2.3. The Sum over Histories

The main appeal of the path integral approach is that classical action is used to describe quantum processes. Thus, eq(2.50) is preferred as the starting point for interpreting or extending the formulism.

Reverting to the discrete form, we set

$$\int_{-\infty}^{\infty} d\,q_j \;\to\; \sum_{n_j=-\infty}^{\infty} \epsilon_q \tag{2.53}$$

where ϵ_q is the infinitesimal measure for the q_j integrations.

Eq(2.50) thus becomes

$$\langle q_b,\, t_b \mid q_a,\, t_a \rangle \approx \left(\frac{m}{2\,\pi\, i\,\hbar\,\epsilon} \right)^{N/2} \sum_{n_1=-\infty}^{\infty} \epsilon_q \cdots \sum_{n_{N-1}=-\infty}^{\infty} \epsilon_q \tag{2.54}$$

$$\times \exp\left\{ \frac{i}{\hbar} \sum_{j=0}^{N-1} \epsilon\, \mathcal{L}\left[(n_{j+1} - n_j)\, \frac{\epsilon_q}{\epsilon},\; n_j\, \epsilon_q \right] \right\}$$

where the position of the particle at time t_j is given by $q_j = n_j\, \epsilon_q$ with

$$q_a = q_0 = n_0\, \epsilon_q \qquad\qquad q_b = q_N = n_N\, \epsilon_q$$

For a given set of $\{ n_j \}$ values, we have a set of $\{ q_j \}$ values that depicts the path (or history) of the particle going from q_a at t_a to q_b at t_b. The contribution of this path to the propagator is weighted by the value of $e^{i\,S/\hbar}$ on it.

The propagator itself is therefore a weighted sum of all possible paths (or histories) of the particle, i.e.,

$$Z \propto \sum_{\text{paths}} e^{i\,S/\hbar} \tag{2.55}$$

Warning: Although the physical concepts are straight-forward & appealing, the mathematical implementation sometimes runs into difficulties that are not yet resolved. See Swanson's text.

We mention here only that the oscillatory behavior of $e^{i\,S/\hbar}$ in eq(2.55) requires the inclusion of "un-ruly" paths so that a well-behaved measure is not available.

One way to get around this is

1. Make an analytic contiuation (or Wick rotation) to imaginary time $\tau = i\,t$.
2. Evaluate

$$Z \propto \sum_{\text{paths}} e^{-S_E/\hbar} \qquad\qquad -S_E(\tau) = i\,S(t)$$

3. Make the analytic contiuation back to real time.

For example, under the Wick rotation ($t = -i\,\tau$), eqs(2.50-2) become

$$Z = \int_{q_a}^{q_b} \overline{\mathcal{D}}\, q\, e^{-S_E/\hbar} \tag{2.60}$$

$$\overline{\mathcal{D}}\, q = \lim_{N\to\infty} \left(\frac{m}{2\,\pi\,\hbar\,\epsilon_\tau} \right)^{N/2} d\,q_1 \ldots d\,q_{N-1} \qquad (\epsilon_\tau = i\,\epsilon) \tag{2.62}$$

$$\mathcal{L} = -\frac{1}{2}\, m \left(\frac{d\,q}{d\,\tau} \right)^2 - V(q)$$

$$S_E = -i\,S = (-i)^2 \int_{t_a}^{t_b} d\,\tau\, \mathcal{L}$$

$$= \int_{t_a}^{t_b} d\tau \left[\frac{1}{2} m \left(\frac{dq}{d\tau} \right)^2 + V(q) \right] \qquad (2.61)$$

Note the desirable feature that (non-classical) paths of large S_E values are suppressed by $e^{-S_E/\hbar}$.

τ & S_E are often called Euclidean time & action, respectively.

For $V = 0$, the Schrodinger eq. in Euclidean time becomes

$$-\hbar \frac{\partial}{\partial \tau} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi \qquad (2.63)$$

which is just the eq. of diffusion governing the Brownian motion.

Consider a point particle of mass m constrained to move on a circle of radius R. The classical real-time Lagrangian is

$$\mathcal{L}(\dot{\theta}, \theta) = \frac{1}{2} m R^2 \dot{\theta}^2 - V(\theta) \qquad (2.64)$$

where θ is the polar angle of the particle. $\mathcal{L}$ must be single-valued so that V must be periodic, i.e.,

$$V(\theta + 2\pi n) = V(\theta) \qquad \forall\ n \in \mathbb{Z} \qquad (2.64a)$$

Using

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m R^2 \dot{\theta}$$

the classical Hamiltonian becomes

$$H(p_\theta, \theta) = \frac{p_\theta^2}{2 m R^2} + V(\theta) \qquad (2.65)$$

Upon quantization, the position eigenstates are $|\theta\rangle$ with completeness

$$\int d\theta\ |\theta\rangle\langle\theta| = I \qquad (2.66)$$

The canonical quantization rule gives

$$[\Theta, P_\theta] = i\hbar$$

Using

$$\left[x, \frac{\partial}{\partial x} \right] = -1$$

we have

$$P_\theta = \frac{\hbar}{i} \frac{\partial}{\partial \theta}$$

Alternatively,

$$\langle\theta| [\Theta, P_\theta] |\theta'\rangle = i\hbar \langle\theta|\theta'\rangle = i\hbar\,\delta(\theta - \theta')$$
$$= (\theta - \theta') \langle\theta| P_\theta |\theta'\rangle$$

$$\rightarrow \qquad \langle\theta| P_\theta |\theta'\rangle = \frac{\hbar}{i} \frac{\partial}{\partial \theta} \langle\theta|\theta'\rangle \qquad (2.67a)$$

The momentum eigenstates are given by

$$P_\theta |p\rangle = p |p\rangle$$

$$\rightarrow \qquad \frac{\hbar}{i} \frac{\partial}{\partial \theta} \langle\theta|p\rangle = p \langle\theta|p\rangle$$

$$\langle\theta|p\rangle = c\, e^{ip\theta/\hbar}$$

$\langle\theta|p\rangle$ must be single-valued.

$$\rightarrow \qquad e^{ip2\pi/\hbar} = 1$$

$$\therefore \qquad p = p_n = n\hbar \qquad n = \mathbb{Z}$$

The momentum eigenstates are therefore the discrete set

$$\langle \theta \mid p_n \rangle = c\, e^{i p_n \theta/\hbar} = c\, e^{i n \theta}$$

Orthonormality requires

$$\begin{aligned}\delta_{n m} &= \langle p_n \mid p_m \rangle \\ &= \mid c \mid^2 \int_0^{2\pi} d\theta\, e^{i(m-n)\theta} \\ &= \mid c \mid^2\, 2\pi\, \delta_{n m} \qquad (2.69)\end{aligned}$$

$$\rightarrow \qquad c = \frac{1}{\sqrt{2\pi}}$$

$$\langle \theta \mid p_n \rangle = \frac{1}{\sqrt{2\pi}} e^{i n \theta} \qquad (2.68)$$

Completeness is

$$\sum_{n=-\infty}^{\infty} \mid p_n \rangle \langle p_n \mid \ = I \qquad (2.70a)$$

$$\rightarrow \qquad \sum_{n=-\infty}^{\infty} \langle \theta \mid p_n \rangle \langle p_n \mid \theta' \rangle = \langle \theta \mid \theta' \rangle = \delta(\theta - \theta')$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{i n(\theta - \theta')} \qquad (2.70)$$

Using

$$\begin{aligned}\mathcal{L}(p_\theta, \theta) &= p_\theta \dot{\theta} - H(p_\theta, \theta) \\ &= p_\theta \dot{\theta} - \frac{p_\theta^2}{2 m R^2} - V(\theta)\end{aligned}$$

eq(2.44) bcomes

$$\langle \theta_b, t_b \mid \theta_a, t_a \rangle \approx \int_0^{2\pi} d\theta_1 \ldots d\theta_{N-1} \int_{-\infty}^{\infty} \frac{d p_0}{2\pi\hbar} \ldots \frac{d p_{N-1}}{2\pi\hbar}$$

$$\times \exp\left\{ \frac{i}{\hbar} \epsilon \left[\sum_{j=0}^{N-1} p_j \dot{\theta}_j - \frac{p_j^2}{2 m R^2} - V(\theta_j) \right] \right\}$$

where

$$\epsilon = \frac{t_b - t_a}{N} \qquad t_j = t_a + j\epsilon \qquad p_j = p_\theta(t_j)$$

$$\theta_j = \theta(t_j) \qquad \theta_0 = \theta_a \quad \& \quad \theta_N = \theta_b$$

Writing the p_j integral as a Riemann sum, we set

$$p_j = n_j \epsilon_p \qquad \text{where} \qquad \epsilon_p \to 0 \qquad n_j \in \mathbb{Z} \qquad (2.71a)$$

so that

$$\int_{-\infty}^{\infty} \frac{d p_j}{2\pi\hbar} f(p_j) \approx \frac{\epsilon_p}{2\pi\hbar} \sum_{n_j=-\infty}^{\infty} f(n_j \epsilon_p)$$

$$\rightarrow \qquad \langle \theta_b, t_b \mid \theta_a, t_a \rangle \approx \int_0^{2\pi} d\theta_1 \ldots d\theta_{N-1} \left(\frac{\epsilon_p}{2\pi\hbar} \right)^N \sum_{n_0=-\infty}^{\infty} \ldots \sum_{n_{N-1}=-\infty}^{\infty} \qquad (2.71b)$$

$$\times \exp\left\{ \frac{i}{\hbar} \epsilon \left[\sum_{j=0}^{N-1} n_j \epsilon_p \frac{\theta_{j+1} - \theta_j}{\epsilon} - \frac{n_j^2 \epsilon_p^2}{2 m R^2} - V(\theta_j) \right] \right\}$$

By setting $\epsilon_p = \hbar$, we have

$$p_j = n_j \hbar \tag{2.71c}$$

$$\langle \theta_b, t_b \mid \theta_a, t_a \rangle \approx \int_0^{2\pi} d\theta_1 \ldots d\theta_{N-1} \left(\frac{1}{2\pi}\right)^N \sum_{n_0=-\infty}^{\infty} \ldots \sum_{n_{N-1}=-\infty}^{\infty} \tag{2.71}$$

$$\times \exp\left\{ \frac{i}{\hbar} \epsilon \left[\sum_{j=0}^{N-1} n_j \hbar \frac{\theta_{j+1} - \theta_j}{\epsilon} - \frac{n_j^2 \hbar^2}{2 m R^2} - V(\theta_j) \right] \right\}$$

Swanson’s argument using eq(2.71) to suggest the quantization of p_θ is faulty since eq(2.71c) was imposed by hand.