3.3. Continuum Techniques

We'll consider

$$\langle q_b, t_b \mid q_a, t_a \rangle = \int_{q_a}^{q_b} \overline{\mathcal{D}}\, q \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt\, \mathcal{L}(\dot q, q) \right\}$$

$$\mathcal{L}(\dot q, q) = \frac{1}{2} m \dot q^2 - V(q)$$

Let

$$q_j = q_c(t_j) + x_j \qquad (3.39)$$

with $q_a = q_c(t_a) \qquad q_b = q_c(t_b)$

→ $$x_a = x_b = 0 \qquad (3.39a)$$

$$d\, q_j = d\, x_j$$

∴ $$\langle q_b, t_b \mid q_a, t_a \rangle = \int_{q_a}^{q_b} \overline{\mathcal{D}}\, x \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt\, \mathcal{L}(\dot q_c, q_c, \dot x, x) \right\} \qquad (3.39b)$$

Using

$$\dot q = \dot q_c + \dot x$$

$$\dot q^2 = \dot q_c^2 + \dot x^2 + 2\, \dot q_c\, \dot x$$

$$V(q) = V(q_c) + x\, V'(q_c) + \frac{1}{2} x^2\, V''(q_c) + \dots$$

we have

$$\mathcal{L}(\dot q, q) = \frac{1}{2} m \dot q_c^2 - V(q_c) + \frac{1}{2} m \left(\dot x^2 + 2\, \dot q_c\, \dot x \right) - x\, V'(q_c) - \frac{1}{2} x^2\, V''(q_c) + \dots$$

Using

$$\mathcal{L}(\dot q_c, q_c) = \frac{1}{2} m \dot q_c^2 - V(q_c)$$

→ $$\frac{\partial \mathcal{L}(\dot q_c, q_c)}{\partial \dot q_c} = m\, \dot q_c \qquad \frac{\partial \mathcal{L}(\dot q_c, q_c)}{\partial q_c} = -V'(q_c)$$

we have

$$\mathcal{L}(\dot q, q) = \mathcal{L}(\dot q_c, q_c) + \dot x \frac{\partial \mathcal{L}(\dot q_c, q_c)}{\partial \dot q_c} + x \frac{\partial \mathcal{L}(\dot q_c, q_c)}{\partial q_c} + \mathcal{L}_R(\dot x, x) \qquad (3.40a)$$

where

$$\mathcal{L}_R(\dot x, x) = \frac{1}{2} m \dot x^2 - \frac{1}{2} x^2\, V''(q_c) + \dots \qquad (3.40b)$$

contains terms quadratic or higher in x or $\dot x$. Therefore,

$$\mathcal{L}_R(\dot x, x) = \mathcal{L}(\dot x, x) \qquad (3.40c)$$

if V is quadratic or less.

$$S = \int_{t_a}^{t_b} dt \left[\mathcal{L}(\dot q_c, q_c) + \dot x \frac{\partial \mathcal{L}(\dot q_c, q_c)}{\partial \dot q_c} + x \frac{\partial \mathcal{L}(\dot q_c, q_c)}{\partial q_c} + \mathcal{L}_R(\dot x, x) \right] \qquad (3.40)$$

Using eq(3.39a), we have

$$\int_{t_a}^{t_b} dt\, \dot x \frac{\partial \mathcal{L}(\dot q_c, q_c)}{\partial \dot q_c} = -\int_{t_a}^{t_b} dt\, x \frac{d}{dt} \frac{\partial \mathcal{L}(\dot q_c, q_c)}{\partial \dot q_c}$$

$$= -\int_{t_a}^{t_b} dt\, x \frac{\partial \mathcal{L}(\dot q_c, q_c)}{\partial q_c} \qquad \text{[Euler-Lagrange eq. used.]}$$

Eq(3.40) thus reduces to

$$S = \int_{t_a}^{t_b} dt \left[\mathcal{L}(\dot{q}_c, q_c) + \mathcal{L}_R(\dot{x}, x)\right] \tag{3.41}$$

while eq(3.39b) becomes

$$\langle q_b, t_b \mid q_a, t_a \rangle = \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt\, \mathcal{L}(\dot{q}_c, q_c) \right\} \times \int_0^0 \overline{\mathcal{D}}\, x \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt\, \mathcal{L}_R(\dot{x}, x) \right\} \tag{3.42}$$

Quadratic Lagrangians

In the special case where $\mathcal{L}$ is quadratic, eq(3.40c) gives

$$\begin{aligned}\langle q_b, t_b \mid q_a, t_a \rangle &= \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt\, \mathcal{L}(\dot{q}_c, q_c) \right\} \\ &\quad \times \int_0^0 \overline{\mathcal{D}}\, x \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt\, \mathcal{L}(\dot{x}, x) \right\} \\ &= e^{i S_c/\hbar} \left\langle 0, t_b \,\middle|\, 0, t_a \right\rangle \end{aligned} \tag{3.43}$$

where $\langle 0, t_b \mid 0, t_a \rangle$ is just a function of t_b & t_a and

$$S_c = \int_{t_a}^{t_b} dt\, \mathcal{L}(\dot{q}_c, q_c)$$

is the action evaluated along the classical path.

In evaluating $\langle 0, t_b \mid 0, t_a \rangle$, we notice that x is periodic with period $T = t_b - t_a$. Hence, we introduce the Fourier series

$$x_j = x(t_j) = \sum_{n=1}^{N-1} a_n \sin \frac{n\pi}{T} (t_j - t_a) \tag{3.45}$$

so that

$$x(t_b) = x(t_a) = 0$$

Hence,

$$d x_1 \ldots d x_{N-1} = J\, d a_1 \ldots d a_{N-1}$$

$$\begin{aligned}\overline{\mathcal{D}}\, x &= \left(\frac{N m}{2 \pi i \hbar T}\right)^{N/2} d x_1 \ldots d x_{N-1} \\ &= \left(\frac{N m}{2 \pi i \hbar T}\right)^{N/2} J\, d a_1 \ldots d a_{N-1} = \mathcal{D}\, a\end{aligned}$$

where

$$\begin{aligned}J &= \frac{\partial(x_1, \ldots, x_{N-1})}{\partial(a_1, \ldots a_{N-1})} = \det \left| \frac{\partial x_j}{\partial a_n} \right| \\ &= \det \left| \sin \frac{n\pi}{T} (t_j - t_a) \right| \end{aligned} \tag{3.46}$$

is the Jacobian of the transformation.

Instead of a direct calculation of J, we'll inferred it indirectly.

Since $\mathcal{L}$ is quadratic, we have

$$S_R = \int_{t_a}^{t_b} dt\, \mathcal{L}_R(\dot{x}, x) = \int_{t_a}^{t_b} dt\, \mathcal{L}(\dot{x}, x)$$

$$= \int_{t_a}^{t_b} d\,t \frac{1}{2} m \dot{x}^2$$

$$= \int_{t_a}^{t_b} d\,t \frac{1}{2} m \sum_{k,n=1}^{N-1} a_k a_n \frac{n k \pi^2}{T^2} \cos \frac{n\pi}{T}(t-t_a) \cos \frac{k\pi}{T}(t-t_a)$$

$$= \frac{\pi}{T} \int_0^{\pi} d\,\theta \frac{1}{2} m \sum_{k,n=1}^{N-1} a_k a_n n k \cos n\,\theta \cos k\,\theta$$

where

$$\theta = \frac{\pi}{T}(t-t_a)$$

Using

$$\int_0^{\pi} d\,\theta \cos n\,\theta \cos k\,\theta = \frac{\pi}{2} \delta_{n k}$$

we have

$$S_R = \frac{m \pi^2}{4T} \sum_{n=1}^{N-1} n^2 a_n^2 \qquad (3.47)$$

Using eq(3.4), we have

$$\langle 0, t_b \mid 0, t_a \rangle = \sqrt{\frac{m}{2\pi i \hbar T}}$$

$$= \int_0^0 \overline{\mathcal{D}} a \exp\left\{ \frac{i}{\hbar} S_R \right\}$$

$$= \left(\frac{N m}{2 \pi i \hbar T} \right)^{N/2} \int_0^{\infty} d\,a_1 \ldots d\,a_{N-1} J \exp\left\{ \frac{i}{\hbar} \frac{m\pi^2}{4T} \sum_{n=1}^{N-1} n^2 a_n^2 \right\}$$

Since

$$\int_0^{\infty} d\,a_n \exp\left\{ \frac{i}{\hbar} \frac{m\pi^2}{4T} n^2 a_n^2 \right\} = \sqrt{\frac{2 i \hbar T}{m \pi n^2}}$$

we have

$$\sqrt{\frac{m}{2\pi i \hbar T}} = J \left(\frac{N m}{2\pi i \hbar T} \right)^{N/2} \left(\frac{2 i \hbar T}{m \pi} \right)^{(N-1)/2} \prod_{n=1}^{N-1} \frac{1}{n}$$

$$= J \left(\frac{m}{2\pi i \hbar T} \right)^{1/2} \frac{N^{N/2}}{2\,\pi^{N-1} (N-1)!}$$

$\rightarrow \qquad J = N^{-N/2} \pi^{N-1} (N-1)! \qquad (3.50)$

which, in view of eq(3.46), is surprisely independent of T !

Harmonic Oscillator

Another quadratic Lagrangian is that of a harmonic oscillator.

$$\mathcal{L} = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \qquad (3.51)$$

with

$$q_c(t) = A \sin[\omega (t - t_a) + \delta] \qquad (3.52)$$

$$\dot{q}_c(t) = \omega A \cos[\omega (t - t_a) + \delta]$$

Imposing the B.C.

$$q_c(t_a) = q_a \qquad\qquad q_c(t_b) = q_b$$

we have

$$q_a = A \sin\delta \qquad\qquad q_b = A\sin(\omega T + \delta)$$

$$\rightarrow \qquad A = \frac{q_a}{\sin\delta} \tag{3.53a}$$

$$\& \qquad q_b = A(\sin\omega T\cos\delta + \cos\omega T\sin\delta)$$
$$= q_a(\sin\omega T\cot\delta + \cos\omega T)$$

$$\therefore \qquad \cot\delta = \frac{q_b}{q_a\sin\omega T} - \cot\omega T \tag{3.53b}$$
$$= \frac{q_b - q_a\cos\omega T}{q_a\sin\omega T}$$

Thus,

$$q_c(t) = \frac{q_a}{\sin\delta}\left[\sin\omega(t-t_a)\cos\delta + \cos\omega(t-t_a)\sin\delta\right]$$
$$= q_a\left[\sin\omega(t-t_a)\,\frac{q_b - q_a\cos\omega T}{q_a\sin\omega T} + \cos\omega(t-t_a)\right]$$
$$= \frac{1}{\sin\omega T}\left[\sin\omega(t-t_a)(q_b - q_a\cos\omega T) + q_a\sin\omega T\cos\omega(t-t_a)\right]$$
$$= \frac{1}{\sin\omega T}\left[q_b\sin\omega(t-t_a) + q_a\sin\omega(t_b-t)\right] \tag{3.53c}$$

$$\rightarrow \qquad \dot{q}_c(t) = \frac{\omega}{\sin\omega T}\left[q_b\cos\omega(t-t_a) - q_a\cos\omega(t_b-t)\right] \tag{3.53d}$$

$$\therefore \qquad \mathcal{L}_c = \frac{1}{2}m\dot{q}_c^2 - \frac{1}{2}m\omega^2 q_c^2$$
$$= \frac{m\omega^2}{2\sin^2\omega T}\left[q_b^2\cos 2\omega(t-t_a) + q_a^2\cos 2\omega(t_b-t) - 2q_a q_b\cos\omega(2t-t_b-t_a)\right]$$

$$\rightarrow \qquad S_c = \int_{t_a}^{t_b} dt\,\mathcal{L}_c$$
$$= \frac{m\omega}{4\sin^2\omega T}\left(q_b^2\sin 2\omega T + q_a^2\sin 2\omega T - 4q_a q_b\sin\omega T\right)$$
$$= \frac{m\omega}{2\sin\omega T}\left[\left(q_b^2 + q_a^2\right)\cos\omega T - 2q_a q_b\right] \tag{3.54}$$

With reference to eq(3.43), the next step is to calculate $\langle 0, t_b \mid 0, t_a\rangle$ for the oscillator. Following the procedure depicted in eqs(3.45-50), we have

$$x_j = x(t_j) = \sum_{n=1}^{N-1} a_n \sin\frac{n\pi}{T}(t_j - t_a)$$

with $\quad x(t_b) = x(t_a) = 0$

In evaluating

$$S_R = \int_{t_a}^{t_b} dt\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2\right)$$

the kinetic energy term is given by eq(3.47). For the potential term, we have

$$-\frac{1}{2}m\omega^2\int_{t_a}^{t_b} dt\sum_{k,n=1}^{N-1} a_k a_n \sin\frac{n\pi}{T}(t-t_a)\sin\frac{k\pi}{T}(t-t_a)$$

$$= -\frac{m\,\omega^2\,T}{2\,\pi}\int_0^{\pi} d\,\theta \sum_{k,\,n=1}^{N-1} a_k\,a_n \sin n\,\theta\,\sin k\,\theta$$

$$= -\frac{m\,\omega^2\,T}{4}\sum_{n=1}^{N-1} a_n^2$$

$$\rightarrow \qquad S_R = \frac{m}{4}\sum_{n=1}^{N-1}\left(\frac{n^2\,\pi^2}{T} - \omega^2\,T\right) a_n^2 \qquad (3.55)$$

$$\therefore \qquad \langle\, 0,\, t_b \mid 0,\, t_a \,\rangle = \left(\frac{N\,m}{2\,\pi\,i\,\hbar\,T}\right)^{N/2}\int_{-\infty}^{\infty} d\,a_1 \ldots d\,a_{N-1}\,J \qquad (3.56)$$

$$\times\, \exp\left\{\frac{i}{\hbar}\,\frac{m}{4}\sum_{n=1}^{N-1}\left(\frac{n^2\,\pi^2}{T} - \omega^2\,T\right) a_n^2\right\}$$

where J is given by eq(3.50).

Eq(3.56) demonstrates the pitfall of real-time path integrals. For $\omega = \frac{n\,\pi}{T}$, the exponent vanishes & the integral for a_n diverges. The remedy is to make a switch to the imaginary time $\mathbb{T} = i\,T$ so that

$$\langle\, 0,\, t_b \mid 0,\, t_a \,\rangle = \left(\frac{N\,m}{2\,\pi\,\hbar\,\mathbb{T}}\right)^{N/2}\int_{0}^{\infty} d\,a_1 \ldots d\,a_{N-1}\,J \qquad (3.57)$$

$$\times\, \exp\left\{-\frac{m}{4\,\hbar}\sum_{n=1}^{N-1}\left(\frac{n^2\,\pi^2}{\mathbb{T}} + \omega^2\,\mathbb{T}\right) a_n^2\right\}$$

$$= \left(\frac{N\,m}{2\,\pi\,\hbar\,\mathbb{T}}\right)^{N/2} N^{-N/2}\,\pi^{N-1}(N-1)!\left(\frac{2\,\pi\,\hbar\,\mathbb{T}}{m}\right)^{(N-1)/2}\prod_{n=1}^{N-1}\frac{1}{\sqrt{n^2\,\pi^2 + \omega^2\,\mathbb{T}^2}}$$

$$= \left(\frac{m}{2\,\pi\,\hbar\mathbb{T}}\right)^{1/2}\pi^{N-1}(N-1)!\prod_{n=1}^{N-1}\frac{1}{\sqrt{n^2\,\pi^2 + \omega^2\,\mathbb{T}^2}}$$

$$= \left(\frac{m}{2\,\pi\,\hbar\mathbb{T}}\right)^{1/2}\prod_{n=1}^{N-1}\frac{1}{\sqrt{1 + \frac{\omega^2\,\mathbb{T}^2}{n^2\,\pi^2}}} \qquad (3.58)$$

Using [see I.S.Gradshteyn & I.M.Ryzhik, “Table of Integrals, Series, & Products”, eq(1.431.2)]

$$\sinh x = x\prod_{n=1}^{\infty}\left(1 + \frac{x^2}{n^2\,\pi^2}\right) \qquad (3.59)$$

we have, as $N \to \infty$,

$$\langle\, 0,\, t_b \mid 0,\, t_a \,\rangle = \left(\frac{m}{2\,\pi\,\hbar\,\mathbb{T}}\right)^{1/2}\sqrt{\frac{\omega\,\mathbb{T}}{\sinh\omega\,\mathbb{T}}}$$

$$= \sqrt{\frac{m\,\omega}{2\,\pi\,\hbar\,\sinh\omega\,\mathbb{T}}} \qquad (3.60)$$

$$= \sqrt{\frac{m\,\omega}{2\,\pi\,\hbar\,i\,\sinh\omega\,T}} \qquad (3.61)$$

See Swanson’s text for re-phrasing the fore-going calculations in more fancy terms like Wick’s rotation & anaytic continuation.

Harmonic Oscillator with Source

Another quadratic Lagrangian is that of a harmonic oscillator with a t-dependent source.

$$\mathcal{L} = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 + J(t)\, q \qquad (3.62a)$$

The Euler-Lagrange eq. is easily found to be

$$\ddot{q} + \omega^2 q = \frac{J(t)}{m} \qquad (3.26b)$$

with B.C.

$$q(t_a) = q_a \qquad q(t_b) = q_b \qquad (3.26c)$$

Define the green's function by

$$\ddot{\Delta}(t, s) + \omega^2 \Delta(t, s) = -\delta(t - s)$$

with B.C.

$$\Delta(t_b, s) = \Delta(t_a, s) = 0$$

The solution to eq(3.26b) is

$$q_c(t) = q_h(t) + q_p(t)$$

where q_h is the homogeneous solution satisfying the B.C. eq(3.26c). Using eq(3.53c), we have

$$q_h(t) = \frac{1}{\sin \omega T} \left[q_b \sin \omega\,(t - t_a) + q_a \sin \omega\,(t_b - t) \right] \qquad (3.26d)$$

q_p is the particular solution given by

$$q_p(t) = -\frac{1}{m} \int_{t_a}^{t_b} d\,s\, \Delta(t, s)\, J(s) \qquad (3.26e)$$

with B.C.

$$q_p\left(t_b\right) = q_p\left(t_a\right) = 0 \qquad (3.26f)$$

According to the recipe given by G.Arfken, "Mathematical Methods for Physicists",

$$\Delta(t, s) = \begin{cases} \alpha \sin \omega(t - t_a) & \text{for } \; t_a < t < s \\ \beta \sin \omega(t_b - t) & \text{for } \; s < t < t_b \end{cases}$$

Continuity of Δ at $t = s$ gives

$$\alpha \sin \omega(s - t_a) = \beta \sin \omega(t_b - s)$$

Discontinuity of $\dot{\Delta}$ at $t = s$ gives

$$-\beta\, \omega \cos \omega\,(t_b - s) - \alpha\, \omega \cos \omega\,(s - t_a) = -1$$

$$\rightarrow \qquad \beta = \frac{1}{\omega\,[\cos \omega\,(t_b - s) + \cot \omega\,(s - t_a) \sin \omega(t_b - s)\,]}$$

$$= \frac{\sin \omega\,(s - t_a)}{\omega\,[\sin \omega\,(s - t_a) \cos \omega\,(t_b - s) + \cos \omega\,(s - t_a) \sin \omega(t_b - s)\,]}$$

$$= \frac{\sin \omega\,(s - t_a)}{\omega \sin \omega\, T}$$

$$\alpha = \frac{\sin \omega(t_b - s)}{\omega \sin \omega\, T}$$

$$\therefore \qquad \Delta(t,\ s) = \begin{cases} \dfrac{\sin\omega(t-t_a)\sin\omega(t_b-s)}{\omega\sin\omega T} & \text{for } t_a < t < s \\ \dfrac{\sin\omega(t_b-t)\sin\omega(s-t_a)}{\omega\sin\omega T} & \text{for } s < t < t_b \end{cases} \qquad (3.26g)$$

$$\mathcal{L}_c = \mathcal{L}_h + J(t)\, q_h + \frac{1}{2} m\, \dot{q}_p^2 - \frac{1}{2} m\, \omega^2 q_p^2 + J(t)\, q_p$$

where

$$\mathcal{L}_h = \frac{1}{2} m\, \dot{q}_h^2 - \frac{1}{2} m\, \omega^2 q_h^2$$

Using the B.C. eq(3.26f) & integrating by part, we have

$$\int_{t_a}^{t_b} d\,t\ \dot{q}_p^2 = -\int_{t_a}^{t_b} d\,t\ q_p\, \ddot{q}_p$$

$$= -\int_{t_a}^{t_b} d\,t\ q_p \left[-\omega^2 q_p + \frac{J(t)}{m} \right]$$

$$\rightarrow \qquad S_c = \int_{t_a}^{t_b} d\,t\ \mathcal{L}_c = \int_{t_a}^{t_b} d\,t \left\{ \mathcal{L}_h + J(t)\, q_h + \frac{1}{2} J(t)\, q_p \right\} \qquad (3.26h)$$

From eq(3.54) we have

$$\int_{t_a}^{t_b} d\,t\ \mathcal{L}_h = \frac{m\,\omega}{2\sin\omega T} \left[\left(q_b^2 + q_a^2 \right) \cos\omega T - 2\, q_a\, q_b \right]$$

Using eq(3.26d), we have

$$\int_{t_a}^{t_b} d\,t\ J(t)\, q_h = \frac{1}{\sin\omega T} \int_{t_a}^{t_b} d\,t\ J(t) \left[q_b \sin\omega\,(t - t_a) + q_a \sin\omega\,(t_b - t) \right]$$

Using eq(3.26e) gives

$$\frac{1}{2} \int_{t_a}^{t_b} d\,t\ J(t)\, q_p = -\frac{1}{2\,m} \int_{t_a}^{t_b} d\,t\ d\,s\ J(t)\, \Delta(t,\ s)\, J(s)$$

$$= -\frac{1}{m} \int_{t_a}^{t_b} d\,t\ d\,s\ \theta(t-s)\, J(t)\, \Delta(t,\ s)\, J(s)$$

where we've made use of the general symmetry of a green's function $\Delta(t,\ s) = \Delta(s,\ t)$.

Using eq(3.26g), we have

$$\frac{1}{2} \int_{t_a}^{t_b} d\,t\ J(t)\, q_p$$

$$= -\frac{1}{m\,\omega\sin\omega T} \int_{t_a}^{t_b} d\,t\ d\,s\ \theta(t-s)\, J(t)\, J(s) \sin\omega(t_b - t) \sin\omega\,(s - t_a)$$

Finally, putting these results into eq(3.26h) gives Swanson's eq(3.62).