4.4. Harmonic Oscillator Coherent States

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 Q^2 \qquad (4.71a)$$

Let

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(Q + i\frac{P}{m\omega}\right) \qquad a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(Q - i\frac{P}{m\omega}\right) \qquad (4.71)$$

$$\rightarrow \qquad [a, a^+] = \frac{m\omega}{2\hbar}\cdot\frac{-2i}{m\omega}[Q, P] = 1 \qquad (4.72)$$

Similarly,

$$a^+ a = \frac{m\omega}{2\hbar}\left\{Q^2 + \left(\frac{P}{m\omega}\right)^2 + \frac{i}{m\omega}[Q, P]\right\}$$

$$= \frac{1}{\hbar\omega}\left(\frac{P^2}{2m} + \frac{1}{2}m\omega^2\right) - \frac{1}{2}$$

$$\rightarrow \qquad H = \hbar\omega\left(a^+ a + \frac{1}{2}\right) \qquad (4.73)$$

Using

$$[a^+ a, a^+] = [a^+ a, a^+] = a^+[a, a^+] = a^+ \qquad (4.74a)$$

$$[a^+ a, a] = [a^+ a, a] = [a^+ a]a = -a \qquad (4.75a)$$

we have

$$[H, a^+] = \hbar\omega a^+ \qquad (4.74)$$

$$[H, a] - \hbar\omega a \qquad (4.75)$$

Let $|n\rangle$ be the normalized eigenstates of $N = a^+ a$ so that

$$N|n\rangle = n|n\rangle \qquad (4.80)$$

Since N is hermitian, n must be a real number and $\{|n\rangle\}$ is an orthonormal complete set, i.e.,

$$\langle n|m\rangle = \delta_{nm} \qquad (4.79)$$

$$\sum_n |n\rangle\langle n| = I \qquad (4.79a)$$

Eq(4.74a) then gives

$$[N, a^+]|n\rangle = a^+|n\rangle$$

$$= (Na^+ - a^+ n)|n\rangle$$

$$\rightarrow \qquad Na^+|n\rangle = (n+1)a^+|n\rangle$$

i.e., $a^+|n\rangle = \alpha_n|n+1\rangle$ (a)

where α_n is a constant.

Similarly, eq(4.75) gives

$$[N, a]|n\rangle = -a|n\rangle$$

$$= (Na - an)|n\rangle$$

$$\rightarrow \qquad Na|n\rangle = (n-1)a|n\rangle$$

i.e., $a|n\rangle = \beta_n|n-1\rangle$ (b)

where β_n is a constant.

Thus, a^+ (a) is a ladder operator that pushes a state one step up (down) the ladder of eigenvalues.

Assuming N is bounded below & denote the ground state as $|g\rangle$, we must have

$$a|g\rangle = 0$$

$\rightarrow \quad N \mid g\rangle = a^+ a \mid g\rangle = 0$

Hence, $\mid g\rangle = \mid 0\rangle$ so that

$$a \mid 0\rangle = 0 \qquad (4.76)$$

Using eq(a) repeatedly on $\mid 0\rangle$, we have

$$(a^+)^n \mid 0\rangle \propto \mid n\rangle \qquad (c)$$

which also shows that n must be a positive integer so that N is called the number operator.

Eqs(a & b) give

$$\langle n \mid a^+ a \mid n\rangle = \langle n-1 \mid \beta_n^* \beta_n \mid n-1\rangle$$

$$\rightarrow \quad \beta_n^* \beta_n = n$$

Assuming β_n to be real, we have

$$\beta_n = \sqrt{n} \quad \rightarrow \quad a \mid n\rangle = \sqrt{n} \left| n-1 \right\rangle \qquad (4.75a)$$

$$\rightarrow \quad a^+ a \mid n\rangle = \sqrt{n}\, a^+ \left| n-1 \right\rangle = \sqrt{n}\, \alpha_{n-1} \left| n \right\rangle = n \mid n\rangle$$

$$\therefore \quad \alpha_{n-1} = \sqrt{n} \quad \rightarrow \quad a^+ \mid n\rangle = \sqrt{n+1} \left| n+1 \right\rangle \qquad (4.75b)$$

Eq(c) thus becomes

$$\mid n\rangle = \frac{(a^+)^n}{\sqrt{n!}} \left| 0 \right\rangle \qquad (4.75c)$$

Eq(4.73) shows that H & N share the same eigenstates with

$$H \mid n\rangle = \hbar\,\omega \left(n + \frac{1}{2}\right) \left| n \right\rangle \qquad (4.78)$$

If we interpret n as the number of quanta present, then a^+ (a) represents the creation (annihilation) of a quantum of energy $\hbar\,\omega$.

Coherent States

The coherent state $\mid \lambda\rangle$, where λ is any complex number, is defined as

$$\mid \lambda\rangle = e^{\lambda a^+ - \lambda^* a} \left| 0 \right\rangle \qquad (4.81)$$

Since

$$\left[a, \left[a^+, a\right]\right] = [a, -1] = 0$$

$$\left[a^+, \left[a^+, a\right]\right] = \left[a^+, -1\right] = 0$$

we can use the BCH formula

$$e^{A+B} = e^A\, e^B\, e^{-[A,B]/2} \qquad (1.66)$$

to get

$$e^{\lambda a^+ - \lambda^* a} = e^{\lambda a^+}\, e^{-\lambda^* a}\, e^{-\lambda \lambda^*/2} \qquad (4.81a)$$

Eq(4.81) thus becomes

$$\mid \lambda\rangle = e^{-\lambda\lambda^*/2}\, e^{\lambda a^+}\, e^{-\lambda^* a} \left| 0 \right\rangle$$

$$= e^{-\lambda\lambda^*/2}\, e^{\lambda a^+} \sum_{n=0}^{\infty} \frac{(-\lambda^*)^n}{n!} a^n \left| 0 \right\rangle$$

$$= e^{-\lambda\lambda^*/2}\, e^{\lambda a^+} \left| 0 \right\rangle$$

$$= e^{-\lambda\lambda^*/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (a^+)^n \left| 0 \right\rangle$$

$$= e^{-\lambda\lambda^*/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \left| n \right\rangle \qquad \text{[Eq(4.75c) used.]} \qquad (4.82)$$

$$\rightarrow \qquad a \mid \lambda \rangle = e^{-\lambda\lambda^*/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} a \left| n \right\rangle$$

$$= e^{-\lambda\lambda^*/2} \sum_{n=1}^{\infty} \frac{\lambda^n}{\sqrt{(n-1)!}} \left| n-1 \right\rangle \text{[Eq(4.75a) used.]}$$

$$= \lambda\, e^{-\lambda\lambda^*/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \left| n \right\rangle$$

$$= \lambda \mid \lambda \rangle \qquad (4.82a)$$

i.e., the coherent state is the eigenstate of the annihilation operator.

Note that one cannot construct an eigenstate for the creation operator in this manner because the ground state is destroyed by a^+.

Eq(4.82) gives

$$\mid \lambda \rangle \langle \lambda \mid = e^{-\lambda\lambda^*} \sum_{m,n=0}^{\infty} \frac{\lambda^{*m} \lambda^n}{\sqrt{m!\, n!}} \mid n \rangle \langle m \mid$$

Setting

$$\lambda = r\, e^{i\theta} \qquad \lambda^* = r\, e^{-i\theta}$$

we have

$$J = \det \begin{vmatrix} \frac{\partial \lambda^*}{\partial r} & \frac{\partial \lambda^*}{\partial \theta} \\ \frac{\partial \lambda}{\partial r} & \frac{\partial \lambda}{\partial \theta} \end{vmatrix} = \det \begin{vmatrix} e^{-i\theta} & -i\, r\, e^{-i\theta} \\ e^{i\theta} & i\, r\, e^{i\theta} \end{vmatrix} = 2\, i\, r$$

$$\rightarrow \qquad \int d\lambda^*\, d\lambda\, e^{-\lambda\lambda^*} \lambda^{*m} \lambda^n = 2\, i \int_0^{2\pi} d\theta \int_0^{\infty} dr\; e^{-r^2}\, r^{m+n+1}\, e^{i(n-m)\theta}$$

$$= 4\pi\, i\, \delta_{nm} \int_0^{\infty} dr\; e^{-r^2}\, r^{m+n+1}$$

$$= 2\pi\, i\, \delta_{nm}\, \Gamma\!\left(\frac{m+n+2}{2}\right)$$

$$= 2\pi\, i\, \delta_{nm}\, \Gamma(n+1)$$

$$= 2\pi\, i\, \delta_{nm}\, n!$$

$$\therefore \qquad \int d\lambda^*\, d\lambda \mid \lambda \rangle \langle \lambda \mid = 2\pi\, i \sum_{n=0}^{\infty} \mid n \rangle \langle n \mid$$

i.e.
$$\int \frac{d\lambda^*\, d\lambda}{2\pi\, i} \mid \lambda \rangle \langle \lambda \mid = I \qquad (4.83)$$

The coherent states are not orthogonal. In fact

$$\langle \sigma \mid \lambda \rangle = e^{-(\lambda\lambda^* + \sigma\sigma^*)/2} \sum_{m,n=0}^{\infty} \frac{\sigma^{*m} \lambda^n}{\sqrt{m!\, n!}} \langle m \mid n \rangle$$

$$= e^{-(\lambda\lambda^* + \sigma\sigma^*)/2} \sum_{n=0}^{\infty} \frac{(\sigma^* \lambda)^n}{n!}$$

$$= e^{-(\lambda\lambda^* + \sigma\sigma^*)/2}\, e^{\sigma^* \lambda} \qquad (4.84)$$

Setting $\sigma = \lambda$, we see that $\mid \lambda \rangle$ is normalized to unity.

$$\langle \lambda \mid \lambda \rangle = 1 \qquad (4.84a)$$

$$\langle \sigma \mid a^{+} \mid \lambda \rangle = e^{-(\lambda\lambda^{*}+\sigma\sigma^{*})/2} \sum_{m,n=0}^{\infty} \frac{\sigma^{*m}\lambda^{n}}{\sqrt{m!\,n!}} \langle m \mid a^{+} \mid n \rangle$$
$$= e^{-(\lambda\lambda^{*}+\sigma\sigma^{*})/2} \sum_{m,n=0}^{\infty} \frac{\sigma^{*m}\lambda^{n}}{\sqrt{m!\,n!}} \sqrt{n+1}\,\langle m \mid n+1 \rangle$$
$$= e^{-(\lambda\lambda^{*}+\sigma\sigma^{*})/2} \sum_{n=0}^{\infty} \frac{\sigma^{*n+1}\lambda^{n}}{\sqrt{(n+1)!\,n!}} \sqrt{n+1}$$
$$= \sigma^{*} e^{-(\lambda\lambda^{*}+\sigma\sigma^{*})/2} \sum_{n=0}^{\infty} \frac{(\sigma^{*}\lambda)^{n}}{n!}$$
$$= \sigma^{*} \langle \sigma \mid \lambda \rangle \qquad (4.85)$$

Similarly,

$$\langle \sigma \mid (a^{+})^{2} \mid \lambda \rangle = e^{-(\lambda\lambda^{*}+\sigma\sigma^{*})/2} \sum_{m,n=0}^{\infty} \frac{\sigma^{*m}\lambda^{n}}{\sqrt{m!\,n!}} \langle m \mid (a^{+})^{2} \mid n \rangle$$
$$= e^{-(\lambda\lambda^{*}+\sigma\sigma^{*})/2} \sum_{m,n=0}^{\infty} \frac{\sigma^{*m}\lambda^{n}}{\sqrt{m!\,n!}} \sqrt{(n+2)(n+1)}\,\langle m \mid n+2 \rangle$$
$$= e^{-(\lambda\lambda^{*}+\sigma\sigma^{*})/2} \sum_{n=0}^{\infty} \frac{\sigma^{*n+2}\lambda^{n}}{\sqrt{(n+2)!\,n!}} \sqrt{(n+2)(n+1)}$$
$$= \sigma^{*2} e^{-(\lambda\lambda^{*}+\sigma\sigma^{*})/2} \sum_{n=0}^{\infty} \frac{(\sigma^{*}\lambda)^{n}}{n!}$$
$$= \sigma^{*2} \langle \sigma \mid \lambda \rangle \qquad (4.85a)$$

Finally,

$$\langle \sigma \mid N \mid \lambda \rangle = \langle \sigma \mid a^{+} a \mid \lambda \rangle$$
$$= \sigma^{*} \lambda \langle \sigma \mid \lambda \rangle \qquad (4.86)$$

$$\langle \sigma \mid N^{2} \mid \lambda \rangle = \langle \sigma \mid a^{+} a\, a^{+} a \mid \lambda \rangle$$
$$= \lambda\sigma^{*} \langle \sigma \mid a\, a^{+} \mid \lambda \rangle$$
$$= \lambda\sigma^{*} \langle \sigma \mid a^{+} a + 1 \mid \lambda \rangle$$
$$= \lambda\sigma^{*} (\lambda\sigma^{*} + 1) \langle \sigma \mid \lambda \rangle \qquad (4.86a)$$

Eq(4.71) can be inverted to give

$$Q = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^{+}) \qquad P = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^{+} - a) \quad (4.86b)$$

$$\rightarrow \qquad \langle \sigma \mid Q \mid \lambda \rangle = \sqrt{\frac{\hbar}{2m\omega}}\,(\lambda + \sigma^{*}) \langle \sigma \mid \lambda \rangle$$
$$\langle \sigma \mid Q^{2} \mid \lambda \rangle = \frac{\hbar}{2m\omega} \langle \sigma \mid a^{2} + 2a^{+} a + 1 + a^{+2} \mid \lambda \rangle$$
$$= \frac{\hbar}{2m\omega} (\lambda^{2} + 2\sigma^{*}\lambda + 1 + \sigma^{*2}) \langle \sigma \mid \lambda \rangle$$

Therefore, the uncertainty $(\Delta Q)_{\lambda}$ of Q with respect to the state $\mid \lambda \rangle$ is

$$(\Delta Q)_{\lambda}^{2} \equiv \langle \lambda \mid \{ Q - \langle \lambda \mid Q \mid \lambda \rangle \}^{2} \mid \lambda \rangle$$
$$= \langle \lambda \mid Q^{2} \mid \lambda \rangle - \langle \lambda \mid Q \mid \lambda \rangle^{2}$$

$$= \frac{\hbar}{2 m \omega} \langle \lambda \mid \lambda \rangle$$
$$= \frac{\hbar}{2 m \omega}$$

Similarly,

$$\langle \sigma \mid P \mid \lambda \rangle = i \sqrt{\frac{m \hbar \omega}{2}} (\sigma^* - \lambda) \langle \sigma \mid \lambda \rangle$$
$$\left\langle \sigma \mid P^2 \mid \lambda \right\rangle = -\frac{m \hbar \omega}{2} \left\langle \sigma \mid a^{+2} - 2 a^+ a - 1 + a^2 \mid \lambda \right\rangle$$
$$= -\frac{m \hbar \omega}{2} \left(\sigma^{*2} - 2 \sigma^* \lambda - 1 + \lambda^2\right) \langle \sigma \mid \lambda \rangle$$

Therefore, the uncertainty $(\Delta P)_\lambda$ of P with respect to the state $\mid \lambda \rangle$ is

$$(\Delta P)_\lambda^2 \equiv \left\langle \lambda \mid \left\{ P - \langle \lambda \mid P \mid \lambda \rangle \right\}^2 \mid \lambda \right\rangle$$
$$= \left\langle \lambda \mid P^2 \mid \lambda \right\rangle - \langle \lambda \mid P \mid \lambda \rangle^2$$
$$= \frac{m \hbar \omega}{2} \langle \lambda \mid \lambda \rangle$$
$$= \frac{m \hbar \omega}{2}$$

$$\rightarrow \qquad (\Delta P)_\lambda (\Delta Q)_\lambda = \frac{\hbar}{2}$$

i.e., the coherent state is a state of minimum uncertainty.

More importantly, for $m \omega = 1$, the uncertainties of both P & Q have the same small value $\sqrt{\hbar}$.

Thus, the coherent states are the closest a quantum system can behave like its classical counterpart.

As the strong suit of the path integral formulism is its linkage between classical & quantum behavior, a path integral based on the coherent states should accentuate the advantage.

Thus, we consider the propagator

$$Z_{a b} = \langle \lambda_b, t_b \mid \lambda_a, t_a \rangle = \left\langle \lambda_b \mid e^{-i H (t_b - t_a) / \hbar} \mid \lambda_a \right\rangle \qquad (4.87a)$$

where

$$\mid \lambda, t \rangle \equiv e^{i H t / \hbar} \mid \lambda \rangle \qquad \& \qquad \mid \lambda \rangle \equiv \mid \lambda, 0 \rangle \qquad (4.87)$$

Using eq(4.83), we have

$$\langle \lambda, t \mid \lambda', t' \rangle = \int \frac{d \lambda_1^* \, d \lambda_1}{2 \pi i} \langle \lambda, t \mid \lambda_1, t_1 \rangle \langle \lambda_1, t_1 \mid \lambda', t' \rangle \qquad (4.87b)$$

With the help of eq(4.87b), we can break up the time interval of $Z_{a b}$ into N segments & get

$$\langle \lambda_b, t_b \mid \lambda_a, t_a \rangle = \int_{\lambda_a}^{\lambda_b} \mathcal{D} \lambda \prod_{j=0}^{N-1} \left\langle \lambda_{j+1}, t_{j+1} \mid \lambda_j, t_j \right\rangle \qquad (4.88)$$

where the measure is

$$\mathcal{D} \lambda = \prod_{j=1}^{N-1} \frac{d \lambda_j^* \, d \lambda_j}{2 \pi i} \qquad (4.89)$$

Reminder: the domain of integration for $d \lambda_j^* \, d \lambda_j$ is the whole complex plane. The symbol $\int_{\lambda_a}^{\lambda_b}$ is just a mnemonic for the B.C.

$$\lambda_N = \lambda_b \; \lambda_0 = \lambda_a$$

Using eq(4.87a), we have

$$\langle \lambda_{j+1}, t_{j+1} \mid \lambda_j, t_j \rangle = \langle \lambda_{j+1} \mid e^{-i H \epsilon / \hbar} \mid \lambda_j \rangle \qquad \epsilon = t_{j+1} - t_j \to 0$$

$$\approx \left\langle \lambda_{j+1} \left| \left(1 - \frac{i}{\hbar} H \epsilon\right) \right| \lambda_j \right\rangle$$

In anticipation to applications to more general quadratic hamiltonians, we set

$$H = \hbar \omega a^+ a = \hbar \omega N$$

$$\to \quad \langle \lambda_{j+1}, t_{j+1} \mid \lambda_j, t_j \rangle \approx \langle \lambda_{j+1} \mid (1 - i \omega \epsilon) \mid \lambda_j \rangle$$

$$\approx (1 - i \omega \epsilon \lambda_{j+1}^* \lambda_j) \langle \lambda_{j+1} \mid \lambda_j \rangle \qquad \text{[Eq(4.86) used.]}$$

$$\approx \exp(-i \omega \epsilon \lambda_{j+1}^* \lambda_j) \langle \lambda_{j+1} \mid \lambda_j \rangle$$

Using eq(4.84), we have

$$\langle \lambda_{j+1}, t_{j+1} \mid \lambda_j, t_j \rangle$$

$$\approx \exp(-i \omega \epsilon \lambda_{j+1}^* \lambda_j) \exp\left[-\frac{1}{2} (\lambda_j \lambda_j^* + \lambda_{j+1} \lambda_{j+1}^*) + \lambda_{j+1}^* \lambda_j\right] \qquad (4.90)$$

Using

$$\sum_{j=0}^{N-1} \frac{1}{2} (\lambda_j \lambda_j^* + \lambda_{j+1} \lambda_{j+1}^*) = \frac{1}{2} (\lambda_N \lambda_N^* - \lambda_0 \lambda_0^*) + \sum_{j=0}^{N-1} \lambda_j \lambda_j^*$$

eq(4.88) becomes

$$\langle \lambda_b, t_b \mid \lambda_a, t_a \rangle = \exp\left[-\frac{1}{2} (\lambda_b \lambda_b^* - \lambda_a \lambda_a^*)\right] \qquad (4.91)$$

$$\times \int_{\lambda_a}^{\lambda_b} \mathcal{D} \lambda \exp\left[\sum_{j=0}^{N-1} (-\lambda_j \lambda_j^* + \lambda_{j+1}^* \lambda_j - i \omega \epsilon \lambda_{j+1}^* \lambda_j)\right]$$

As $\epsilon \to 0$, we have

$$\lambda_{j+1}^* \lambda_j \approx \lambda^*(t) \lambda(t)$$

$$-\lambda_j \lambda_j^* + \lambda_{j+1}^* \lambda_j \approx \epsilon \dot{\lambda}^*(t) \lambda(t)$$

so that

$$\langle \lambda_b, t_b \mid \lambda_a, t_a \rangle = \exp\left[-\frac{1}{2} (\lambda_b \lambda_b^* - \lambda_a \lambda_a^*)\right] \qquad (4.92a)$$

$$\times \int \mathcal{D} \lambda \exp\left[\int_{t_a}^{t_b} dt \left(\dot{\lambda}^* \lambda - i \omega \lambda^* \lambda\right)\right]$$

Using

$$\int_{t_a}^{t_b} dt \, \dot{\lambda}^* \lambda = \lambda_b^* \lambda_b - \lambda_a^* \lambda_a - \int_{t_a}^{t_b} dt \, \lambda^* \dot{\lambda}$$

eq(4.92a) becomes

$$\langle \lambda_b, t_b \mid \lambda_a, t_a \rangle = \exp\left[\frac{1}{2} (\lambda_b \lambda_b^* - \lambda_a \lambda_a^*)\right] \qquad (4.92)$$

$$\times \int_{\lambda_a}^{\lambda_b} \mathcal{D} \lambda \exp\left[-\int_{t_a}^{t_b} dt \left(\lambda^* \dot{\lambda} + i \omega \lambda^* \lambda\right)\right]$$

Using

$$\int_{t_a}^{t_b} dt \frac{d}{dt} (\lambda^* \lambda) = \lambda_b^* \lambda_b - \lambda_a^* \lambda_a$$

$$= \int_{t_a}^{t_b} d\,t \left(\dot{\lambda}^* \lambda + \lambda^* \dot{\lambda}\right) \qquad (4.93)$$

eq(4.92) becomes

$$\langle \lambda_b, t_b \mid \lambda_a, t_a \rangle = \int_{\lambda_a}^{\lambda_b} \mathcal{D}\,\lambda \exp\left\{ \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} \left(\dot{\lambda}^* \lambda - \lambda^* \dot{\lambda}\right) - i\,\omega\,\lambda^* \lambda \right] \right\} \quad (4.94)$$

Analogous to eq(4.71), we set

$$\lambda = \sqrt{\frac{m\,\omega}{2\,\hbar}} \left(q + i\,\frac{p}{m\,\omega}\right) \qquad \lambda^* = \sqrt{\frac{m\,\omega}{2\,\hbar}} \left(q - i\,\frac{p}{m\,\omega}\right) \qquad (4.95)$$

$$\rightarrow \qquad J = \det \begin{vmatrix} \frac{\partial \lambda^*}{\partial q} & \frac{\partial \lambda^*}{\partial p} \\ \frac{\partial \lambda}{\partial q} & \frac{\partial \lambda}{\partial p} \end{vmatrix} = \frac{m\,\omega}{2\,\hbar} \det \begin{vmatrix} 1 & -\frac{i}{m\,\omega} \\ 1 & \frac{i}{m\,\omega} \end{vmatrix} = \frac{i}{\hbar}$$

Eq(4.89) thus becomes

$$\mathcal{D}\,\lambda = \prod_{j=1}^{N-1} \frac{d\,p_j\,d\,q_j}{2\,\pi\,\hbar} \equiv \mathcal{D}\,p\,\mathcal{D}\,q \qquad (4.97)$$

Using

$$\dot{\lambda}^* \lambda - \lambda^* \dot{\lambda} = \frac{m\,\omega}{2\,\hbar}\left[\left(\dot{q} - i\,\frac{\dot{p}}{m\,\omega}\right)\left(q + i\,\frac{p}{m\,\omega}\right) - \left(q - i\,\frac{p}{m\,\omega}\right)\left(\dot{q} + i\,\frac{\dot{p}}{m\,\omega}\right) \right]$$

$$= \frac{i}{\hbar}\left(\dot{q}\,p - \dot{p}\,q\right)$$

$$\lambda^* \lambda = \frac{m\,\omega}{2\,\hbar}\left(q^2 + \frac{p^2}{m^2\,\omega^2}\right)$$

eq(4.94) becomes

$$\langle \lambda_b, t_b \mid \lambda_a, t_a \rangle = \int_{q_a,p_a}^{q_b,p_b} \mathcal{D}\,p\,\mathcal{D}\,q\,e^{i\,S/\hbar} \qquad (4.96a)$$

where

$$S(p, q) = \int_{t_a}^{t_b} d\,t \left[\frac{1}{2}\left(\dot{q}\,p - \dot{p}\,q\right) - \frac{m\,\omega^2}{2}\left(q^2 + \frac{p^2}{m^2\,\omega^2}\right) \right]$$

$$= \int_{t_a}^{t_b} d\,t \left[\frac{1}{2}\left(\dot{q}\,p - \dot{p}\,q\right) - \frac{1}{2}\,m\,\omega^2\,q^2 - \frac{p^2}{2\,m} \right] \qquad (4.96)$$

$$= \int_{t_a}^{t_b} d\,t \left[\frac{1}{2}\left(\dot{q}\,p - \dot{p}\,q\right) - H \right]$$

$$= -\frac{1}{2}\left(q_b\,p_b - q_a\,p_a\right) + \int_{t_a}^{t_b} d\,t\left(\dot{q}\,p - H\right)$$

$$= -\frac{1}{2}\left(q_b\,p_b - q_a\,p_a\right) + \int_{t_a}^{t_b} d\,t\,\mathcal{L}$$

Eq(4.96a) becomes

$$\langle \lambda_b, t_b \mid \lambda_a, t_a \rangle = \exp\left[\frac{i}{2\,\hbar}\left(q_a\,p_a - q_b\,p_b\right)\right] \qquad (4.98)$$

$$\times \int_{q_a,p_a}^{q_b,p_b} \mathcal{D}\,p\,\mathcal{D}\,q \exp\left[\frac{i}{\hbar} \int_{t_a}^{t_b} d\,t\,\mathcal{L} \right]$$

Do Ex.4.12.