4.1. Natural Units

$$\hbar = 1.0545\times10^{-27}\text{ erg-s } = 1.0545\times10^{-34}\text{ J-s}$$
$$c = 2.9979\times10^{10}\text{ cm/s } = 2.9979\times10^{8}\text{ m/s}$$

Natural units:

$$\hbar = c = 1$$

Since 2 of the 3 fundamental units are fixed, all dynamical quantities are measured in 1 unit, say, $[\text{length}] = \text{cm}$, or $[\text{energy}] = \text{ev}$.

For example

$$[\text{mass}] = \left[\text{mass}\times c^2\right] = [\text{energy}] = \text{ev}$$

Therefore, mass of electron is

$$\begin{aligned} m_e &= 9.1094\times10^{-28}\,g \\ &= \left(9.1094\times10^{-28}\,g\right)\times\left(2.9979\times10^{10}\,\text{cm}/s\right)^2 = 8.187\times10^{-7}\text{ erg} \\ &= \frac{8.187\times10^{-7}\text{ erg}}{1.6022\times10^{-12}\text{ erg / ev}} = 0.511\text{ Mev} \end{aligned}$$

Also,

$$\begin{aligned} m_e &= \frac{m_e\,c}{\hbar} = \frac{\left(9.1094\times10^{-28}\,g\right)\left(2.9979\times10^{10}\,\text{cm}/s\right)}{1.0545\times10^{-27}\,g\,\text{cm}^2/s} = 2.5896\times10^{10}\,\text{cm}^{-1} \\ &= \frac{m_e\,c^2}{\hbar} = \left(2.5896\times10^{10}\,\text{cm}^{-1}\right)\left(2.9979\times10^{10}\,\text{cm}/s\right) = 7.7634\times10^{20}\,s^{-1} \end{aligned}$$

$$\begin{aligned} \frac{1}{m_e} &= \frac{\hbar}{m_e\,c} = \frac{1}{2.5896\times10^{10}\,\text{cm}^{-1}} = 3.8616\times10^{-11}\,\text{cm} \\ &= \frac{\hbar}{m_e\,c^2} = \frac{3.8616\times10^{-11}\,\text{cm}}{2.9979\times10^{10}\,\text{cm}/s} = 1.2881\times10^{-21}\,s \end{aligned}$$

Let the generic unit of length be L. Then all dynamic quantities have units in powers of L. For example

$$[\text{time}] = \left[\frac{\text{length}}{c}\right] = L$$
$$[\text{energy}] = \left[\frac{\hbar}{\text{time}}\right] = \left[\frac{\hbar\,c}{\text{length}}\right] = L^{-1}$$
$$[\text{mass}] = \left[\frac{\text{energy}}{c^2}\right] = L^{-1}$$
$$[\text{momentum}] = [\text{mass}\times c] = L^{-1}$$
$$[\text{angular momentum}] = [\text{momentum}\times\text{length}] = L^0$$
$$[\text{velocity}] = [c] = 1 = L^0$$
$$[\text{force}] = \left[\frac{\text{momentum}}{\text{time}}\right] = L^{-2}$$

Charge of electron is e. From Coulomb's law in Lorentz-Heaviside units:

$$F = \frac{q^2}{4\pi r^2} \qquad \left(\frac{q}{\sqrt{4\pi}}\text{ in statcoulombs}\right) \qquad (4.1)$$

$$\rightarrow \qquad \left[\frac{q}{\sqrt{4\pi}}\right] = \text{statC} = \left[\sqrt{\text{Force} \times \text{Length}^2}\,\right] = \sqrt{g\,\frac{\text{cm}^3}{s^2}}$$

$$C = 3 \times 10^9 \text{ statC}$$

$$= 3 \times 10^9 \left(\frac{10^{-3}\,\text{Kg} \times 10^{-6}\,m^3}{s^2}\right)^{1/2}$$

$$= 3 \times 10^{9/2} \sqrt{\frac{\text{Kg}\,m^3}{s^2}}$$

$$\frac{e}{\sqrt{4\pi}} = 4.8032 \times 10^{-10}\text{ statC} = 1.6011 \times 10^{-19}\,C$$

$$\rightarrow \qquad \frac{e^2}{4\pi} = \left(1.6011 \times 10^{-19}\right)^2 9 \times 10^9\,\frac{\text{kg}\,m^3}{s^2}$$

$$= 2.3071 \times 10^{-28}\,\frac{\text{kg}\,m^3}{s^2} \qquad (4.2)$$

The fine structure constant

$$\frac{e^2}{4\pi\hbar c} = \frac{2.3071 \times 10^{-28}\,\text{kg}\,m^3/s^2}{\left(1.0545 \times 10^{-34}\,\text{kg}\,m^2/s\right)\left(2.9979 \times 10^8\,m/s\right)}$$

$$= \frac{1}{137.04}$$

is dimensionless.

Hence, in natural unit, e is dimensionless with

$$\frac{e^2}{4\pi} = \frac{1}{137.04} \qquad (4.3)$$

From the Lorentz force

$$\boldsymbol{F} = e\,\boldsymbol{E} + e\,\boldsymbol{v} \times \boldsymbol{B}$$

we have

$$[\boldsymbol{E}] = [\text{force}] = L^{-2}$$

$$[\boldsymbol{B}] = \left[\frac{\text{force}}{\text{velocity}}\right] = L^{-2}$$

From the potentials

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \qquad \boldsymbol{E} = -\dot{\boldsymbol{A}} - \nabla \times \Phi$$

we have

$$[\boldsymbol{A}] = [\boldsymbol{B}]\,L = L^{-1}$$

$$[\Phi] = [\boldsymbol{E}]\,L = L^{-1}$$

To write Newton's gravitational law

$$F = -G\,\frac{m_1\,m_2}{r^2} \qquad (4.4)$$

Using

$$\left[\frac{\hbar}{\text{mass} \times c}\right] = [\text{length}] = L$$

$$[\text{Force}] = \left[\frac{\text{energy}}{\text{length}}\right] = \left[\left(\text{mass} \times c^2\right)\frac{\text{mass} \times c}{\hbar}\right]$$

we have

$$\left[\text{Force}\,\frac{\hbar}{c^3}\right] = \left[\text{mass}^2\right]$$

$$\rightarrow \qquad \left[G\,\frac{\hbar}{c^3}\right] = L^2$$

Using $G = 6.6726 \times 10^{-8}\,\frac{\text{cm}^3}{g\,s^2}$, we have, in natural unit

$$G = \left(6.6726 \times 10^{-8}\,\frac{\text{cm}^3}{g\,s^2}\right) \times \frac{1.0545 \times 10^{-27}\,\left(g\,\text{cm}^2/s\right)}{\left(2.9979 \times 10^{8}\,\text{cm}/s\right)^3}$$

$$= 2.6116 \times 10^{-66}\,\text{cm}^2$$