3.2. Motion with a Source

This section consider path integrals for Lagrangians with a source function $J(t)$, i.e.

$$\mathcal{L}(\dot{q}, q, t) = \frac{1}{2} m \dot{q}^2 - V(q) + J(t)\, q \tag{3.14}$$

For $V = 0$, we have

$$\mathcal{L}_J^0(p, q) = p\,\dot{q} - \frac{p^2}{2m} + J(t)\, q \tag{3.15}$$

$$\langle q_b, t_b \mid q_a, t_a \rangle_J = \int_{q_a}^{q_b} \mathcal{D}\, p\, \mathcal{D}\, q \exp\left[\frac{i}{\hbar} \int_{t_a}^{t_b} d\,t\, \mathcal{L}_J^0(p, q) \right] \tag{3.16a}$$

$$= \int_{-\infty}^{\infty} d\,q_1 \ldots d\,q_{N-1} \frac{d\,p_0}{2\pi\hbar} \ldots \frac{d\,p_{N-1}}{2\pi\hbar} \tag{3.16}$$

$$\times \exp\left\{ \frac{i}{\hbar} \sum_{j=0}^{N-1} \left[p_j \left(q_{j+1} - q_j \right) - \epsilon \frac{p_j^2}{2m} + \epsilon\, J\left(t_j\right) q_j \right] \right\}$$

Using

$$\int_{-\infty}^{\infty} \frac{d\,q_j}{2\pi\hbar} \exp\left\{ \frac{i}{\hbar} \left[p_{j-1} - p_j + \epsilon\, J(t_j) \right] q_j \right\} = \delta\left[p_{j-1} - p_j + \epsilon\, J(t_j) \right]$$

the q_j integrals produce $N-1$ delta functions. Taking note that parts of the $j = 0$ & $j = N-1$ terms are not involved in the integrations, we have

$$Z_J^0(q_a, t_a, q_b, t_b) = \langle q_b, t_b \mid q_a, t_a \rangle_J^0$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\,p_0 \ldots d\,p_{N-1} \tag{3.17}$$

$$\times\ \delta[\,p_0 - p_1 + \epsilon\, J(t_1)\,] \ldots \delta[\,p_{N-2} - p_{N-1} + \epsilon\, J(t_{N-1})\,]$$

$$\times \exp\left\{ \frac{i}{\hbar} \left[p_{N-1}\, q_N + [-p_0 + \epsilon\, J(t_0)]\, q_0 - \epsilon \sum_{j=0}^{N-1} \frac{p_j^2}{2m} \right] \right\}$$

Doing the p_0 integral merely involves replacing every p_0 with $p_1 - \epsilon\, J(t_1)$. Hence

$$Z_J^0 = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\,p_1 \ldots d\,p_{N-1}$$

$$\times\ \delta[\,p_1 - p_2 + \epsilon\, J(t_2)\,] \ldots \delta[\,p_{N-2} - p_{N-1} + \epsilon\, J(t_{N-1})\,]$$

$$\times \exp\left\{ \frac{i}{\hbar} \left[p_{N-1}\, q_N + [-p_1 + \epsilon\, J(t_1) + \epsilon\, J(t_0)]\, q_0 \right.\right.$$

$$\left.\left. - \frac{\epsilon}{2m} [p_1 - \epsilon\, J(t_1)]^2 - \epsilon \sum_{j=1}^{N-1} \frac{p_j^2}{2m} \right] \right\}$$

Similarly, integrating p_1 gives

$$
\begin{aligned}
Z_J^0 = & \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\,p_2\ \dots\ d\,p_{N-1} \\
& \times\ \delta[\,p_2 - p_3 + \epsilon\, J(t_3)\,] \dots \delta[\,p_{N-2} - p_{N-1} +\ \epsilon\, J(t_{N-1})\,] \\
& \times \exp\left\{\frac{i}{\hbar}\left[\,p_{N-1}\, q_N + [\,-p_2 + \epsilon\, J\,(t_2) +\ \epsilon\, J(t_1) + \epsilon\, J(t_0)]\, q_0\right.\right. \\
& \left.\left. - \frac{\epsilon}{2\,m}[\,p_2 - \epsilon\, J\,(t_2) -\ \epsilon\, J(t_1)\,]^2 - \frac{\epsilon}{2\,m}[\,p_2 - \epsilon\, J(t_2)\,]^2 - \epsilon \sum_{j=2}^{N-1} \frac{p_j^2}{2\,m}\,\right]\right\} \\
= & \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\,p_2\ \dots\ d\,p_{N-1} \\
& \times\ \delta[\,p_2 - p_3 + \epsilon\, J(t_3)\,] \dots \delta[\,p_{N-2} - p_{N-1} +\ \epsilon\, J(t_{N-1})\,] \\
& \times \exp\left\{\frac{i}{\hbar}\left[\,p_{N-1}\, q_N + \left[-p_2 + \epsilon \sum_{j=0}^{2} J(t_j)\right] q_0\right.\right. \\
& \left.\left. - \frac{\epsilon}{2\,m}\sum_{j=0}^{2}\left[\,p_2 - \epsilon \sum_{k=j+1}^{2} J(t_k)\right]^2 - \epsilon \sum_{j=3}^{N-1}\frac{p_j^2}{2\,m}\,\right]\right\}
\end{aligned}
$$

The pattern is now clear. Thus, after the p_{n-1} integration, we have

$$
\begin{aligned}
Z_J^0 = & \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\,p_n\ \dots\ d\,p_{N-1} \\
& \times\ \delta[\,p_n - p_{n+1} + \epsilon\, J(t_{n+1})\,] \dots \delta[\,p_{N-2} - p_{N-1} +\ \epsilon\, J(t_{N-1})\,] \\
& \times \exp\left(\frac{i}{\hbar}\left\{\,p_{N-1}\, q_N + \left[-p_n + \epsilon \sum_{j=0}^{n} J(t_j)\right] q_0\right.\right. \\
& \left.\left. - \frac{\epsilon}{2\,m}\sum_{j=0}^{n}\left[\,p_n - \epsilon \sum_{k=j+1}^{n} J(t_k)\right]^2 - \epsilon \sum_{j=n+1}^{N-1}\frac{p_j^2}{2\,m}\,\right\}\right)
\end{aligned}
$$

After the p_{N-2} integration, we have

$$
\begin{aligned}
Z_J^0 = & \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\,p_{N-1} \\
& \times \exp\left(\frac{i}{\hbar}\left\{\,p_{N-1}\, q_N + \left[-p_{N-1} + \epsilon \sum_{j=0}^{N-1} J(t_j)\right] q_0\right.\right. \\
& \left.\left. - \frac{\epsilon}{2\,m}\sum_{j=0}^{N-1}\left[\,p_{N-1} - \epsilon \sum_{k=j+1}^{N-1} J(t_k)\right]^2\,\right\}\right) \\
= & \exp\left[\frac{i}{\hbar}\, q_0\, \epsilon \sum_{j=0}^{N-1} J(t_j)\right]\ \int_{-\infty}^{\infty}\frac{d\,p_{N-1}}{2\pi\hbar} \qquad (3.18) \\
& \times \exp\left(\frac{i}{\hbar}\left\{\,p_{N-1}(q_N - q_0) - \frac{\epsilon}{2\,m}\sum_{j=0}^{N-1}\left[\,p_{N-1} - \epsilon \sum_{k=j+1}^{N-1} J(t_k)\right]^2\right\}\right)
\end{aligned}
$$

Setting $t = t_{j+1}$, we have

$$
\lim_{\epsilon\to 0} \epsilon \sum_{k=j+1}^{N-1} J(t_k) = \int_t^{t_b} d\,\tau\, J(\tau) = \int_{t_a}^{t_b} d\,\tau\, \theta(\tau - t)\, J(\tau)\ \equiv I(t) \qquad (3.19)
$$

$$\lim_{\epsilon\to 0}\epsilon\sum_{k=0}^{N-1}J(t_k)=\int_{t_a}^{t_b}d\,\tau\,J(\tau)=I\,(t_a) \qquad (3.19a)$$

With $p=p_{N-1}$, eq(3.18a) thus bcomes

$$Z_J^0=\exp\Big[\frac{i}{\hbar}\,q_0\,I\,(t_a)\Big]\int_{-\infty}^{\infty}\frac{d\,p}{2\,\pi\,\hbar} \qquad (3.20a)$$

$$\times\exp\left(\frac{i}{\hbar}\left\{p\,(q_b-q_a)-\frac{1}{2\,m}\int_{t_a}^{t_b}d\,t\,[\,p-I(t)\,]^2\right\}\right)$$

Expanding $[\,p-I(t)\,]^2$ & collecting terms, we have

$$Z_J^0=\exp\left\{\frac{i}{\hbar}\left[q_0\,I(t_a)-\frac{1}{2\,m}\int_{t_a}^{t_b}d\,t\,I(t)^2\right]\right\}\int_{-\infty}^{\infty}\frac{d\,p}{2\,\pi\,\hbar} \qquad (3.21)$$

$$\times\exp\left(\frac{i}{\hbar}\left\{-\frac{1}{2\,m}\,p^2(t_b-t_a)+p\left[q_b-q_a+\frac{1}{m}\int_{t_a}^{t_b}d\,t\,I(t)\right]\right\}\right)$$

Using eq(1.107)

$$\int_{-\infty}^{\infty}d\,x\,e^{-i\,\alpha\,x^2\pm i\,\beta\,x}=\sqrt{\frac{\pi}{i\,\alpha}}\,\exp\!\left(i\,\frac{\beta^2}{4\,\alpha}\right)$$

to evaluate the Gaussian integral , we set

$$\alpha=\frac{t_b-t_a}{2\,m\,\hbar} \qquad \beta=\frac{1}{\hbar}\left[q_b-q_a+\frac{1}{m}\int_{t_a}^{t_b}d\,t\,I(t)\right]$$

to get

$$Z_J^0=\exp\left\{\frac{i}{\hbar}\left[q_a\,I(t_a)-\frac{1}{2\,m}\int_{t_a}^{t_b}d\,t\,I(t)^2\right]\right\}$$

$$\times\sqrt{\frac{m}{2\,\pi\,i\,\hbar\,(t_b-t_a)}}\,\exp\!\left(\frac{i\,m}{2\,(t_b-t_a)\,\hbar}\left[q_b-q_a+\frac{1}{m}\int_{t_a}^{t_b}d\,t\,I(t)\right]^2\right)$$

$$=\sqrt{\frac{m}{2\,\pi\,i\,\hbar\,(t_b-t_a)}}\,\exp\left\{\frac{i}{\hbar}\left[q_a\,I(t_a)-\frac{1}{2\,m}\int_{t_a}^{t_b}d\,t\,I(t)^2+\frac{m\,(q_b-q_a)^2}{2\,(t_b-t_a)}\right.\right. \qquad (3.22)$$

$$\left.\left.+\frac{q_b-q_a}{t_b-t_a}\int_{t_a}^{t_b}d\,t\,I(t)+\frac{1}{2\,m\,(t_b-t_a)}\left(\int_{t_a}^{t_b}d\,t\,I(t)\right)^2\right]\right\}$$

Although eq(3.22) looks formidible, it can, in principle, be easily obtained using path integral techniques (see Ex. 3.2 & 3.3).

For a constant source $J(t)=f$,

$$I(t)=(t-t_a)\,f$$

$$\int_{t_a}^{t_b}d\,t\,I(t)=f\int_{t_a}^{t_b}d\,t\,(t-t_a)=f\int_0^{t_b-t_a}d\,\tau\,\tau=\frac{1}{2}\,f\,(t_b-t_a)^2$$

$$\int_{t_a}^{t_b}d\,t\,I(t)^2=f^2\int_{t_a}^{t_b}d\,t\,(t-t_a)^2=\frac{1}{3}\,f^2\,(t_b-t_a)^3$$

& eq(3.22) reduces to

$$Z_J^0 = \sqrt{\frac{m}{2\pi i \hbar (t_b - t_a)}} \exp\left\{\frac{i}{\hbar}\left[q_a(t_b - t_a) f - \frac{(t_b - t_a)^3}{6m} f^2\right.\right.$$
$$\left.\left. + \frac{m(q_b - q_a)^2}{2(t_b - t_a)} + \frac{1}{2}(q_b - q_a)(t_b - t_a) f + \frac{(t_b - t_a)^3}{8m} f^2\right]\right\}$$
$$= \sqrt{\frac{m}{2\pi i \hbar (t_b - t_a)}} \exp\left\{\frac{i}{\hbar}\left[\frac{m(q_b - q_a)^2}{2(t_b - t_a)}\right.\right. \tag{3.23}$$
$$\left.\left. + \frac{1}{2}(q_b + q_a)(t_b - t_a) f - \frac{(t_b - t_a)^3}{24m} f^2\right]\right\}$$

To simplify the notations, we write eq(3.16a) as

$$Z_{ab}^0[J] = \langle q_b, t_b \mid q_a, t_a \rangle_J = \int_{q_a}^{q_b} \mathcal{D} p \mathcal{D} q \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} dt\, \mathcal{L}_J^0(p, q)\right] \tag{3.24a}$$

$$\rightarrow \quad \frac{\hbar}{i}\frac{\delta Z_{ab}^0[J]}{\delta J(t)} = \int_{q_a}^{q_b} \mathcal{D} p \mathcal{D} q \left(\int_{t_a}^{t_b} d\tau \frac{\delta \mathcal{L}_J^0(p, q)}{\delta J(t)}\right) \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} d\tau\, \mathcal{L}_J^0(p, q)\right]$$
$$= \int_{q_a}^{q_b} \mathcal{D} p \mathcal{D} q\; q(t) \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} d\tau\, \mathcal{L}_J^0(p, q)\right]$$
$$= \int_{q_a}^{q_b} \mathcal{D} p \mathcal{D} q \exp\left[\frac{i}{\hbar}\int_{t_a}^{t} d\tau\, \mathcal{L}_J^0(p, q)\right] q(t) \exp\left[\frac{i}{\hbar}\int_{t}^{t_b} d\tau\, \mathcal{L}_J^0(p, q)\right]$$
$$= \left\langle q_b \mid e^{i H_J^0 (t_b - t)/\hbar} Q(t) e^{i H_J^0 (t - t_a)/\hbar} \mid q_a \right\rangle_J$$
$$= \langle q_b, t_b \mid Q(t) \mid q_a, t_a \rangle_J \tag{3.24}$$

Hence, expection value of $Q(t)$ with respect to the sourceless states is given by

$$\langle q_b, t_b \mid Q(t) \mid q_a, t_a \rangle = \frac{\hbar}{i}\frac{\delta Z_{ab}^0[J]}{\delta J(t)}\bigg|_{J=0} \tag{3.25a}$$

From eq(3.19a), we have

$$\frac{\delta I(t')}{\delta J(t)} = \int_{t_a}^{t_b} d\tau\, \theta(\tau - t')\, \delta(\tau - t)$$
$$= \theta(t - t') \tag{3.25}$$

Hence, eq(3.22) gives

$$\frac{\hbar}{i}\frac{\delta Z_{ab}^0[J]}{\delta J(t)} = Z_{ab}^0[J]\left\{q_a\, \theta(t - t_a) - \frac{1}{m}\int_{t_a}^{t_b} dt'\, I(t')\, \theta(t - t')\right. \tag{3.25b}$$
$$+ \frac{q_b - q_a}{t_b - t_a}\int_{t_a}^{t_b} dt'\, \theta(t - t')$$
$$\left. + \frac{1}{m(t_b - t_a)}\int_{t_a}^{t_b} dt''\, I(t'')\int_{t_a}^{t_b} dt'\, \theta(t - t')\right\}$$

Setting $J(t) = 0$ means all $I(t) = 0$.

$$\therefore \quad \frac{\hbar}{i}\frac{\delta Z_{ab}^0[J]}{\delta J(t)}\bigg|_{J=0} = Z_{ab}^0[0]\left\{q_a\, \theta(t - t_a) + \frac{q_b - q_a}{t_b - t_a}\int_{t_a}^{t_b} dt'\, \theta(t - t')\right\}$$

For $t_a < t < t_b$, we have

$$\langle q_b, t_b \mid Q(t) \mid q_a, t_a \rangle = Z_{ab}^0[0]\left\{q_a + \frac{q_b - q_a}{t_b - t_a}(t - t_a)\right\} \tag{3.26}$$

Eq(3.25b) has the form

$$Z = e^{i\mathbb{S}/\hbar} \qquad \to \qquad \frac{\hbar}{i}\frac{\delta Z}{\delta J} = Z\,\frac{\delta \mathbb{S}}{\delta J}$$

Therefore

$$\left(\frac{\hbar}{i}\right)^2 \frac{\delta^2 Z}{\delta J\,\delta J'} = Z\left(\frac{\delta \mathbb{S}}{\delta J}\,\frac{\delta \mathbb{S}}{\delta J'} + \frac{\hbar}{i}\,\frac{\delta^2 \mathbb{S}}{\delta J\,\delta J'}\right) \tag{3.26a}$$

Hence,

$$\begin{aligned}
\left(\frac{\hbar}{i}\right)^2 \frac{\delta^2 Z^0_{a\,b}[J]}{\delta J(s)\,\delta J(t)} = Z^0_{a\,b}[J]\Bigg\{ & q_a\,\theta\,(t - t_a) - \frac{1}{m}\int_{t_a}^{t_b} d\,t'\,I(t')\,\theta(t-t') \\
& + \frac{q_b - q_a}{t_b - t_a}\int_{t_a}^{t_b} d\,t'\,\theta(t-t') \\
& + \frac{1}{m(t_b - t_a)}\int_{t_a}^{t_b} d\,t''\,I(t'')\int_{t_a}^{t_b} d\,t'\,\theta(t-t')\Bigg\} \\
\times \Bigg\{ & q_a\,\theta\,(s - t_a) - \frac{1}{m}\int_{t_a}^{t_b} d\,t'\,I(t')\,\theta(s-t') \\
& + \frac{q_b - q_a}{t_b - t_a}\int_{t_a}^{t_b} d\,t'\,\theta(s-t') \\
& + \frac{1}{m(t_b - t_a)}\int_{t_a}^{t_b} d\,t''\,I(t'')\int_{t_a}^{t_b} d\,t'\,\theta(s-t')\Bigg\} \\
+ \frac{\hbar}{i}\,Z^0_{a\,b}[J]\Bigg\{ & -\frac{1}{m}\int_{t_a}^{t_b} d\,t'\,\theta(s-t')\,\theta(t-t') \\
& + \frac{1}{m(t_b - t_a)}\int_{t_a}^{t_b} d\,t''\,\theta\,(s-t'')\int_{t_a}^{t_b} d\,t'\,\theta(t-t')\Bigg\}
\end{aligned} \tag{3.27a}$$

$$\begin{aligned}
\therefore \qquad \left(\frac{\hbar}{i}\right)^2 \frac{\delta^2 Z^0_{a\,b}[J]}{\delta J(s)\,\delta J(t)}\Bigg|_{J=0} = Z^0_{a\,b}[0]\Bigg\{ & q_a\,\theta\,(t - t_a) + \frac{q_b - q_a}{t_b - t_a}\int_{t_a}^{t_b} d\,t'\,\theta(t-t')\Bigg\} \\
\times \Bigg\{ & q_a\,\theta(s - t_a) + \frac{q_b - q_a}{t_b - t_a}\int_{t_a}^{t_b} d\,t'\,\theta(s-t')\Bigg\} \\
+ \frac{\hbar}{i}\,Z^0_{a\,b}[0]\Bigg\{ & -\frac{1}{m}\int_{t_a}^{t_b} d\,t'\,\theta(s-t')\,\theta(t-t') \\
& + \frac{1}{m(t_b - t_a)}\int_{t_a}^{t_b} d\,t''\,\theta\,(s-t'')\int_{t_a}^{t_b} d\,t'\,\theta(t-t')\Bigg\}
\end{aligned}$$

For $t_a < s < t < t_b$, we have

$$\begin{aligned}
&\langle\, q_b, t_b \mid Q(t)\,Q(s) \mid q_a, t_a \,\rangle \\
&= Z^0_{a\,b}[0]\Bigg\{ \left[q_a + \frac{q_b - q_a}{t_b - t_a}\,(t - t_a)\right]\left[q_a + \frac{q_b - q_a}{t_b - t_a}\,(s - t_a)\right] \\
&\qquad + \frac{\hbar}{i}\left[-\frac{1}{m}\,(s - t_a) + \frac{1}{m(t_b - t_a)}\,(s - t_a)\,(t - t_a)\right]\Bigg\} \\
&= Z^0_{a\,b}[0]\Bigg\{ \left[q_a + \frac{q_b - q_a}{t_b - t_a}\,(t - t_a)\right]\left[q_a + \frac{q_b - q_a}{t_b - t_a}\,(s - t_a)\right] \\
&\qquad + i\,\hbar\,\frac{1}{m(t_b - t_a)}\,(t_b - t)\,(s - t_a)\Bigg\}
\end{aligned}$$

For $t_a < t < s < t_b$, we have

$$\langle q_b, t_b \mid Q(t)\, Q(s) \mid q_a, t_a \rangle$$
$$= Z^0_{ab}[0] \left\{ \left[q_a + \frac{q_b - q_a}{t_b - t_a}(t - t_a)\right]\left[q_a + \frac{q_b - q_a}{t_b - t_a}(s - t_a)\right] + \frac{\hbar}{i}\left[-\frac{1}{m}(t - t_a) + \frac{1}{m(t_b - t_a)}(s - t_a)(t - t_a)\right]\right\}$$
$$= Z^0_{ab}[0] \left\{ \left[q_a + \frac{q_b - q_a}{t_b - t_a}(t - t_a)\right]\left[q_a + \frac{q_b - q_a}{t_b - t_a}(s - t_a)\right] + i\hbar \frac{1}{m(t_b - t_a)}(t_b - s)(t - t_a)\right\}$$

Combining these 2 results, we have

$$\langle q_b, t_b \mid T[Q(s)\, Q(t)] \mid q_a, t_a \rangle$$
$$= Z^0_{ab}[0] \left\{ \left[q_a + \frac{q_b - q_a}{t_b - t_a}(t - t_a)\right]\left[q_a + \frac{q_b - q_a}{t_b - t_a}(s - t_a)\right] + \frac{i\hbar}{m(t_b - t_a)}\left[\theta(s - t)(t_b - s)(t - t_a) + \theta(t - s)(t_b - t)(s - t_a)\right]\right\} \quad (3.28)$$

$Z^0_{ab}[J]$ is therefore the generating functional for time-ordered products of a free particle.

Eq(3.28) contains a classical (quantum) term that's free of (proportional to) $\hbar$.
The quantum term is called the connected piece in perturbation theory (see Chap. 8).

In general, the classical term can be removed by using the generating functional for connected pieces

$$W_{ab}[J] = \frac{\hbar}{i} \ln Z_{ab}[J] \quad (3.29)$$

For example

$$\frac{\delta W_{ab}[J]}{\delta J(t)} = \frac{\hbar}{i} \frac{1}{Z_{ab}[J]} \frac{\delta Z_{ab}[J]}{\delta J(t)}$$
$$= \frac{1}{Z_{ab}[J]} \langle q_b, t_b \mid Q(t) \mid q_a, t_a \rangle_J$$
$$\equiv \langle Q(t) \rangle_J \quad (3.30)$$

is the expectation value of $Q(t)$ with source present.

$$\frac{\hbar}{i} \frac{\delta^2 W_{ab}[J]}{\delta J(s)\, \delta J(t)} = \left(\frac{\hbar}{i}\right)^2 \frac{\delta}{\delta J(s)} \left(\frac{1}{Z_{ab}[J]} \frac{\delta Z_{ab}[J]}{\delta J(t)}\right)$$
$$= \left(\frac{\hbar}{i}\right)^2 \left\{ \frac{1}{Z_{ab}[J]} \frac{\delta^2 Z_{ab}[J]}{\delta J(s)\, \delta J(t)} - \left(\frac{1}{Z_{ab}[J]}\right)^2 \frac{\delta Z_{ab}[J]}{\delta J(s)} \frac{\delta Z_{ab}[J]}{\delta J(t)} \right\} \quad (3.31)$$

Setting

$$\langle T\{ Q(s)\, Q(t) \} \rangle_J = \frac{1}{Z_{ab}[J]} \langle q_b, t_b \mid T\{Q(s)\, Q(t)\} \mid q_a, t_a \rangle_J$$
$$= \left(\frac{\hbar}{i}\right)^2 \frac{1}{Z_{ab}[J]} \frac{\delta^2 Z_{ab}[J]}{\delta J(s)\, \delta J(t)}$$

eq(3.31) becomes

$$\frac{\hbar}{i} \frac{\delta^2 W_{ab}[J]}{\delta J(s)\, \delta J(t)} = \langle T\{ Q(s)\, Q(t) \} \rangle_J - \langle Q(s) \rangle_J \langle Q(t) \rangle_J \quad (3.33)$$

Comparing with eq(3.28), we see that eq(3.33) contains only the connected pieces, just as claimed in the introduction of $W_{a\,b}[J]$ in eq(3.29). In fact, using eq(3.26a), it can be shown that this feature persists for all higher order derivatives for a general Lagrangian.

For the free particle case, eq(3.27a) shows that the connected piece in $\dfrac{\delta^2 W^0_{a\,b}[J]}{\delta J(s)\,\delta J(t)}$ is independent of J. Therefore,

$$\frac{\delta^n W^0_{a\,b}[J]}{\delta J(t_1)\ldots\delta J(t_n)} = 0 \qquad\qquad \forall\ n \geq 3$$

Consider now a more general Lagrangian

$$\mathcal{L}_J(p,\,q) = p\,\dot{q} - \frac{p^2}{2\,m} - V(q) + J(t)\,q \tag{3.34}$$

Caution: not all V can be successfully quantized so the following derivations apply only to those that can be.

$$Z_{a\,b}[J] = \langle\, q_b,\, t_b \mid q_a,\, t_a\,\rangle_J$$

$$= \int_{q_a}^{q_b} \mathcal{D}\,p\,\mathcal{D}\,q \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\left[p\,\dot{q} - \frac{p^2}{2\,m} - V(q) + J(t)\,q\right]\right\} \tag{3.35}$$

$$= \int_{q_a}^{q_b} \mathcal{D}\,p\,\mathcal{D}\,q\ \exp\left\{-\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,V(q)\right\}$$

$$\times\ \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\left[p\,\dot{q} - \frac{p^2}{2\,m} + J(t)\,q\right]\right\}$$

$$= \exp\left\{-\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,V\!\left(\frac{\hbar}{i}\,\frac{\delta}{\delta J(t)}\right)\right\}\int_{q_a}^{q_b} \mathcal{D}\,p\,\mathcal{D}\,q \tag{3.37}$$

$$\times\ \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\left[p\,\dot{q} - \frac{p^2}{2\,m} + J(t)\,q\right]\right\}$$

$$= \exp\left\{-\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,V\!\left(\frac{\hbar}{i}\,\frac{\delta}{\delta J(t)}\right)\right\} Z^0_{a\,b}[J] \tag{3.38}$$

By expanding $\exp\left\{-\frac{i}{\hbar}\int_{t_a}^{t_b} d\,t\,V\!\left(\frac{\hbar}{i}\,\frac{\delta}{\delta J(t)}\right)\right\}$ as a power series, eq(3.38) presents a perturbation series for calculating $Z_{a\,b}[J]$.

See Swanson’s text for fine points.