2.1. Quantum Mechanics

In quantum mechanics, the set of all possible quantum states of the system is a Hilbert space $\mathcal{H}$.
The classical Cartesian coordinates q_i & momenta p_i become operators Q_i & P_i on $\mathcal{H}$, respectively.
Quantization is achieved by the commutation relations [see also eq(2.9)]

$$[Q_j, P_k] \equiv Q_j P_k - P_k Q_j = i \hbar \delta_{jk} \qquad (2.1)$$

Note:

1. For an N-particle system in d-D space, $i = 1, \ldots, d N$.
2. Quantization in non-Cartesian coordinates is a complicated problem that should be addressed in terms of differential geometry.

A Hilbert space is roughly a vector space with an inner product.
Vectors in $\mathcal{H}$ are called states. In the Dirac bra-ket notation, a state ϕ is denoted as a ket $|\phi\rangle$.
The inner product of states ϕ & ψ is denoted as

$$\langle \phi | \psi \rangle = \langle \psi | \phi \rangle^* \qquad (2.3)$$

The bra $\langle \psi | = | \psi \rangle^+$ is a member of the dual space of $\mathcal{H}$.

The eigenstates of $\boldsymbol{Q}$ & $\boldsymbol{P}$ as given by

$$\boldsymbol{Q} | \boldsymbol{q} \rangle = \boldsymbol{q} | \boldsymbol{q} \rangle \qquad \boldsymbol{P} | \boldsymbol{p} \rangle = \boldsymbol{p} | \boldsymbol{p} \rangle \qquad (2.2)$$

are complete

$$\int d\boldsymbol{q} | \boldsymbol{q} \rangle \langle \boldsymbol{q} | = I \qquad \int d\boldsymbol{p} | \boldsymbol{p} \rangle \langle \boldsymbol{p} | = I \qquad (2.5)$$

& orthogonal with delta-normalization

$$\langle \boldsymbol{q} | \boldsymbol{q}' \rangle = \delta(\boldsymbol{q} - \boldsymbol{q}') \qquad \langle \boldsymbol{p} | \boldsymbol{p}' \rangle = \delta(\boldsymbol{p} - \boldsymbol{p}') \qquad (2.4)$$

$\langle \boldsymbol{q} |$ Eq(2.1) $| \boldsymbol{q}' \rangle$ gives

$$\langle \boldsymbol{q} | [Q_j, P_k] | \boldsymbol{q}' \rangle = i \hbar \delta_{jk} \langle \boldsymbol{q} | \boldsymbol{q}' \rangle$$

$$\rightarrow \qquad \langle \boldsymbol{q} | q_j P_k - P_k q_j' | \boldsymbol{q}' \rangle = i \hbar \delta_{jk} \delta(\boldsymbol{q} - \boldsymbol{q}')$$

$$= (q_j - q_j') \langle \boldsymbol{q} | P_k | \boldsymbol{q}' \rangle$$

where we've assumed $\boldsymbol{Q}$ & $\boldsymbol{P}$ are hermitain so that

$$\langle \boldsymbol{q} | \boldsymbol{Q} = \langle \boldsymbol{q} | \boldsymbol{q} \qquad \langle \boldsymbol{p} | \boldsymbol{P} = \langle \boldsymbol{p} | \boldsymbol{P}$$

Assuming the domain of integration includes the point x', we have

$$\int dx \frac{\partial}{\partial x} \delta(x - x') = 0$$

&

$$\int dx\, f(x)\,(x - x') \frac{\partial}{\partial x} \delta(x - x') = -\int dx\, \delta(x - x') \left[\frac{df}{dx} (x - x') + f(x) \right]$$

$$= -\int dx\, \delta(x - x')\, f(x')$$

$$= -f(x')$$

$$\rightarrow \qquad (x - x') \frac{\partial}{\partial x} \delta(x - x') = -\delta(x - x')$$

Setting $x \leftrightarrow x'$, we have

$$(x - x') \frac{\partial}{\partial x'} \delta(x - x') = \delta(x - x') \qquad [\, \delta(x - x') = \delta(x' - x) \,]$$

Hence,

$$\langle \boldsymbol{q} | P_j | \boldsymbol{q}' \rangle = \frac{\hbar}{i} \frac{\partial}{\partial q_j} \delta(\boldsymbol{q} - \boldsymbol{q}')$$

$$= \frac{\hbar}{i} \frac{\partial}{\partial q_j} \langle \boldsymbol{q} \mid \boldsymbol{q}' \rangle \qquad (2.6)$$

$$= -\frac{\hbar}{i} \frac{\partial}{\partial q_j'} \langle \boldsymbol{q} \mid \boldsymbol{q}' \rangle$$

$\therefore$ $\langle \boldsymbol{q} \mid P_j \mid \boldsymbol{p} \rangle = p_j \langle \boldsymbol{q} \mid \boldsymbol{p} \rangle$ [Eq(2.2) used.]

$$= \int d\boldsymbol{q}' \langle \boldsymbol{q} \mid P_j \mid \boldsymbol{q}' \rangle \langle \boldsymbol{q}' \mid \boldsymbol{p} \rangle \qquad \text{[Eq(2.5) used.]}$$

$$= -\int d\boldsymbol{q}' \frac{\hbar}{i} \frac{\partial \delta(\boldsymbol{q}-\boldsymbol{q}')}{\partial q_j'} \langle \boldsymbol{q}' \mid \boldsymbol{p} \rangle \qquad \text{[Eq(2.6) used.]}$$

$$= \int d\boldsymbol{q}' \delta(\boldsymbol{q}-\boldsymbol{q}') \frac{\hbar}{i} \frac{\partial}{\partial q_j'} \langle \boldsymbol{q}' \mid \boldsymbol{p} \rangle \qquad \text{[Integration by part]}$$

$$= \frac{\hbar}{i} \frac{\partial}{\partial q_j} \langle \boldsymbol{q} \mid \boldsymbol{p} \rangle$$

$\to$ $\frac{\hbar}{i} \frac{\partial}{\partial \boldsymbol{q}} \langle \boldsymbol{q} \mid \boldsymbol{p} \rangle = \boldsymbol{p} \langle \boldsymbol{q} \mid \boldsymbol{p} \rangle$

$$\langle \boldsymbol{q} \mid \boldsymbol{p} \rangle = C e^{i \boldsymbol{p} \cdot \boldsymbol{q} / \hbar}$$

$\langle \boldsymbol{q} \mid$ Eq(2.5) $\mid \boldsymbol{q}' \rangle$ gives

$$\int d\boldsymbol{p} \langle \boldsymbol{q} \mid \boldsymbol{p} \rangle \langle \boldsymbol{p} \mid \boldsymbol{q}' \rangle = \langle \boldsymbol{q} \mid \boldsymbol{q}' \rangle$$

$\to$ $\int d\boldsymbol{p} \mid C \mid^2 e^{i \boldsymbol{p} \cdot (\boldsymbol{q}-\boldsymbol{q}') / \hbar} = \delta(\boldsymbol{q}-\boldsymbol{q}')$

$$= \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^n} e^{i \boldsymbol{p} \cdot (\boldsymbol{q}-\boldsymbol{q}') / \hbar} \qquad (2.8)$$

$\therefore$ $\mid C \mid^2 = \frac{1}{(2\pi\hbar)^n}$ (n = dimension of $\boldsymbol{p}$.)

$$\langle \boldsymbol{q} \mid \boldsymbol{p} \rangle = \frac{1}{(2\pi\hbar)^{n/2}} e^{i \boldsymbol{p} \cdot \boldsymbol{q} / \hbar} \qquad (2.7)$$

Checking for consistency, $\langle \boldsymbol{p} \mid$ Eq(2.5) $\mid \boldsymbol{p}' \rangle$ gives

$$\int d\boldsymbol{q} \langle \boldsymbol{p} \mid \boldsymbol{q} \rangle \langle \boldsymbol{q} \mid \boldsymbol{p}' \rangle = \langle \boldsymbol{p} \mid \boldsymbol{p}' \rangle$$

$\to$ $\int d\boldsymbol{q} \frac{e^{i (\boldsymbol{p}'-\boldsymbol{p}) \cdot \boldsymbol{q} / \hbar}}{(2\pi\hbar)^n} = \delta(\boldsymbol{p}-\boldsymbol{p}')$

as expected.

Warning: Eq(2.7) in Swanson's book is a typo.

However, if one insists on using

$$\langle \boldsymbol{q} \mid \boldsymbol{p} \rangle = \frac{1}{(2\pi\hbar)^n} e^{i \boldsymbol{p} \cdot \boldsymbol{q} / \hbar}$$

then one must set

$$\langle \boldsymbol{p} \mid \boldsymbol{p}' \rangle = \int d\boldsymbol{q} \langle \boldsymbol{p} \mid \boldsymbol{q} \rangle \langle \boldsymbol{q} \mid \boldsymbol{p}' \rangle$$

$$= \int d\boldsymbol{q} \frac{e^{i (\boldsymbol{p}'-\boldsymbol{p}) \cdot \boldsymbol{q} / \hbar}}{(2\pi\hbar)^{2n}}$$

$$= \frac{1}{(2\pi\hbar)^n}\,\delta(\boldsymbol{p}-\boldsymbol{p}')$$

instead of eq(2.4). Also,

$$\int d\boldsymbol{p}\,\langle \boldsymbol{q} \mid \boldsymbol{p}\rangle\langle \boldsymbol{p} \mid \boldsymbol{q}'\rangle = \int d\boldsymbol{p}\,\frac{e^{i\,\boldsymbol{p}\cdot(\boldsymbol{q}-\boldsymbol{q}')/\hbar}}{(2\pi\hbar)^{2n}} = \frac{1}{(2\pi\hbar)^n}\,\delta(\boldsymbol{q}-\boldsymbol{q}')$$

so that the completeness of $|\,\boldsymbol{p}\rangle$ takes the form

$$\int \frac{d\boldsymbol{p}}{(2\pi\hbar)^n}\,|\,\boldsymbol{p}\rangle\langle \boldsymbol{p}\,| = I \qquad (2.5a)$$

instead of eq(2.5).

One guideline for quantizing a classical system is to replace the Poisson bracket with the corresponding commutator:

$$\{A(p,q),\, B(p,q)\}_{p,q} \;\to\; \frac{1}{i\hbar}[A(P,Q),\, B(P,Q)] \qquad (2.9)$$

of which eq(2.1) is a special case.

Eq(1.49) then becomes

$$\frac{dO}{dt} = \frac{1}{i\hbar}[O, H] + \frac{\partial O}{\partial t} \qquad (2.10)$$

Closer examination shows that this corresponds to the Heisenberg picture formulism & is true only if H is hermitian [see eq(2.21)].

In Newtonian mechanics, motion of a system is a path (or curve) in phase space and time is a parameter that describes the progress along the path.

In the Schrodinger picture of non-relativistic quantum mechanics, evolution of the system is a path in the Hilbert space and time is a parameter that describes the progress along the path. This evolution is governed by the Schrodinger eq.

$$i\hbar\frac{\partial}{\partial t}|\,\psi(t)\rangle_S = H\,|\,\psi(t)\rangle_S \qquad (2.11)$$

where $H = H(\boldsymbol{P}, \boldsymbol{Q}, t)$ is the hamiltonian.

In the q-representation, one chooses $\{\,|\,\boldsymbol{q}\rangle\}$ as basis. The $\boldsymbol{q}^{\text{th}}$ component of eq(2.11) is given by

$$i\hbar\frac{\partial}{\partial t}\langle \boldsymbol{q} \mid \psi(t)\rangle_S = \langle \boldsymbol{q} \mid H \mid \psi(t)\rangle_S$$

$$= H\left(\frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{q}}, \boldsymbol{q}, t\right)\langle \boldsymbol{q} \mid \psi(t)\rangle_S$$

Defining the wave function as

$$\psi(\boldsymbol{q}, t) = \langle \boldsymbol{q} \mid \psi(t)\rangle_S$$

one recovers the more familiar form

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{q}, t) = H\left(\frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{q}}, \boldsymbol{q}, t\right)\psi(\boldsymbol{q}, t) \qquad (2.12)$$

The expection value of observable $O(\boldsymbol{P}, \boldsymbol{Q})$ when the system is in state $|\,\psi\rangle$ is given by

$$\langle O\rangle = \frac{\langle \psi \mid O \mid \psi\rangle}{\langle \psi \mid \psi\rangle}$$

Unless otherwise stated, we shall assume $|\,\psi\rangle$ normalized, i.e.

$$\langle \psi \mid \psi\rangle = 1$$

so that

$$\langle O\rangle = \langle \psi \mid O \mid \psi\rangle \qquad (2.13a)$$

In the q-representation, one has

$$\langle O\rangle = \int d\,\boldsymbol{q}\, d\,\boldsymbol{q}'\langle \psi \mid \boldsymbol{q}\rangle\langle \boldsymbol{q} \mid O \mid \boldsymbol{q}'\rangle\langle \boldsymbol{q}' \mid \psi\rangle$$

For a 1-particle system or if there's no interaction between particles,

$$\langle \boldsymbol{q} \mid O \mid \boldsymbol{q}'\rangle = \delta(\boldsymbol{q}-\boldsymbol{q}')\, O\!\left(\frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{q}'}, \boldsymbol{q}'\right)$$

$$\rightarrow \qquad \langle O\rangle = \int d\,\boldsymbol{q}\,\psi^*(\boldsymbol{q})\, O\!\left(\frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{q}}, \boldsymbol{q}\right)\psi(\boldsymbol{q}) \qquad (2.13b)$$

In the Schrodinger picture, evolution is embodied by ψ so that

$$\langle O(t)\rangle = {}_S\langle \psi(t) \mid O \mid \psi(t)\rangle_S$$

$$= \int d\,\boldsymbol{q}\,\psi^*(\boldsymbol{q}, t)\, O\!\left(\frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{q}}, \boldsymbol{q}\right)\psi(\boldsymbol{q}, t) \qquad (2.13)$$

Eq(2.13) suggests the interpretation of $\mid \psi(\boldsymbol{q}, t) \mid^2$ as a probability density.

If no particle is added or removed from the system, we must have

$${}_S\langle \psi(t) \mid \psi(t)\rangle_S = 1$$

$$= \int d\,\boldsymbol{q} \mid \psi(\boldsymbol{q}, t) \mid^2 \qquad (2.14)$$

i.e., the wave function is normalized at all times.

The evolution operator U is defined by

$$\mid \psi(t)\rangle_S = U(t, t_0) \mid \psi(t_0)\rangle_S$$

If H is time-independent, it is easy to check that

$$\mid \psi(t)\rangle_S = \exp\!\left(-\frac{i}{\hbar} H(t-t_0)\right)\Big| \psi(t_0)\Big\rangle_S \qquad (2.15)$$

$$U(t, t_0) = \exp\!\left(-\frac{i}{\hbar} H(t-t_0)\right)$$

If H is time-dependent, one can solve eq(2.11) by iteration:

$$\mid \psi(t)\rangle_S = \mid \psi(t_0)\rangle_S - \frac{i}{\hbar}\int_{t_0}^{t} d\,t_1\, H(t_1)\Big| \psi(t_1)\Big\rangle_S$$

$$= \mid \psi(t_0)\rangle_S - \frac{i}{\hbar}\int_{t_0}^{t} d\,t_1\, H(t_1)\left(\Big| \psi(t_0)\Big\rangle_S - \frac{i}{\hbar}\int_{t_0}^{t_1} d\,t_2\, H(t_2)\Big| \psi(t_2)\Big\rangle_S \right)$$

$$= \left(1 - \frac{i}{\hbar}\int_{t_0}^{t} d\,t_1\, H(t_1)\right)\Big| \psi(t_0)\Big\rangle_S + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} d\,t_1\, H(t_1)\int_{t_0}^{t_1} d\,t_2\, H(t_2)\Big| \psi(t_2)\Big\rangle_S$$

$$= \left(1 - \frac{i}{\hbar}\int_{t_0}^{t} d\,t_1\, H(t_1) + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} d\,t_1\, H(t_1)\int_{t_0}^{t_1} d\,t_2\, H(t_2)\right)\Big| \psi(t_0)\Big\rangle_S$$

$$+\left(-\frac{i}{\hbar}\right)^3 \int_{t_0}^{t} d\,t_1\, H(t_1)\int_{t_0}^{t_1} d\,t_2\, H(t_2)\int_{t_0}^{t_2} d\,t_3\, H(t_3)\Big| \psi(t_3)\Big\rangle_S$$

$$= \left\{ \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} d\,t_1\, H(t_1) \ldots \int_{t_0}^{t_{n-1}} d\,t_n\, H(t_n)\right\}\Big| \psi(t_0)\Big\rangle_S$$

$$\rightarrow \qquad U(t, t_0) = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} d\,t_1\, H(t_1) \ldots \int_{t_0}^{t_{n-1}} d\,t_n\, H(t_n)$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} d\,t_1\, H(t_1) + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} d\,t_1\, H(t_1)\int_{t_0}^{t_1} d\,t_2\, H(t_2) + \ldots$$

The time-order operator T arranges a product of operators so that each operator has a later time than every one on its right. For example,

$$T[A(t)\,B(t')] = \begin{cases} A(t)\,B(t') & \text{if } t > t' \\ B(t')\,A(t) & \text{if } t' > t \end{cases}$$
$$= \theta(t - t')\,A(t)\,B(t') + \theta(t' - t)\,B(t')\,A(t) \qquad (2.20)$$

$$T[A(t_1)\,B(t_2)\,C(t_3)] = \begin{cases} A(t_1)\,B(t_2)\,C(t_3) & \text{if } t_1 > t_2 > t_3 \\ B(t_2)\,A(t_1)\,C(t_3) & \text{if } t_2 > t_1 > t_3 \\ A(t_1)\,C(t_3)\,B(t_2) & \text{if } t_1 > t_3 > t_2 \\ C(t_3)\,A(t_1)\,B(t_2) & \text{if } t_3 > t_1 > t_2 \\ B(t_2)\,C(t_3)\,A(t_1) & \text{if } t_2 > t_3 > t_1 \\ C(t_3)\,B(t_2)\,A(t_1) & \text{if } t_3 > t_2 > t_1 \end{cases}$$

In general, $T[A_1(t_1) \ldots A_n(t_n)]$ has $n!$ possible outcomes of the form

$$A_{j_1}(t_{j_1}) \ldots A_{j_n}(t_{j_n}) \qquad \text{with} \qquad t_{j_1} > \ldots > t_{j_n}$$

Hence,

$$\int_{t_0}^{t} d\,t_1\,H(t_1) \ldots \int_{t_0}^{t_{n-1}} d\,t_n\,H(t_n) = \frac{1}{n!}\int_{t_0}^{t} d\,t_1 \;\ldots \int_{t_0}^{t} d\,t_n\,T[H(t_1) \ldots H(t_n)]$$
$$= \frac{1}{n!}\,T\left(\int_{t_0}^{t} d\,\tau\,H(\tau)\right)^n$$

so that

$$|\,\psi(t)\rangle_S = T\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} d\,\tau\,H(\tau)\right)|\,\psi(t_0)\rangle_S \qquad (2.16)$$

$$U(t, t_0) = T\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} d\,\tau\,H(\tau)\right)$$
$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} d\,t_1\,H(t_1) + \frac{1}{2!}\left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} d\,t_1 \int_{t_0}^{t} d\,t_2\,T[H(t_1)\,H(t_2)] + \ldots$$

Eq(2.15) implies U satisfies

$$i\hbar\frac{\partial}{\partial t}U(t, t') = H(t)\,U(t, t') \qquad (2.16a)$$
$$-i\hbar\frac{\partial}{\partial t}U^+(t, t') = U^+(t, t')\,H^+(t)$$

& has the group properties

$$U(t, t) = 1$$
$$U(t, t')\,U(t', t'') = U(t, t'')$$

so that

$$U(t, t')\,U(t', t) = 1$$

$$\rightarrow \qquad U^{-1}(t, t') = U(t', t)$$
$$= 1 - \frac{i}{\hbar}\int_{t}^{t'} d\,t_1\,H(t_1) + \left(-\frac{i}{\hbar}\right)^2 \int_{t}^{t'} d\,t_1\,H(t_1)\int_{t}^{t_1} d\,t_2\,H(t_2) + \ldots$$
$$= 1 + \frac{i}{\hbar}\int_{t'}^{t} d\,t_1\,H(t_1) + \left(\frac{i}{\hbar}\right)^2 \int_{t'}^{t} d\,t_1\,H(t_1)\int_{t_1}^{t} d\,t_2\,H(t_2) + \ldots$$
$$= 1 + \frac{i}{\hbar}\int_{t'}^{t} d\,t_1\,H(t_1) + \frac{1}{2!}\left(\frac{i}{\hbar}\right)^2 \int_{t'}^{t} d\,t_1 \int_{t'}^{t} d\,t_2\,\tilde{T}[H(t_1)\,H(t_2)] + \ldots$$
$$= \tilde{T}\exp\left(\frac{i}{\hbar}\int_{t'}^{t} d\,\tau\,H(\tau)\right)$$

where $\tilde{T}$ is the reversed-time-order operator that arranges a product of operators so that each operator has a later time than every one on its left.

If H is hermitian, then

$$\tilde{T}[H(t_1)\dots H(t_n)] = \{T[H(t_1)\dots H(t_n)]\}^+$$

$$\to \qquad U^{-1}(t,t') = \left[T\exp\left(-\frac{i}{\hbar}\int_{t'}^{t} d\tau\, H(\tau)\right)\right]^+ = U^+(t,t')$$

i.e., U is unitary.

Let

$$\mathcal{I} = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} d\omega\, \frac{e^{i\omega t}}{\omega - i\varepsilon}$$

$$\mathcal{J}_C = \oint_C dz\, \frac{e^{izt}}{z - i\varepsilon}$$

Since e^{izt} is an entire function, the only pole in $\mathcal{J}_C$ is $z = i\varepsilon$.

Let C_+ be the counterclockwise contour that consists of the $\text{Re}\, z = \omega$ axis & the semicircle on the upper half plane that connects the ends of the ω-axis.

For $t > 0$, $e^{izt} = 0$ on the semicircle & the pole $z = i\varepsilon$ is inside C_+. Hence,

$$\mathcal{I} = \mathcal{J}_{C_+} = 2\pi i\, e^{-\varepsilon t} = 2\pi i$$

Let C_- be the clockwise contour that consists of the $\text{Re}\, z = \omega$ axis & the semicircle on the lower half plane that connects the ends of the ω-axis. Then For $t < 0$, $e^{izt} = 0$ on the semicircle & the pole $z = i\varepsilon$ is outside C_-. Hence,

$$\mathcal{I} = \mathcal{J}_{C_-} = 0$$

$$\therefore \qquad \theta(t) = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i}\, \frac{e^{i\omega t}}{\omega - i\varepsilon} \qquad (2.18)$$

$$\to \qquad \frac{\partial}{\partial t}\theta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t} = \delta(t) \qquad (2.19)$$

In the Heisenberg picture, the states are related to its counterpart in the Schrodinger picture by

$$|\,\psi\rangle_H = U(t_0, t)\,|\,\psi(t)\rangle_S = |\,\psi(t_0)\rangle_S$$

so that $|\,\psi\rangle_H$ is time-independent.

Of particular interest is the case of time-independent H. Setting $t_0 = 0$ gives

$$|\,\psi\rangle_H = e^{iHt/\hbar}\,\left|\,\psi(t)\right\rangle_S = |\,\psi(0)\rangle_S$$

Since matrix elements are measurable quantities, they must be independent of the picture one chooses. For any classical quantity $O(\boldsymbol{p}, \boldsymbol{q}, t)$, we set $O_S(t) \equiv O_S(\boldsymbol{P}, \boldsymbol{Q}, t)$ and

$$_H\langle\psi\,|\,O_H(t)\,|\,\phi\rangle_H = {}_S\langle\psi(t)\,|\,O_S(t)\,|\,\phi(t)\rangle_S \qquad (2.22)$$

$$= {}_H\left\langle\psi\,\middle|\,U^+(t,t_0)\,O_S(t)\,U(t,t_0)\,|\,\phi\right\rangle_H$$

$$\to \qquad O_H(t) = U^+(t,t_0)\,O_S(t)\,U(t,t_0)$$

In particular, for a time-independent & hermitian H with $t_0 = 0$, we have

$$O_H(t) = e^{iHt/\hbar}\,O_S(t)\,e^{-iHt/\hbar} \qquad (2.24)$$

$$\to \qquad O_H(0) = O_S(0)$$

Using eq(2.16a), we have

$$\frac{d\,O_H(t)}{dt} = \frac{\partial U^+(t,t_0)}{\partial t}\,O_S(t)\,U(t,t_0) + U^+(t,t_0)\,O_S(t)\,\frac{\partial U(t,t_0)}{\partial t} + U^+(t,t_0)\,\frac{\partial O_S(t)}{\partial t}\,U(t,t_0)$$

$$= \frac{1}{i\hbar}\,U^+(t,t_0)\left(-H^+(t)\,O_S(t) + O_S(t)\,H(t)\right)U(t,t_0) + U^+(t,t_0)\,\frac{\partial O_S(t)}{\partial t}\,U(t,t_0)$$

$$= \frac{1}{i\hbar}\left(-H_H^{+}(t)\, O_H(t) - O_H(t)\, H_H(t)\right) + \left(\frac{\partial O(t)}{\partial t}\right)_H$$

where

$$\left(\frac{\partial O(t)}{\partial t}\right)_H = U^{+}(t, t_0)\, \frac{\partial O_S(t)}{\partial t}\, U(t, t_0)$$

If H is hermitian, we have

$$\frac{d\, O_H(t)}{d\, t} = \frac{1}{i\hbar}[\, O_H(t)\,,\, H_H(t)\,] + \left(\frac{\partial O(t)}{\partial t}\right)_H \qquad (2.21)$$

which agrees with eq(2.10).

The transition amplitude from $|\, \phi\, (t_a) \rangle$ to $|\, \psi\, (t_b) \rangle$ is defined as

$$Z(\, \psi\, (t_b),\, \phi\, (t_a)\,) = \langle\, \psi\, \mid U(t_b, t_a) \mid \phi\, \rangle \qquad (2.25)$$

Thus, state $|\, \phi\, \rangle$ is allowed to evolve from t_a to t_b. Its overlap with state $|\, \psi \rangle$ then gives Z.

If H is t-independent,

$$Z(\, \psi\, (t_b),\, \phi\, (t_a)\,) = \left\langle \psi \,\middle|\, e^{-i\, H\, (t_b - t_a)\, /\, \hbar} \,\middle|\, \phi \right\rangle$$

The probability of such a transition is

$$P = |\, Z\, |^2 \qquad (2.25a)$$

The main object of interest in the path integral formulism is Z.