4.3. Symmetry & Generating Functionals

Noether's Theorem

An action is called cyclic in q_j if

$$\frac{\partial \mathcal{L}}{\partial q_j} = 0 \tag{4.37}$$

The related Euler-Lagrange eq.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \tag{4.37a}$$

simplifies to

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} = \frac{d p_j}{dt} = 0 \tag{4.38}$$

so that the momentum conjugate to q_j is conserved.

Eq(4.37) implies

$$\mathcal{L}(q_j + c,\ \dot{q}_j) = \mathcal{L}(q_j,\ \dot{q}_j) \qquad \forall \text{ constant } c \tag{4.39}$$

which means the action is translational invariant.

Thus, translational symmetry implies momentum conservation.

More generally, consider the symmetry of the action with respect to a continuous coordinate transformation. This means the action is invariant against the infinitesimal transformation

$$q_j \to q_j + \delta q_j^s \tag{4.39a}$$

The corresponding change (or no change) in the action is

$$\begin{aligned} 0 = \delta S &= \int_{t_a}^{t_b} dt \left(\frac{\partial \mathcal{L}}{\partial q_j}\delta q_j^s + \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\delta \dot{q}_j^s \right) \\ &= \int_{t_a}^{t_b} dt \left[\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) \delta q_j^s + \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{d\,\delta q_j^s}{dt} \right] \qquad \text{[Eq(4.37a) used.]} \\ &= \int_{t_a}^{t_b} dt \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\delta q_j^s \right) \qquad (4.41) \\ &= \left. \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\delta q_j^s \right|_{t_a}^{t_b} \end{aligned}$$

Note that unlike variations about the classical path in the derivation of the E-L eq., there is no restriction that δq_j^s vanish at the end points t_a & t_b.

Since t_a, t_b are arbitrary, we have

$$\frac{dG}{dt} = 0 \tag{4.42}$$

where

$$G = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\delta q_j^s \tag{4.43}$$

is called the charge corresponding to the transformation eq(4.39a). Eq(4.42) is known as Noether's theorem.

In case the action is not invariant against the transformation, then eq(4.41) becomes

$$\delta S = \int_{t_a}^{t_b} dt\, F(t) \neq 0 \tag{4.44}$$

where

$$F(t) = \frac{d\,G}{d\,t} \tag{4.45}$$

Thus, we can still define the charge G, but it's no longer conserved.

Canonical Transformations

Another type of continuous symmetry is that against a variation of the integration domain:

$$S = \int_{t_a}^{t_b} d\,t\,\mathcal{L} \;\;\rightarrow\;\; S' = \int_{t_a+\delta t}^{t_b+\delta t} d\,t\,\mathcal{L} \tag{4.46}$$

It is easy to show that

$$\delta\,S = 0 \rightarrow \qquad \frac{\partial\,\mathcal{L}}{\partial\,t} = 0$$

→ H is conserved along a classical trajectory.

Note that for an arbitrary function F, the transformation

$$\mathcal{L} \;\rightarrow\; \mathcal{L} + \frac{d\,F}{d\,t} \tag{4.47}$$

leads to

$$S \;\rightarrow\; S + F(t_b) - F(t_b)$$

so that the E-L eq, & hence the dynamics of the system, is unchanged.

This leads to a class of transformations that involves both q_j & p_j. The defining property of these so called canonical transformations is that they leave the Poisson brackets form invariant, which means the relevant jacobian is 1. Consequently, the phase space volume is also preserved.

Quantum Anomaly

In quantum mechanics, symmetry transformations may involve both states & operators while physical properties are represented by transition elements (amplitudes). Hence, transformations depicted by eq(4.47) can lead to changes of the physical properties of the system.

In the path integral formulation, classical component of the system is represented by the action & the quantum component by the integral measure. Obviously, both action & measure need to be invariant for a transformation to qualify as a symmetry. Transformation of the measure is represented by the jacobian of the transformation. A classical symmetry that does not survive quantization due to a changed jacobian is called an anomaly.

For example, the classical Lagrangian

$$\mathcal{L} = \frac{1}{2\,\omega}\,\frac{\dot{q}^2}{q^2} \tag{4.48}$$

is invariant under the time-independent rescaling

$$q \;\rightarrow\; q' = e^{\lambda}\,q \qquad\qquad \lambda = \text{const.}$$

But $\mathcal{D}\,q = J\,\mathcal{D}\,q' = e^{-(N-1)\,\lambda}\,\mathcal{D}\,q'$

$$p = \frac{\dot{q}}{\omega\,q^2} \qquad\qquad p' = \frac{\dot{q}'}{\omega\,q'^{\,2}} = e^{-\lambda}\,p$$

→ $\mathcal{D}\,p = J_p\,\mathcal{D}\,p' = e^{N\,\lambda}\,\mathcal{D}\,p'$

∴ $\mathcal{D}\,p\,\mathcal{D}\,q = e^{\lambda}\,\mathcal{D}\,p'\,\mathcal{D}\,q'$

However, the measure is scale invariant for the partition function owing to an extra q integral for taking the trace.

Ehrenfest's Theorem

As already mentioned, the structure of the measure reveals many quantum properties of the system.

For example, if the q integrals have limits $\pm\infty$, one can prove a form of Ehrenfest's theorem which states that the expectation values of an observable obey the classical Hamilton's equation of motion.

To this end, let

$$q_j \to q'_j = q_j + f_j \qquad \text{where } f_j = \text{const.}$$

$$\therefore \qquad \mathcal{D}q = \mathcal{D}q'$$

If the q_j integrals have limits $\pm\infty$, they remain unchanged under the transformation.

In the continuum limit, the transformation becomes

$$q(t) \to q'(t) = q(t) + f(t)$$

Since q is fixed at the end points, we must have $f(t_a) = f(t_b) = 0$.

Using

$$\dot{q}(t) \to \dot{q}'(t) = \dot{q}(t) + \dot{f}(t) \qquad \& \qquad \delta q(t) = f(t)$$

we have, to 1st order in δq,

$$\mathcal{L}(q, \dot{q}) \to \mathcal{L}(q', \dot{q}') \approx \mathcal{L}(q, \dot{q}) + \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) f(t)$$

Invariance of the transform means

$$\int_{q_a}^{q_b} \overline{\mathcal{D}}q \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} dt\, \mathcal{L}(q, \dot{q})\right] = \int_{q_a}^{q_b} \overline{\mathcal{D}}q' \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} dt\, \mathcal{L}(q', \dot{q}')\right]$$

$$= \int_{q_a}^{q_b} \overline{\mathcal{D}}q \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} dt\, \mathcal{L}(q, \dot{q})\right]$$

$$\times\left\{1 + \frac{i}{\hbar}\int_{t_a}^{t_b} dt \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) f(t) + \ldots\right\}$$

$$\to \qquad \int_{q_a}^{q_b} \overline{\mathcal{D}}q \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} dt\, \mathcal{L}(q, \dot{q})\right] \frac{i}{\hbar}\int_{t_a}^{t_b} dt \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) f(t) = 0$$

Since f is arbitrary as long as it's small, we can choose

$$f(t) = \epsilon\, \delta(t - t') \text{ with } \qquad \epsilon \to 0 \;\; \& \; t' \text{ arbitrary}$$

which gives

$$\int_{q_a}^{q_b} \overline{\mathcal{D}}q \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)_{t=t'} \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} dt\, \mathcal{L}(q, \dot{q})\right] = 0 \qquad (4.50)$$

$$\to \qquad \left\langle \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} \right\rangle = 0$$

i.e., the observable $\langle q \rangle$ satisfies the E-L eq.

Spontaneous Symmetry Breaking

For non-linear systems, it is possible that the true ground state does not have the symmetry of the system. This is called spontaneous symmetry breaking. Again, path integrals are useful in the study of this area.

Rotation

Next, we demonstrate the effects of symmetry on the transition elements & time-ordered products using rotational invariance as an example.

A rotation is a transformation of the Cartesian coordinates

$$x'_j = R_{jk}\, x_k$$

such that the norm of the position vector is preserved:

$$r'^2 = x'_i\, x'_i = R_{ij}\, x_j\, R_{ik}\, x_k = x_j\, x_j = r^2 \qquad (4.51a)$$

→ $$R_{ij}\, R_{ik} = \delta_{jk} \qquad (4.51)$$

or $\boldsymbol{R}\,\boldsymbol{R}^T = \boldsymbol{I}$

i.e., the matrix $\boldsymbol{R}$ is orthogonal.

For an infinitesimal rotation,

$$R_{ij} \approx \delta_{ij} + \delta R_{ij}$$

Eq(4.51) becomes

$$\left(\delta_{ij} + \delta R_{ij}\right)\left(\delta_{ik} + \delta R_{ik}\right) = \delta_{jk}$$

$$= \delta_{jk} + \delta R_{jk} + \delta R_{kj} + O(\delta R)^2 \qquad (4.52)$$

→ $$\delta R_{jk} = -\delta R_{kj} \qquad (4.52a)$$

or $\delta\boldsymbol{R}^T = -\delta\boldsymbol{R}$

i.e., $\delta\boldsymbol{R}$ is anti-symmetric. In 3-D space, $\delta\boldsymbol{R}$ has only 3 independent elements & we can write

$$\delta R_{jk} = \epsilon_{jki}\, \delta\theta_i \qquad (4.52b)$$

where θ_i is the rotational angle about the x_i-axis.

For an infinitesimal rotation,

$$x_j \;\to\; x'_j = x_j + \delta R_{jk}\, x_k \qquad (4.53)$$

By eq(4.51), the inverse transformation is

$$x_j \;\to\; x'_j = x_j + \delta R_{kj}\, x_k = x_j - \delta R_{jk}\, x_k \qquad (4.54)$$

The generating functional for a 3-D particle moving in a central potential is

$$Z_{ab}[J] = \int_{\boldsymbol{x}_a}^{\boldsymbol{x}_b} \overline{\mathcal{D}}\boldsymbol{x}\, \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} m\, \dot{x}_j\, \dot{x}_j - V(r) - J_j\, x_j \right] \right\} \qquad (4.55)$$

where the measure is [see eq(2.52)]

$$\overline{\mathcal{D}}\boldsymbol{x} = \lim_{N\to\infty} \left(\frac{m}{2\pi i \hbar \epsilon} \right)^{3N/2} \prod_{k=1}^{N-1} d\boldsymbol{x}(t_k) \qquad (4.56)$$

We shall consider only the case $\boldsymbol{x}_a = \boldsymbol{x}_b = 0$.

First of all, consider the case $J = 0$. Under a rotation

$$Z_{00}[0] = \int_0^0 \overline{\mathcal{D}}\boldsymbol{x}\, \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} m\, \dot{x}_j\, \dot{x}_j - V(r) \right] \right\}$$

$$\to\; Z'_{00}[0] = \int_0^0 \overline{\mathcal{D}}\boldsymbol{x}'\, \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} m\, \dot{x}'_j\, \dot{x}'_j - V(r') \right] \right\}$$

Using eq(4.51a), we have

$$Z_{00}[0] = Z'_{00}[0] \qquad (4.57)$$

For $J \neq 0$, the $\boldsymbol{J}\cdot\boldsymbol{x}$ term is unchanged if $\boldsymbol{J}$ is similarly rotated, i.e.,

$$J_j \;\to\; J'_j = J_j + \delta R_{jk}\, J_k = J_j + \delta\theta_i\, \epsilon_{ijk}\, J_k$$

⇒ $$J_i\, x_i \;\to\; J'_i\, x'_i = \left(J_i - \delta R_{ij}\, J_j\right)\left(x_i + \delta R_{ik}\, x_k\right)$$

$$= J_i\, x_i + \delta R_{ik}\, J_i\, x_k + \delta R_{ij}\, J_j\, x_i + O(\delta R)^2$$

Using

$$\delta R_{ik}\, J_i\, x_k + \delta R_{ij}\, J_j\, x_i = \delta R_{ji}\, J_j\, x_i + \delta R_{ij}\, J_j\, x_i$$

$$= -\delta R_{ij}\, J_j\, x_i + \delta R_{ij}\, J_j\, x_i$$

$$= 0$$

we have

$$J_i\, x_i \quad \to \quad J'_i\, x'_i = J_i\, x_i + O(\delta R)^2$$

$$\therefore \qquad Z'_{00}[J'] = Z_{00}[J] \qquad\qquad \text{to 1st order in } \delta R.$$

where

$$Z'_{00}[J'] = \int_0^0 \overline{\mathcal{D}}\, \boldsymbol{x}' \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} m\, \dot{x}'_j\, \dot{x}'_j - V(r) - J'_j\, x'_j \right] \right\}$$

Since the jacobian is $\det R = 1$, a change of variable $\boldsymbol{x}' \to \boldsymbol{x}$ gives

$$Z'_{00}[J'] = \int_0^0 \overline{\mathcal{D}}\, \boldsymbol{x} \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} m\, \dot{x}_j\, \dot{x}_j - V(r) - J'_j\, x_j \right] \right\}$$

$$= \int_0^0 \overline{\mathcal{D}}\, \boldsymbol{x} \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} m\, \dot{x}_j\, \dot{x}_j - V(r) - J_j\, x_j \right] \right\}$$

$$\times \left\{ 1 - \frac{i}{\hbar} \delta\theta_i\, \epsilon_{i\,j\,k} \int_{t_a}^{t_b} d\,t\, J_k\, x_j + \ldots \right\}$$

$$= Z_{00}[J] + \delta\theta_i\, \epsilon_{i\,j\,k} \int_{t_a}^{t_b} d\,t\, J_k\,(t)\, \frac{\delta\, Z_{00}[J]}{\delta\, J_j(t)}$$

Hence,

$$0 = Z'_{00}[J'] - Z_{00}[J] = \delta\theta_i\, \epsilon_{i\,j\,k} \int_{t_a}^{t_b} d\,t\, J_k(t)\, \frac{\delta\, Z_{00}[J]}{\delta\, J_j(t)} \qquad (4.59)$$

Since $\delta\theta_i$ are arbitrary, we have

$$\epsilon_{i\,j\,k} \int_{t_a}^{t_b} d\,t\, J_k(t)\, \frac{\delta\, Z_{00}[J]}{\delta\, J_j(t)} = 0 \qquad (4.59a)$$

which serves as a generator for relations between time-ordered products.

For example, $\dfrac{\delta}{\delta\, J_n(t')}$ eq(4.59 a) gives

$$\epsilon_{i\,j\,k} \int_{t_a}^{t_b} d\,t \left\{ \delta_{k\,n}\, \delta(t - t')\, \frac{\delta\, Z_{00}[J]}{\delta\, J_j(t)} + J_k(t)\, \frac{\delta^2\, Z_{00}[J]}{\delta\, J_n(t')\, \delta\, J_j(t)} \right\} = 0$$

$$\to \qquad \epsilon_{i\,j\,n}\, \frac{\delta\, Z_{00}[J]}{\delta\, J_j(t')} + \epsilon_{i\,j\,k} \int_{t_a}^{t_b} d\,t\, J_k(t)\, \frac{\delta^2\, Z_{00}[J]}{\delta\, J_n(t')\, \delta\, J_j(t)} = 0 \qquad (4.60)$$

For the general propagator $Z_{a\,b}[J]$, eq(4.59a) generalizes to

$$\epsilon_{i\,j\,k} \left\{ \int_{t_a}^{t_b} d\,t\, J_k(t)\, \frac{\delta\, Z_{a\,b}[J]}{\delta\, J_j(t)} + \left(x_k^a\, \frac{\partial}{\partial\, x_j^a} + x_k^b\, \frac{\partial}{\partial\, x_j^b} \right) Z_{a\,b}[J] \right\} = 0 \qquad (4.61)$$

the proof of which is left as an exercise.

Gauge Transformation

$$\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2} m\, \dot{x}_j\, \dot{x}_j - V(\boldsymbol{x}) - e\, \phi(\boldsymbol{x}) + \frac{e}{c} A_j(x)\, \dot{x}_j \qquad (4.62)$$

As is well known, the EM fields are invariant if the potentials $(\phi, \boldsymbol{A})$ undergo a gauge transformation

$$A_j \to A_j + \frac{\partial\, \Lambda}{\partial\, x_j} \qquad\qquad \phi \to \phi - \frac{\partial\, \Lambda}{c\, \partial\, t} \qquad (4.63)$$

which means

$$\mathcal{L} \to \mathcal{L} + \frac{e}{c}\left(\frac{\partial \Lambda}{\partial t} + \frac{\partial \Lambda}{\partial x_j}\dot{x}_j\right) = \mathcal{L} + \frac{e}{c}\frac{d\Lambda}{dt}$$

$$\therefore \qquad S \to S + \frac{e}{c}[\Lambda(x_b, t_b) - \Lambda(x_a, t_a)] \qquad (4.64)$$

Hence, the classical equations of motion are also invariant under the transformation.

However, the quantum transition amplitude (or propagator) acquires a non-trivial phase factor

$$Z_{ab} \to \exp\left\{i\frac{e}{\hbar c}[\Lambda(x_b, t_b) - \Lambda(x_a, t_a)]\right\} Z_{ab} \qquad (4.65)$$

Recalling that

$$Z_{ab} = \langle x_b, t_b \mid x_a, t_a \rangle$$

we see that Z_{ab} is invariant if every wave function also acquire a phase factor

$$\langle x \mid \psi(t) \rangle \to \exp\left\{i\frac{e}{\hbar c}\Lambda(x_b, t_b)\right\}\langle x \mid \psi(t) \rangle \qquad (4.66)$$

Symmetry Charge in Quantum Mechanics

In quantum mechanics, eq(4.43) becomes an operator equation

$$G = P_j\, \delta Q_j^s \qquad (4.67)$$

Using

$$\left[Q_k, \delta Q_j^s\right] = 0$$

we have

$$[Q_k, G] = \left[Q_k, P_j\, \delta Q_j^s\right] = i\hbar\, \delta_{kj}\, \delta Q_j^s$$
$$= i\hbar\, \delta Q_j^s \qquad (4.68)$$

Hence, G generates an infinitesimal displacement along the direction of the symmetry coordinate. G is therefore called the generator of the symmetry.

If G is conserved, it commutes with H. Its eigenvalues can be used as a quantum number for labelling the energy eigenstates. In other words, G is a member of the mutually commuting set of operators that specify the system.

For the 3-D rotations, the conserved charge is the angular momentum

$$G_j = \varepsilon_{jkm}\, Q_k\, P_m \qquad (4.69)$$

where the unimportant mass is set to 1.

G_j can be used as generators of a Lie algebra defined by the Lie brackets

$$\left[G_j, G_k\right] = i\hbar\, \varepsilon_{jkm}\, G_m \qquad (4.70)$$

which are just the commutators between the components of the angular momentum. See Chap.7 for further discussion.