = \int_{-L}^{L} dx_1 \ldots dx_n\; f(x_1, x_2) \ldots f(x_n, x_1) \qquad (1.77)$$

Let $\{\varphi_n(x)\}$ be an orthonormal basis , i.e.,

$$\langle \varphi_m \mid \varphi_n \rangle = \int dx\, \varphi_m^*(x)\, \varphi_n(x) = \delta_{mn} \qquad (1.79)$$

$$\sum_n \mid \varphi_n \rangle \langle \varphi_n \mid = I \qquad \to \qquad \sum_n \varphi_n(x)\, \varphi_n^*(y) = \delta(x-y) \qquad (1.79a)$$

A matrix is an operator so that in general we have

$$f = \sum_{nm} \mid \varphi_n \rangle f_{nm} \langle \varphi_m \mid$$

For the special case

$$f = \sum_n \mid \varphi_n \rangle f_n \langle \varphi_n \mid \qquad \text{where} \quad f_n = f_n^* \in R$$

we have

$$\begin{aligned} f(x, y) &= \sum_n \langle x \mid \varphi_n \rangle f_n \langle \varphi_n \mid y \rangle \\ &= \sum_n f_n\, \varphi_n(x)\, \varphi_n^*(y) \qquad (1.78) \end{aligned}$$

Since

$$f^*(x, y) = \sum_n f_n\, \varphi_n^*(x)\, \varphi_n(y) = f(y, x)$$

f is analogous to a hermitian matrix.

Using eq(1.79a), we have

$$f - I = \sum_n |\ \varphi_n \rangle (f_n - 1) \langle \varphi_n \ |$$

$$f(x, y) - \delta(x - y) = \sum_n (f_n - 1)\ \varphi_n(x)\ \varphi_n^*(y) \qquad (1.80)$$

Using eqs(1.64-5), we have

$$\det f = \exp(\operatorname{Tr} \ln f) = \exp\left(\sum_{m=1}^{\infty} \frac{(-)^{m+1}}{m} \operatorname{Tr} (f - I)^m \right) \qquad (1.81)$$

Using eqs(1.77 & 1.80) we have

$$\operatorname{Tr}(f - I) = \int d\,x\,[f(x, x) - \delta(x - x)] = \sum_n (f_n - 1) \int d\,x\ \varphi_n(x)\ \varphi_n^*(x) = \sum_n (f_n - 1)$$

$$\begin{aligned}\operatorname{Tr} (f - I)^2 &= \int d\,x_1\, d\,x_2\, [f(x_1, x_2) - \delta(x_1 - x_2)][f(x_2, x_1) - \delta(x_2 - x_1)] \\ &= \sum_{n,m} (f_n - 1) (f_m - 1) \int d\,x_1\, d\,x_2\ \varphi_n(x_1)\ \varphi_n^*(x_2)\ \varphi_m(x_2)\ \varphi_m^*(x_1) \\ &= \sum_{n,m} (f_n - 1) (f_m - 1)\ \delta_{n\,m} \int d\,x_1\ \varphi_n(x_1)\ \varphi_m^*(x_1) \\ &= \sum_n (f_n - 1)^2 \int d\,x_1\ \varphi_n(x_1)\ \varphi_n^*(x_1) \\ &= \sum_n (f_n - 1)^2\end{aligned}$$

& by deduction,

$$\operatorname{Tr} (f - I)^m = \sum_n (f_n - 1)^m \qquad (1.82)$$

Eq(1.81) thus becomes

$$\det f = \exp\left(\sum_{m=1}^{\infty} \frac{(-)^{m+1}}{m} \sum_n (f_n - 1)^m \right) = \exp\left(\sum_n \ln f_n \right) = \prod_n f_n \qquad (1.83)$$

Consider the action of a hermitian differential operator $D(x)$ on a function f. In matrix form, we have

$$D(x)\, f(x) = \int d\,y\ D(x)\ \delta(x - y)\, f(y) = \int d\,y\ D(x, y)\, f(y) \qquad (1.84)$$

which describes the product of the matrix D on the vector f, with the matrix elements of D given by

$$D(x, y) = D(x)\ \delta(x - y)$$

Let $\{ \varphi_n \}$ be complete set of the orthonormal eigenstates of $D(x)$ with

$$D(x)\ \varphi_n(x) = \lambda_n\ \varphi_n(x) \qquad \lambda_n \in R$$

then

$$D(x, y) = D(x) \sum_n \varphi_n(x)\, \varphi_n^*(y)$$
$$= \sum_n \lambda_n\, \varphi_n(x)\, \varphi_n^*(y) \qquad (1.85)$$

Since eq(1.85) is in the form of eq(1.78). Hence, the matrix $D(x, y)$ is also hermitian.

Eq(1.83) then gives [c.f. eq(1.61)]

$$\det D = \prod_n \lambda_n \qquad (1.86)$$