## 3.1. The Free Particle

$$\mathcal{L}(\dot{q}, q) = \frac{1}{2} m \dot{q}^2 \tag{3.1}$$

$$\rightarrow \qquad p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m \dot{q} \qquad \dot{q} = \frac{p}{m}$$

$$H(p, q) = p \frac{p}{m} - \frac{1}{2} m \left(\frac{p}{m}\right)^2 = \frac{p^2}{2m}$$

$$\therefore \qquad H(P, Q) = \frac{P^2}{2m} \tag{3.2}$$

Since $H$ is $t$-independent,

$$\langle q_b, t_b \mid q_a, t_a \rangle = \left\langle q_b \left| e^{-i H t_b/\hbar} e^{i H t_a/\hbar} \right| q_a \right\rangle$$

$$= \left\langle q_b \left| e^{-i H (t_b - t_a)/\hbar} \right| q_a \right\rangle$$

$$= \left\langle q_b \left| e^{-i P^2 (t_b - t_a)/2m\hbar} \right| q_a \right\rangle$$

Using the completeness of $\{ \mid p \rangle \}$, we have

$$\langle q_b, t_b \mid q_a, t_a \rangle = \int_{-\infty}^{\infty} dp \left\langle q_b \left| e^{-i P^2 (t_b - t_a)/2m\hbar} \right| p \right\rangle \langle p \mid q_a \rangle$$

$$= \int_{-\infty}^{\infty} dp \, e^{-i p^2 (t_b - t_a)/2m\hbar} \langle q_b \mid p \rangle \langle p \mid q_a \rangle$$

$$= \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \exp\left\{ -i \frac{p^2}{2m\hbar} (t_b - t_a) + i \frac{p}{\hbar} (q_b - q_a) \right] \Big\} \tag{3.3}$$

Using eq(1.107), we have

$$\langle q_b, t_b \mid q_a, t_a \rangle = \sqrt{\frac{m}{2\pi i \hbar (t_b - t_a)}} \exp\left( \frac{i}{\hbar} \frac{m (q_b - q_a)^2}{2 (t_b - t_a)} \right) \tag{3.4}$$

For the path integral approach, we use eq(2.50) to get

$$\langle q_b, t_b \mid q_a, t_a \rangle = \left( \frac{m}{2\pi i \hbar \epsilon} \right)^{N/2} \int_{-\infty}^{\infty} dq_1 \ldots dq_{N-1} \exp\left( \frac{i}{\hbar} \sum_{j=0}^{N-1} \frac{1}{2\epsilon} m \left( q_{j+1} - q_j \right)^2 \right) \tag{3.5}$$

with

$$q_0 = q_a \qquad q_N = q_b$$

Using [see eq(1.107)]

$$\int_{-\infty}^{\infty} dx \exp\left\{ i \alpha \left( x^2 \pm \beta x \right) \right\} = \sqrt{\frac{i\pi}{\alpha}} \exp\left( -i \frac{\alpha}{4} \beta^2 \right)$$

we have

$$\int_{-\infty}^{\infty} dx \exp\left\{ i \alpha \left[ (x - a)^2 + \frac{n}{m} (x - b)^2 \right] \right\}$$

$$= \int_{-\infty}^{\infty} dx \exp\left\{ i \alpha \left[ \left( 1 + \frac{n}{m} \right) x^2 - 2 \left( a + \frac{n}{m} b \right) x + a^2 + \frac{n}{m} b^2 \right] \right\}$$

$$= \int_{-\infty}^{\infty} dx \exp\left\{ i \left( \frac{m+n}{m} \right) \alpha \left[ x^2 - \frac{2 (m a + n b)}{m+n} x + \frac{m a^2 + n b^2}{m+n} \right] \right\}$$

$$= \sqrt{\frac{i\pi m}{\alpha(m+n)}} \exp\left\{ i\left(\frac{m+n}{m}\right)\alpha\left[-\left(\frac{ma+nb}{m+n}\right)^2 + \frac{ma^2+nb^2}{m+n}\right]\right\}$$

$$= \sqrt{\frac{i\pi m}{\alpha(m+n)}} \exp\left\{ i\,\frac{n\alpha}{m+n}(a-b)^2\right\} \qquad (3.6a)$$

Let

$$\alpha = \frac{m}{2\hbar\epsilon}$$

Starting with the $q_1$ integral, we have

$$\int_{-\infty}^{\infty} d\,q_1 \exp\left\{ i\,\alpha\left[(q_2-q_1)^2+(q_1-q_0)^2\right]\right\} = \sqrt{\frac{i\pi}{2\alpha}} \exp\left[i\,\frac{\alpha}{2}(q_2-q_0)^2\right]$$

$$\to \qquad \int_{-\infty}^{\infty} d\,q_2 \exp\left\{ i\,\alpha\left[(q_3-q_2)^2+\frac{1}{2}(q_2-q_0)^2\right]\right\} = \sqrt{\frac{i\pi}{\alpha}\cdot\frac{2}{3}} \exp\left[i\,\frac{\alpha}{3}(q_3-q_0)^2\right]$$

$\vdots$

$$\int_{-\infty}^{\infty} d\,q_j \exp\left\{ i\,\alpha\left[(q_{j+1}-q_j)^2+\frac{1}{j}(q_j-q_0)^2\right]\right\} = \sqrt{\frac{i\pi}{\alpha}\cdot\frac{j}{j+1}} \exp\left[i\,\frac{\alpha}{j+1}(q_{j+1}-q_0)^2\right] \qquad (3.7)$$

$\vdots$

$$\int_{-\infty}^{\infty} d\,q_{N-1} \exp\left\{ i\,\alpha\left[(q_N-q_{N-1})^2+\frac{1}{N-1}(q_{N-1}-q_0)^2\right]\right\}$$

$$= \sqrt{\frac{i\pi}{\alpha}\cdot\frac{N-1}{N}} \exp\left[i\,\frac{\alpha}{N}(q_N-q_0)^2\right]$$

$$= \sqrt{\frac{2i\pi\hbar\epsilon}{m}\cdot\frac{N-1}{N}} \exp\left[\frac{i\,m}{2\hbar\epsilon N}(q_b-q_a)^2\right]$$

There're $N-1$ integrals of the type eq(3.7) in eq(3.5). Their product gives

$$\left(\frac{2i\pi\hbar\epsilon}{m}\right)^{(N-1)/2} \frac{1}{\sqrt{N}} \exp\left[\frac{i\,m}{2\hbar\epsilon N}(q_b-q_a)^2\right]$$

Eq(3.5) itself thus becomes

$$\langle q_b, t_b \mid q_a, t_a\rangle = \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{1/2} \frac{1}{\sqrt{N}} \exp\left[\frac{i\,m}{2\hbar\epsilon N}(q_b-q_a)^2\right] \qquad (3.8)$$

which is simply eq(3.8).

The (retarded) green's function for the 1-D Schrodinger eq. is defined by

$$\left[i\hbar\frac{\partial}{\partial t} - H\left(\frac{\hbar}{i}\frac{\partial}{\partial x}, x\right)\right] G(x, t, x' t') = i\hbar\,\delta(t-t')\,\delta(x-x') \qquad (3.9)$$

For $H$ time-independent, we have

$$\langle x, t \mid x', t'\rangle = \left\langle x \,\middle|\, e^{-iH(t-t')/\hbar} \,\middle|\, x'\right\rangle$$

$$\to \qquad i\hbar\frac{\partial}{\partial t}\langle x, t \mid x', t'\rangle = \left\langle x \,\middle|\, H e^{-iH(t-t')/\hbar} \,\middle|\, x'\right\rangle$$

$$\therefore \quad i\hbar \frac{\partial}{\partial t}\{\theta(t-t')\langle x, t \mid x', t'\rangle\}$$

$$= i\hbar\,\delta(t-t')\langle x, t \mid x', t\rangle + \theta(t-t')\left\langle x \,\middle|\, H e^{-iH(t-t')/\hbar} \,\middle|\, x'\right\rangle \qquad (3.11)$$

$$= i\hbar\,\delta(t-t')\,\delta(x-x') + \theta(t-t')\,H\left(\frac{\hbar}{i}\frac{\partial}{\partial x}, x\right)\left\langle x \,\middle|\, e^{-iH(t-t')/\hbar} \,\middle|\, x'\right\rangle$$

Hence, $G(x, t, x' t') = \theta(t-t')\langle x, t \mid x', t'\rangle$ (3.10)

Note: the green's function defined by Swanson is the advanced version.
For a discussion of the various types of Green's functions, see, e.g., Chap 2 of "Green's Functions in Quantum Physics" by E.N.Economou.

For the classical motion, the eq. of motion is

$$\ddot{q} = 0 \qquad \text{with B.C.} \qquad q(t_a) = q_a \qquad q(t_b) = q_b$$

$$\rightarrow \quad \dot{q}_c = \frac{q_b - q_a}{t_b - t_a}$$

$$q_c(t) = q_a + \frac{q_b - q_a}{t_b - t_a}(t - t_a) \qquad (3.12)$$

$$\therefore \quad S_c = \int_{t_a}^{t_b} dt\, \frac{1}{2} m \dot{q}_c^2 = \frac{1}{2} m \frac{(q_b - q_a)^2}{t_b - t_a} \qquad (3.13)$$

$$\rightarrow \quad \langle q_b, t_b \mid q_a, t_a\rangle = \sqrt{\frac{m}{2\pi i\hbar(t_b - t_a)}}\,\exp\left(\frac{i}{\hbar} S_c\right)$$

$$= \langle q, t_b \mid q, t_a\rangle \exp\left(\frac{i}{\hbar} S_c\right)$$