3.4. Topological Measure

Consider the path integral of a point mass moving on a circle of radius R [see eq(2.71) of §2.5]

$$\langle \theta_b, t_b \mid \theta_a, t_a \rangle = \int_0^{2\pi} d\,\theta_1 \dots d\,\theta_{N-1} \left(\frac{1}{2\pi} \right)^N \sum_{n_0=-\infty}^{\infty} \dots \sum_{n_{N-1}=-\infty}^{\infty} \tag{3.63}$$

$$\times \exp\left\{ \frac{i}{\hbar} \epsilon \left[\sum_{j=0}^{N-1} n_j \hbar \frac{\theta_{j+1} - \theta_j}{\epsilon} - \frac{n_j^2 \hbar^2}{2 m R^2} - V(\theta_j) \right] \right\}$$

with

$$\theta_0 = \theta_a \qquad \qquad \theta_N = \theta_b$$

Our concern here is how eq(3.63) is equivalent to

$$\langle q_b, t_b \mid q_a, t_a \rangle = \int_{q_a}^{q_b} \overline{\mathcal{D}}\, q\, e^{i S/\hbar} \tag{3.63a}$$

with [see eq(2.64)]

$$\mathcal{L}\left(\dot{\theta}, \theta\right) = \frac{1}{2} m R^2 \dot{\theta}^2 - V(\theta) \tag{3.64}$$

If V is absent, the θ integrals can be performed & the results directly compared.
For $V \neq 0$, we need to do the n summations.

To this end, we notice that the Jacobian theta function of the 3rd kind is defined as

$$\Theta_3(A \mid s) = \sum_{n=-\infty}^{\infty} \exp\left(-i \pi A n^2 + 2 \pi i n s\right) \tag{3.65}$$

which is directly applicable to eq(3.63).

The next step, which is called the Poisson resummation technique, is to rewrite eq(3.65) as

$$\Theta_3(A \mid s) = \frac{1}{\sqrt{i A}} \sum_{j=-\infty}^{\infty} \exp\left[i \frac{\pi}{A} (s + j)^2 \right] \tag{3.66}$$

Proof of eq(3.66) is as follows.

For a periodic function f of period L, eq(1.15) says

$$\int_{-\infty}^{\infty} d\,x f(x)\, \delta(x - y) = \sum_{n=-\infty}^{\infty} f(y + 2 n L)$$

Setting $y = 0$ & $2L = 1$, we have

$$\int_{-\infty}^{\infty} d\,x f(x)\, \delta(x) = \sum_{n=-\infty}^{\infty} f(n)$$

$$= \int_{-\infty}^{\infty} d\,x f(x) \sum_{j=-\infty}^{\infty} e^{i 2 \pi j x} \tag{3.67}$$

Since we're working with the same function f in the above expressions,

$$f(x) = f(n) \Big|_{n=x}$$

Eq(3.65) thus becomes

$$\Theta_3(A \mid s) = \int_{-\infty}^{\infty} d\,x \exp\left(-i \pi A x^2 + 2 \pi i x s\right) \sum_{j=-\infty}^{\infty} e^{i 2 \pi j x}$$

$$= \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} d\,x \exp\left[-i \pi A x^2 + 2 \pi i (s + j) x\right] \tag{3.68}$$

Using [see eq(1.107)]

$$\int_{-\infty}^{\infty} d\,x\, e^{-i\,\alpha\,x^2 \pm i\,\beta\,x} = \sqrt{\frac{\pi}{i\,\alpha}}\ \exp\left(i\,\frac{\beta^2}{4\,\alpha}\right)$$

we set

$$\alpha = \pi\,A \qquad \beta = 2\,\pi\,(s+j) \qquad \to \qquad \frac{\beta^2}{4\,\alpha} = \frac{\pi\,(s+j)^2}{A}$$

$$\therefore \qquad \Theta_3(A \mid s) = \sqrt{\frac{1}{i\,A}}\ \sum_{j=-\infty}^{\infty} \exp\left[i\,\frac{\pi\,(s+j)^2}{A}\right] \qquad \text{QED}$$

Combining eqs(3.65-6), the Poisson resummation means

$$\sum_{n=-\infty}^{\infty} \exp\left(-i\,\pi\,A\,n^2 + 2\,\pi\,i\,n\,s\right) = \frac{1}{\sqrt{i\,A}}\ \sum_{n=-\infty}^{\infty} \exp\left[i\,\frac{\pi\,(s+n)^2}{A}\right] \qquad (3.68a)$$

Applying this to eq(3.63), we set

$$A = \frac{\epsilon\,\hbar}{2\,\pi\,m\,R^2} \qquad s = \frac{\theta_{j+1}-\theta_j}{2\,\pi}$$

$$\to \qquad \langle\,\theta_b,\,t_b \mid \theta_a,\,t_a\,\rangle = \int_0^{2\pi} d\,\theta_1 \ldots d\,\theta_{N-1} \left(\frac{1}{2\,\pi}\right)^N \left(\frac{2\,\pi\,m\,R^2}{i\,\epsilon\,\hbar}\right)^{N/2} \sum_{n_0=-\infty}^{\infty} \cdots \sum_{n_{N-1}=-\infty}^{\infty}$$

$$\times \exp\left\{\sum_{j=0}^{N-1}\left[i\,\pi\,\frac{2\,\pi\,m\,R^2}{\epsilon\,\hbar}\left(\frac{\theta_{j+1}-\theta_j}{2\,\pi} + n_j\right)^2 - \frac{i}{\hbar}\,\epsilon\,V(\theta_j)\right]\right\}$$

$$= \int_0^{2\pi} d\,\theta_1 \ldots d\,\theta_{N-1} \left(\frac{m\,R^2}{2\,\pi\,i\,\epsilon\,\hbar}\right)^{N/2} \sum_{n_0=-\infty}^{\infty} \cdots \sum_{n_{N-1}=-\infty}^{\infty} \qquad (3.69)$$

$$\times \exp\left\{\frac{i}{\hbar}\sum_{j=0}^{N-1}\left[\frac{m\,R^2}{2\,\epsilon}\left(\theta_{j+1}-\theta_j+2\,\pi\,n_j\right)^2 - \epsilon\,V(\theta_j)\right]\right\}$$

For the θ_1 integral, we need to include the $j = 0\ \&\ 1$ terms:

$$\mathcal{I}_1 = \sum_{n_0=-\infty}^{\infty}\sum_{n_1=-\infty}^{\infty}\int_0^{2\pi} d\,\theta_1 \exp\left\{\frac{i}{\hbar}\left[\frac{m\,R^2}{2\,\epsilon}\left(\theta_1-\theta_0+2\,\pi\,n_0\right)^2 + \frac{m\,R^2}{2\,\epsilon}\left(\theta_2-\theta_1+2\,\pi\,n_1\right)^2 - \epsilon\,V(\theta_1)\right]\right\}$$

Now

$$\int_{-\infty}^{\infty} d\,\theta\,f(\theta) = \sum_{n=-\infty}^{\infty}\int_{2\pi n}^{2\pi(n+1)} d\,\theta\ f(\theta)$$

$$= \sum_{n=-\infty}^{\infty}\int_0^{2\pi} d\,(\theta+2\,\pi\,n)\ f(\theta+2\,\pi\,n)$$

$$= \sum_{n=-\infty}^{\infty}\int_0^{2\pi} d\,\theta\ f(\theta+2\,\pi\,n)$$

$$= \sum_{n=-\infty}^{\infty}\int_0^{2\pi} d\,\theta\ f(\theta) \qquad \text{if } f \text{ is periodic with period } 2\,\pi.$$

Since V is periodic with period $2\,\pi$, we have

$$\mathcal{I}_1 = \sum_{n_0=-\infty}^{\infty} \int_{-\infty}^{\infty} d\,\theta_1 \exp\left(\frac{i}{\hbar} \left\{ \frac{m R^2}{2\epsilon} \left[(\theta_1 - \theta_0 + 2\pi n_0)^2 + (\theta_2 - \theta_1)^2 \right] - \epsilon \left[V(\theta_0) + V(\theta_1) \right] \right\} \right) \tag{3.70}$$

where we've also renamed $n_0 + n_1$ as n_0.

Next, we include the θ_2 integration & get

$$\mathcal{I}_2 = \sum_{n_0=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \int_{-\infty}^{\infty} d\,\theta_1 \int_0^{2\pi} d\,\theta_2 \exp\left(\frac{i}{\hbar} \left\{ \frac{m R^2}{2\epsilon} \left[(\theta_1 - \theta_0 + 2\pi n_0)^2 + (\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2 + 2\pi n_2)^2 \right] - \epsilon \left[V(\theta_0) + V(\theta_1) + V(\theta_2) \right] \right\} \right)$$

Using the same extension trick on θ_2, we get

$$\mathcal{I}_2 = \sum_{n_0=-\infty}^{\infty} \int_{-\infty}^{\infty} d\,\theta_1\, d\,\theta_2 \exp\left(\frac{i}{\hbar} \left\{ \frac{m R^2}{2\epsilon} \left[(\theta_1 - \theta_0 + 2\pi n_0)^2 + (\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2 \right] - \epsilon \left[V(\theta_0) + V(\theta_1) + V(\theta_2) \right] \right\} \right) \tag{3.71}$$

Continuing the same treatment with the rest of the θ_j's, eq(3.69) becomes

$$\langle \theta_b, t_b \mid \theta_a, t_a \rangle = \int_{-\infty}^{\infty} d\,\theta_1 \ldots d\,\theta_{N-1} \left(\frac{m R^2}{2\pi i \epsilon \hbar} \right)^{N/2} \sum_{n_0=-\infty}^{\infty} \times \exp\left\{ \frac{i}{\hbar} \left[\frac{m R^2}{2\epsilon} (\theta_1 - \theta_a + 2\pi n_0)^2 - \epsilon V(\theta_0) \right] \right\} \times \exp\left\{ \frac{i}{\hbar} \sum_{j=1}^{N-1} \left[\frac{m R^2}{2\epsilon} (\theta_{j+1} - \theta_j)^2 - \epsilon V(\theta_j) \right] \right\}$$

Setting a new set of B.C.'s

$$\theta_0 = \theta_a - 2\pi n \qquad \theta_N = \theta_b$$

we have

$$\langle \theta_b, t_b \mid \theta_a, t_a \rangle = \sum_{n=-\infty}^{\infty} \int_{\substack{-\infty \\ \theta_0 = \theta_a - 2\pi n \\ \theta_N = \theta_b}}^{\infty} d\,\theta_1 \ldots d\,\theta_{N-1} \left(\frac{m R^2}{2\pi i \epsilon \hbar} \right)^{N/2} \times \exp\left\{ \frac{i}{\hbar} \sum_{j=0}^{N-1} \left[\frac{m R^2}{2\epsilon} (\theta_{j+1} - \theta_j)^2 - \epsilon V(\theta_j) \right] \right\} \tag{3.72}$$

which is indeed of the form eq(3.63a) except for the extra sum over different B.C.s.

This can be traced to the topology of the circle on which each n in eq(3.72) describes a path belonging to a different homotopy class.

The circle is a relatively simple manifold on which paths can be classified simply by their winding numbers. For a general manifold M, one needs its homotopy group $\pi_n(M)$, where n is the dimension of the closed geometric object used to probe M.

Some basic results in homotopy theory are,

$$\pi_n(S_n) = \mathbb{Z} \qquad \pi_n(S_m) = 0 \quad \forall\; n > m \qquad \pi_n(S_1) = 0 \quad \forall\; n > 1 \tag{3.73}$$

See T.Frankel, “The Geometry of Physics”,Chaps 22.

See Swanson’s text for further discussions related to path integrals

For $V = 0$, one can use eq(3.4) to evaluate eq(3.72) to get

$$\langle \theta_b, t_b \mid \theta_a, t_a \rangle = \left(\frac{m R^2}{2 \pi i \hbar (t_b - t_a)} \right)^{1/2} \times \sum_{n=-\infty}^{\infty} \exp\left\{ \frac{i}{\hbar} \left[\frac{m R^2}{2 (t_b - t_a)} (\theta_b - \theta_a + 2 \pi n)^2 \right] \right. \tag{3.74}$$