1.5. Gaussian Integrals

The basic Gaussian integral is

$$I(\alpha) = \int_{-\infty}^{\infty} d\,x\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \qquad \alpha > 0 \text{ is real} \qquad (1.87)$$

Let

$$z = x + i\,y$$

Similarly

$$f(z) = u(x, y) + i\,v(x, y) \qquad u, v \in R$$

f is analytic if it satisfies the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (1.89)$$

Goursat's theorem:
If f is analytic inside the contour C, then

$$\oint_C d\,z\, f(z) = 0 \qquad (1.91)$$

Using Goursat's theorem, one can prove Cauchy's theorem:
If f is analytic inside the contour C, then

$$\oint_C d\,z\, \frac{f(z)}{z - z_0} = \begin{cases} 2\pi\, i\, f(z_0) & \text{if } z_0 \text{ is inside } C \\ 0 & \text{if } z_0 \text{ is outside } C \end{cases} \qquad (1.90)$$

Assuming z_0 is inside C & f is analytic inside C,

$$\oint_C d\,z\, \frac{f(z)}{(z - z_0)^{n+1}} = \sum_{m=0}^{\infty} \frac{f^{(m)}(z_0)}{m!} \oint_C d\,z\,(z - z_0)^{m-n-1}$$

Since $(z - z_0)^{m-n-1}$ is analytic for $z \neq z_0$, we can deform C into a circle so that

$$z - z_0 = r\, e^{i\,\theta} \qquad \forall\, z \in C$$

$$\rightarrow \qquad \oint_C d\,z\,(z - z_0)^{m-n-1} = r^{m-n} \int_0^{2\pi} d\,\theta\, e^{i\,(m-n-1)\,\theta}$$

$$= r^{m-n}\, i \int_0^{2\pi} d\,\theta\, e^{i\,(m-n)\,\theta}$$

$$= 2\pi\, i\, \delta_{m\,n} \qquad (1.92a)$$

$$\therefore \qquad \oint_C d\,z\, \frac{f(z)}{(z - z_0)^{n+1}} = 2\pi\, i\, \frac{f^{(n)}(z_0)}{n!} \qquad (1.92)$$

A non-analytic function f can be expanded as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad (1.93)$$

For any C that encloses z_0,

$$\oint_C d\,z\, f(z) = \sum_{n=-\infty}^{\infty} a_n \oint_C d\,z\,(z - z_0)^n$$

$$= \sum_{n=-\infty}^{\infty} a_n\, 2\pi\, i\, \delta_{n,-1} \qquad \text{[Eq(1.92a) used.]}$$

$$= 2\pi\, i\, a_{-1} \qquad (1.94)$$

a_{-1} is called the residue of f & eq(1.94) the residue theorem.

The Gaussian integral eq(1.87) can be evaluated for α imaginary by means of analytic continuation. Thus, to evaluate

$$\mathcal{I} = \int_0^\infty d\,x\, e^{-i\,\alpha\,x^2} \qquad \text{where } \alpha > 0 \text{ is real}$$

we consider

$$\mathcal{J} = \oint_C d\,z\, e^{-i\,\alpha\,z^2}$$

Noting that $e^{-i\,\alpha\,z^2}$ is analytic everywhere and $I = \mathcal{I}$ if C is the x-axis, we choose C as a triangle such that

1st leg: $z = x$ with $x = 0 \to \infty$
2nd leg: $z = \lim_{x\to\infty} (x + i\,y)$ with $y = 0 \to -\infty$
3rd leg: $z = r(1 - i)$ with $r = \infty \to 0$

Hence

$$\mathcal{J} = 0 = \mathcal{I} + i \lim_{x\to\infty} \int_0^{-\infty} d\,y\, e^{-i\,\alpha\,(x^2 - y^2) + 2\,\alpha\,x\,y} + (1 - i) \int_\infty^0 d\,r\, e^{-2\,\alpha\,r^2}$$

$$\rightarrow \qquad \int_0^\infty d\,x\, e^{-i\,\alpha\,x^2} = (1 - i) \int_0^\infty d\,r\, e^{-2\,\alpha\,r^2}$$

$$= (1 - i)\,\frac{1}{2}\sqrt{\frac{\pi}{2\,\alpha}} \qquad \text{[Eq(1.87) used.]}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{i\,\alpha}} \qquad (1.95)$$

where we've used

$$\frac{1}{2}(1 - i)^2 = -i \qquad \rightarrow \qquad 1 - i = \sqrt{-2\,i}$$

Since $e^{-i\,\alpha\,x^2}$ is even in x, we have

$$I(i\,\alpha) = \int_{-\infty}^\infty d\,x\, e^{-i\,\alpha\,x^2} = 2 \int_0^\infty d\,x\, e^{-i\,\alpha\,x^2} = \sqrt{\frac{\pi}{i\,\alpha}} \qquad (1.96)$$

Let

$$I(\boldsymbol{A}) = \int_{-\infty}^\infty d\,x_1 \ldots d\,x_n \exp\left(-x_i\, a_{i\,j}\, x_j\right) \qquad (1.97)$$

$$= \int_{-\infty}^\infty d\,\boldsymbol{x} \exp\left(-\boldsymbol{x}^T \boldsymbol{A}\,\boldsymbol{x}\right)$$

where $\boldsymbol{A}$ is real & symmetric.

By eq(1.60), $\boldsymbol{A}$ can be diagonalized by an orthogonal transform so that

$$\boldsymbol{O}^T \boldsymbol{A}\,\boldsymbol{O} = \text{diag}(\lambda_1 \ldots \lambda_n) = \boldsymbol{\lambda} \qquad \boldsymbol{O}^{-1} = \boldsymbol{O}^T \qquad (1.97a)$$

$$\boldsymbol{O}^T \boldsymbol{O} = \boldsymbol{I} \qquad \rightarrow \qquad \det \boldsymbol{O}^T \det \boldsymbol{O} = (\det \boldsymbol{O})^2 = 1$$

Let

$$\boldsymbol{y} = \boldsymbol{O}^T \boldsymbol{x} \qquad \rightarrow \qquad \boldsymbol{x} = \boldsymbol{O}\,\boldsymbol{y} \qquad (1.99)$$

$$x_i = O_{i\,j}\, y_j \qquad \rightarrow \qquad d\,x_i = O_{i\,j}\, d\,y_j$$

The jacobian of the transformation is

$$J = \det\left(\frac{\partial\, x_i}{\partial\, y_j}\right) = \det \boldsymbol{O} = \pm 1 \qquad (1.101)$$

→ $$\int d\,x_1 \ldots d\,x_n = \int \mid J \mid \; d\,y_1 \ldots d\,y_n \; = \int d\,y_1 \ldots d\,y_n \qquad (1.100)$$

or $$\int d\,\boldsymbol{x} = \int \mid \det \boldsymbol{O} \mid \; d\,\boldsymbol{y} \; = \int d\,\boldsymbol{y}$$

$$\boldsymbol{x}^T \boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{y}^T \boldsymbol{O}^T \boldsymbol{A}\,\boldsymbol{O}\,\boldsymbol{y} \; = \boldsymbol{y}^T \boldsymbol{\lambda}\,\boldsymbol{y} \; = \lambda_i\,y_i^2 \qquad (1.98a)$$

$$I(\boldsymbol{A}) = \int_{-\infty}^{\infty} d\,\boldsymbol{y}\exp\left(-\boldsymbol{y}^T \boldsymbol{\lambda}\,\boldsymbol{y}\right)$$

$$= \int_{-\infty}^{\infty} d\,y_1 \ldots d\,y_n \exp\left(-\lambda_i\,y_i^2\right)$$

$$= \prod_{i=1}^{n} \left(\frac{\pi}{\lambda_i}\right)^{1/2} \qquad \text{[Eq(1.87) used.]} \qquad (1.103)$$

$$= \sqrt{\frac{\pi^n}{\det \boldsymbol{A}}} \qquad \text{[Eq(1.86) used.]}$$

$$= \sqrt{\pi^n \det \boldsymbol{A}^{-1}}$$

More generally, and in particular for analytic continuation purposes, we set

$$I(\alpha\,\boldsymbol{A}) = \int_{-\infty}^{\infty} d\,\boldsymbol{x}\exp\left(-\alpha\,\boldsymbol{x}^T \boldsymbol{A}\,\boldsymbol{x}\right)$$

$$= \int_{-\infty}^{\infty} d\,x_1 \ldots d\,x_n \exp\left(-\alpha\,x_i\,a_{i\,j}\,x_j\right) \qquad (1.104)$$

$$= \prod_{i=1}^{n} \left(\frac{\pi}{\alpha\,\lambda_i}\right)^{1/2} = \sqrt{\left(\frac{\pi}{\alpha}\right)^n \det \boldsymbol{A}^{-1}} \qquad (1.105a)$$

Setting $\alpha = i$, we have

$$I(i\,\boldsymbol{A}) = \int_{-\infty}^{\infty} d\,\boldsymbol{x}\exp\left(-i\,\boldsymbol{x}^T \boldsymbol{A}\,\boldsymbol{x}\right)$$

$$= \int_{-\infty}^{\infty} d\,x_1 \ldots d\,x_n \exp\left(-i\,x_i\,a_{i\,j}\,x_j\right)$$

$$= \sqrt{(-i\,\pi)^n \det \boldsymbol{A}^{-1}} \qquad (1.105)$$

Let

$$I(\alpha,\,\beta) = \int_{-\infty}^{\infty} d\,x\,e^{-\alpha\,x^2 \pm \beta\,x}$$

$$= \exp\left(\frac{\beta^2}{4\,\alpha}\right) \int_{-\infty}^{\infty} d\,x \exp\left[-\alpha\left(x \pm \frac{\beta}{2\,\alpha}\right)^2\right]$$

$$= \sqrt{\frac{\pi}{\alpha}}\;\exp\left(\frac{\beta^2}{4\,\alpha}\right) \qquad \text{[Eq(1.87) used.]} \qquad (1.106)$$

$$I(i\,\alpha,\,i\,\beta) = \int_{-\infty}^{\infty} d\,x\,e^{-i\,\alpha\,x^2 \pm i\,\beta\,x}$$

$$= \sqrt{\frac{\pi}{i\,\alpha}}\;\exp\left(i\,\frac{\beta^2}{4\,\alpha}\right) \qquad (1.107)$$

Finally, let

$$I\,(\boldsymbol{A},\,\boldsymbol{b}) = \int_{-\infty}^{\infty} d\,x_1 \ldots d\,x_n \exp\left(-x_i\,a_{i\,j}\,x_j \pm b_i\,x_i\right)$$

$$= \int_{-\infty}^{\infty} d\,\boldsymbol{x} \exp\left(-\boldsymbol{x}^T \boldsymbol{A}\, \boldsymbol{x} \pm \boldsymbol{b}^T \boldsymbol{x}\right) \qquad (1.110a)$$

where

$$\boldsymbol{A} = \boldsymbol{A}^T \rightarrow \qquad \boldsymbol{A}^{-1} = \left(\boldsymbol{A}^T\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^T$$

Let

$$\boldsymbol{y} = \boldsymbol{x} \mp \frac{1}{2} \boldsymbol{A}^{-1} \boldsymbol{b} \qquad \rightarrow \qquad \boldsymbol{y}^T = \boldsymbol{x}^T \mp \frac{1}{2} \boldsymbol{b}^T \boldsymbol{A}^{-1}$$

$$\therefore \qquad -\boldsymbol{y}^T \boldsymbol{A}\, \boldsymbol{y} = -\boldsymbol{x}^T \boldsymbol{A}\, \boldsymbol{x} \pm \boldsymbol{b}^T \boldsymbol{x} - \frac{1}{4} \boldsymbol{b}^T \boldsymbol{A}^{-1} \boldsymbol{b}$$

$$\rightarrow \qquad I(\boldsymbol{A}, \boldsymbol{b}) = \exp\left(\frac{1}{4} \boldsymbol{b}^T \boldsymbol{A}^{-1} \boldsymbol{b}\right) \int_{-\infty}^{\infty} d\,\boldsymbol{x} \exp\left(-\boldsymbol{y}^T \boldsymbol{A}\, \boldsymbol{y}\right)$$

$$= \exp\left(\frac{1}{4} \boldsymbol{b}^T \boldsymbol{A}^{-1} \boldsymbol{b}\right) \sqrt{\pi^n \det \boldsymbol{A}^{-1}} \qquad (1.110)$$

Similarly, let

$$z_i = x_i + i\, y_i \qquad i = 1, \ldots, n \qquad \text{or} \qquad \boldsymbol{z} = \boldsymbol{x} + i\, \boldsymbol{y}$$

then

$$\mathcal{I}(\boldsymbol{A}, \boldsymbol{b}) = \int_{-\infty}^{\infty} d\,\boldsymbol{z} \exp\left(-\boldsymbol{z}^+ \boldsymbol{A}\, \boldsymbol{z} + \boldsymbol{b}^+ \boldsymbol{z} + \boldsymbol{z}^+ \boldsymbol{b}\right)$$

$$= \exp\left(\boldsymbol{b}^T \boldsymbol{A}^{-1} \boldsymbol{b}\right) \sqrt{(2\pi)^n \det \boldsymbol{A}^{-1}} \qquad (1.111)$$

where $\boldsymbol{A}^+ = \boldsymbol{A}$.

Note that the addition of the linear terms can be evaluted by the saddle point method:

$$\int_{-\infty}^{\infty} d\,\boldsymbol{x}\, e^{F(\boldsymbol{x})} = c\, e^{F(\boldsymbol{x}_c)} \qquad (1.112)$$

where

$$\left.\frac{\partial F(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x} = \boldsymbol{x}_c} = 0 \qquad (1.113)$$

For example, eq(1.106) gives

$$F(x) = -\alpha\, x^2 \pm \beta\, x$$

$$\frac{\partial F}{\partial x} = -2\,\alpha\, x \pm \beta = 0 \qquad \rightarrow \qquad x_c = \pm \frac{\beta}{2\,\alpha}$$

$$\rightarrow \qquad F(x_c) = -\alpha \frac{\beta^2}{4\,\alpha^2} + \beta \frac{\beta}{2\,\alpha} = \frac{\beta^2}{4\,\alpha}$$

$$\therefore \qquad \int_{-\infty}^{\infty} d\,x\, e^{-\alpha x^2 \pm \beta x} = c \exp\left(\frac{\beta^2}{4\,\alpha}\right)$$

Comparing with the actual value gives

$$c = \int_{-\infty}^{\infty} d\,x\, e^{-\alpha x^2}$$

$$\rightarrow \qquad I(\alpha, \beta) = I(\alpha)\, e^{F(\boldsymbol{x}_c)} \qquad (1.113a)$$

For eq(1.110a),

$$F(\boldsymbol{x}) = -\boldsymbol{x}^T \boldsymbol{A}\, \boldsymbol{x} \pm \boldsymbol{b}^T \boldsymbol{x} = -x_i A_{ij} x_j \pm b_i x_i \qquad \boldsymbol{A}^T = \boldsymbol{A}$$

$$\frac{\partial F(\boldsymbol{x})}{\partial x_k} = -A_{kj} x_j - x_i A_{ik} \pm b_k = -2 A_{kj} x_j \pm b_k = 0$$

$$\rightarrow \qquad \boldsymbol{x}_c = \pm \frac{1}{2} \boldsymbol{A}^{-1} \boldsymbol{b}$$

$$F(\boldsymbol{x}_c) = -\frac{1}{4}\,\boldsymbol{b}^T\,\boldsymbol{A}^{-1}\,\boldsymbol{b} + \frac{1}{2}\,\boldsymbol{b}^T\,\boldsymbol{A}^{-1}\,\boldsymbol{b} \;= \frac{1}{4}\,\boldsymbol{b}^T\,\boldsymbol{A}^{-1}\,\boldsymbol{b}$$

$$\therefore \qquad I\,(\boldsymbol{A},\,\boldsymbol{b}) = I(\boldsymbol{A})\,e^{F(\boldsymbol{x}_c)} = I(\boldsymbol{A})\exp\left(\frac{\mathbf{1}}{4}\,\boldsymbol{b}^T\,\boldsymbol{A}^{-1}\,\boldsymbol{b}\right) \qquad (1.113b)$$

in agreement with eq(1.110).